Modular Curves (TCC) Problem Sheet 3

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This is the last of 3 problem sheets, which will be distributed after lectures 3, 6, and 8 of the course. This problem sheet will be marked out of a total of 25; the number of marks available for each question is indicated.

Work should be submitted, on paper or by email, on or before Tuesday 1st April.

Throughout this sheet, all rings are assumed to be commutative and unital.

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1. [3 points] Let C be the scheme $\operatorname{Spec} k[X,Y]/\{XY=0\}$, where k is a field. Give an example to show that the structure map $C \to \operatorname{Spec} k$ does not satisfy the functorial criterion of smoothness.

2. [3 points] Let $c > 1$ be an integer coprime to 6, and let ${}_c\theta(z,\tau)$ be the meromorphic function defined for $z \in \mathbf{C}$ and $\tau \in \mathcal{H}$ by
$${}_c\theta(z,\tau) = q^{\frac{c^2-1}{12}}(-t)^{\frac{c^2-c}{2}}\gamma_q(t)^{c^2}\gamma_q(t^c)^{-1},$$
where $q = e^{2\pi i z}$, $q = e^{2\pi i \tau}$, and
$$\gamma_q(t) = \prod_{n\ge 0}(1-q^n t)\prod_{n \ge 1}(1-q^n t^{-1}).$$
Show that:
   (a) ${}_c\theta(z,\tau)$ depends only on the class of z modulo $\mathbb{Z} + \mathbb{Z}\tau$;
   (b) for τ fixed, the divisor of ${}_c\theta(z,\tau)$ as a meromorphic function on $E_\tau = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ is $c^2(0) - E_\tau[c]$;
   (c) for any N coprime to c we have
$$\prod_{\substack{y \in E_\tau \\ Ny = z}} {}_c\theta(y,\tau) = {}_c\theta(z,\tau).$$

3. For integers c, N with $(c, 6N) = 1$ let ${}_cg_N(\tau)$ be the meromorphic function on $Y_1(N)$ defined by
$${}_cg_N(\tau) = {}_c\theta(1/N, \tau).$$
   (a) [1 point] Show that the q-expansion coefficients of ${}_cg_N$ lie in $\mathbb{Z}[\zeta_N]$, and satisfy
$$a_n({}_cg_N)^\sigma = a_n\left(\langle \chi(\sigma)\rangle\, {}_cg_N\right)$$
   for all $\sigma \in \operatorname{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$, where χ is the mod N cyclotomic character and $\langle d \rangle$ is the diamond operator.
   (b) [2 points] Show that ${}_cg_N$ generates $\mathbb{Q}(Y_1(N))$ as a field extension of $\mathbb{Q}(Y_0(N))$.
   (c) [2 points] Show that we have an equality of differentials
$$\operatorname{dlog}({}_cg_N) = -2\pi i\left(c^2 F^{(2)}_{1,N} - F^{(2)}_{c,N}\right)\mathrm{d}z,$$
   where $F^{(2)}_{b,N}$, for non-zero $b \in \mathbb{Z}/N\mathbb{Z}$, is the weight 2 Eisenstein series
$$F^{(2)}_{b,N}(q) = \frac{-1}{12} + \sum_{n \ge 1}\left(\sum_{d \mid n}\frac{n}{d}(\zeta_N^{bd} + \zeta_N^{-bd})\right)q^n.$$

4. [1 point] Let X be a quasiprojective S-scheme, for S some scheme, equipped with a finite group G acting on X by S-scheme automorphisms. Is the functor $\underline{Sch/S} \to \underline{Sets}$ defined by

$$Y \mapsto \{\text{homs of } S\text{-schemes } Y \to X \text{ commuting with } G\text{-action}\}$$

representable?

5. Recall that, for S a scheme, $\underline{Sch/S}$ denotes the "slice category" whose objects are pairs consisting of a scheme T and a morphism of schemes $T \to S$ (and whose morphisms are the obvious ones: morphisms of schemes $T \to U$ commuting with the morphism to S). There is a natural "forgetful functor" $\underline{Sch/S} \to \underline{Sch}$.

   (a) [1 point] Show that for any two schemes S, T, there is a canonical bijection between the following two sets:
   - morphisms of schemes $S \to S'$;
   - functors $\underline{Sch/S} \to \underline{Sch/S'}$ commuting with the forgetful map to $\underline{Sch}$.

   (b) [1 point] Show that for any scheme S there is a canonical bijection between
   - elliptic curves over S,
   - functors $\underline{Sch/S} \to \underline{Ell/\mathbb{Z}}$ commuting with the forgetful map to $\underline{Sch}$.

   (* This justifies thinking of $\underline{Ell/\mathbb{Z}}$ as *the category of S'-schemes, for a non-existent S' representing the $S \to \{$elliptic curves over $S\}$*. Can you construct a similar category which is "the universal elliptic curve over $\underline{Ell/\mathbb{Z}}$"? *)

6. Let $\mathcal{P}$ be a moduli problem on $\underline{Ell/R}$, for some ring R, and let $\widetilde{\mathcal{P}}$ be the associated contravariant functor $\underline{Sch/R} \to \underline{Sets}$.

   (a) [2 points] Show that

$$(\mathcal{P} \text{ is representable}) \Leftrightarrow (\mathcal{P} \text{ is rigid and } \widetilde{\mathcal{P}} \text{ is representable}).$$

   (b) [2 points] Assume that there exists an elliptic curve E_0 over $\operatorname{Spec} R$. Let $\mathcal{P}$ be the following (rather pathological) moduli problem:

$$\mathcal{P}(E/S) = \begin{cases} \text{the set with one element,} & \text{if } E \text{ is isomorphic over } S \text{ to } E_0 \times_R S \\ \varnothing, & \text{otherwise.} \end{cases}$$

   Show that $\mathcal{P}$ is not rigid, but $\widetilde{\mathcal{P}}$ is representable (by $\operatorname{Spec} R$ itself).

   [*This counterexample is due to Ofer Gabber and is in section A.4 of Katz–Mazur.*]

7. Recall the series $X(u,q), Y(u,q)$ defined in the section on the Tate curve.

   (a) [2 points] Verify that we have

$$\frac{\frac{\mathrm{d}}{\mathrm{d}u} X(u,q)}{2Y(u,q) + X(u,q)} = \frac{1}{u}.$$

   (b) [1 point] Why do the projective coordinates

$$((1-u)^3 X(u,q) : (1-u)^3 Y(u,q) : (1-u)^3)$$

   not define a morphism of schemes over $R = \mathbb{Z}[[q]]$ from $\mathbf{G}_m/R$ to $\operatorname{Tate}(q)$? *[Hint: Convince yourself that $\mathbb{Z}[u][[q]]$ is not the same ring as $\mathbb{Z}[[q]][u]$.]*

   (c) (*) If you know about formal schemes, convince yourself that this formula *does* define a morphism of formal schemes over R from $\widehat{\mathbf{G}_m}$ to the formal completion of $\operatorname{Tate}(q)$ at infinity, and the pullback of the invariant differential $\frac{\mathrm{d}X}{2Y+X}$ on $\operatorname{Tate}(q)$ is the invariant differential $\frac{\mathrm{d}u}{u}$ on $\widehat{\mathbf{G}_m}$. *[Hint: Convince yourself that $\mathbb{Z}[[u]][[q]]$ **is** the same ring as $\mathbb{Z}[[q]][[u]]$.]*

8. [4 points] List the eight cusps of $Y(3)$, and the action of $\operatorname{Gal}(\mathbb{Q}(\zeta_3)/\mathbb{Q})$ on them.