Modular Curves (TCC) Problem Sheet 1 – Solutions

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This is the first of 3 problem sheets, which will be distributed after lectures 3, 5, and 7 of the course. This problem sheet will be marked out of a total of 25; the number of marks available for each question is indicated.

Work should be submitted, on paper or by email, on or before Friday 14th February.

Throughout this sheet a *level* should be understood to mean a finite-index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

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1. [2 points] Let X be a topological space and G a group acting on X by homeomorphisms (i.e. for any $g \in G$, the map $X \to X$ given by g is a homeomorphism). Equip $G\backslash X$ with the quotient topology. Show that the image of an open subset of X is an open subset of $G\backslash X$. Is the same statement true with 'open' replaced by 'closed'? Give a proof or counterexample as appropriate.

Solution: Let π be the projection map $X \to G\backslash X$. By the definition of the quotient topology, a subset V of $G\backslash X$ is open if and only if $\pi^{-1}V$ is open in X. But for $U \subset X$ open, $\pi^{-1}(\pi(U)) = \bigcup_{g\in G} gU$ is a union of open sets in X, hence it is open.

The corresponding result for closed sets is false, since an arbitrary union of closed sets is not necessarily closed. For instance, let $X = \mathbb{R}$, $G = \mathbb{Z}$ acting by translations, and let U be the subset $\{n + 1/n : n \ge 1 \in \mathbb{Z}\}$.

2. [3 points] Let Γ be a level and let $\gamma_1, \dots, \gamma_r$ be such that $\mathrm{SL}_2\,\mathbb{Z} = \bigsqcup_j \Gamma\gamma_j$. Show that $[\gamma_j i] \in Y(\Gamma)$ is elliptic if and only if $\gamma_j \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \gamma_j^{-1} \in \Gamma$. Hence show that $Y_0(p)$, for p an odd prime, has either 0 or 2 elliptic points of order 2 depending on the congruence class of p modulo 4.

Solution: The stabiliser of $\gamma_j i$ in $\mathrm{SL}_2\,\mathbb{Z}$ is just $\gamma_j \,\mathrm{Stab}_{\mathrm{SL}_2\mathbb{Z}}(i)\gamma_j^{-1}$, which is the cyclic group generated by $\gamma_j \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \gamma_j^{-1}$. The orbit of $\gamma_j i$ is an elliptic point of $Y(\Gamma)$ if and only if this subgroup is contained in Γ, i.e. if and only if $\gamma_j \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \gamma_j^{-1} \in \Gamma$.

In the case $\Gamma = \Gamma_0(p)$ we may take a set of coset representatives to be $\gamma_j = \begin{pmatrix} 1 & 0 \\ j & 1 \end{pmatrix}$ for $0 \le j < p$ and $\gamma_p = \begin{pmatrix} 0 & - \\ 1 & 1 \end{pmatrix} 0$. Then $\gamma_j \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \gamma_j^{-1}$ is $\begin{pmatrix} j & -1 \\ 1+j^2 & -j \end{pmatrix}$, which is in Γ if and only if $j < p$ and $1 + j^2 = 0 \bmod p$. So if $p = 3 \bmod 4$ there are no elliptic points of order 2, while if $p = 1 \bmod 4$ there is at least one.

To see that there are two when $p = 1 \bmod 4$, we must check that the elliptic points $\gamma_j i$ and $\gamma_{-j} i$, where j is the square root of -1 modulo p, are distinct; but if $\gamma_j i$ and $\gamma_{-j} i$ (where $-j$ is taken modulo p) are in the same orbit, then we must have $\gamma_{-j} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \gamma_j \in \Gamma$. But this matrix has bottom left corner $1 - j^2 = 2 \neq 0$.

3. [2 points] Calculate the genus of $X_0(17)$. (You may assume an analogue of the preceding question for points of order 3 as long as it is stated clearly.)

Solution: The analogue of the previous question is as follows: for $p \geq 5$, the curve $Y_0(p)$ has 2 elliptic points if $p = 1 \bmod 3$ and 0 if $p = 2 \bmod 3$.

Now, since 17 is 1 mod 4 and 2 mod 3, we have $\epsilon_2 = 2, \epsilon_3 = 0, \epsilon_\infty = 2$, and $N = [\mathrm{PSL}_2 \mathbb{Z} : \overline{\Gamma}] = 18$ in the notation of Corollary 1.3.5. This gives genus $1 + \frac{18}{12} - \frac{2}{4} - \frac{0}{3} - \frac{2}{2} = 1$.

4. [1 point] Does there exist a level Γ which is neat (i.e. Γ has no elements of finite order and all cusps are regular), but $X(\Gamma)$ has genus 0?

Solution: Yes, such levels exist; $\Gamma_1(5)$ is one such.

5. [3 points] Let $\Gamma = \Gamma_0(5)$. Show that $S_4(\Gamma)$ is one-dimensional, and that if F is a basis vector of $S_4(\Gamma)$, then multiplication by F is an isomorphism $M_k(\Gamma) \to S_{k+4}(\Gamma)$ for all $k \in \mathbb{Z}$.

Solution: We have $\varepsilon_2 = 2$, $\varepsilon_3 = 0$, $\varepsilon_\infty = 2$ and $N = 6$. In particular, the genus is 0.

Moreover, since there are no elliptic points of order 4, the sheaves ω_k satisfy $\omega_{k+4} = \omega_k \otimes \omega_4$ for all $k \in \mathbb{Z}$. So it suffices to show that $\omega_4(-\text{cusps})$ has a nowhere-vanishing holomorphic section. By Riemann-Roch, this holds if and only if $\omega_4(-\text{cusps})$ has degree 0.

However, we know $\omega_2(-\text{cusps}) \cong \Omega^1$ has degree $2g - 2 = -2$, so ω_2 has degree 0 and thus ω_2^2 also degree 0. Comparing local bases at the elliptic points, $\omega_2^2 = \omega_4(-\text{ell.pts})$, and there are 2 elliptic points and 2 cusps, so we have $\deg \omega_4(-\text{cusps}) = \deg \omega_4(-\text{ell.pts}) = 0$ as required.

6. [3 points] Let X be a compact Riemann surface, $\mathcal{L}$ an invertible sheaf on X, and $P_1, \ldots, P_n$ any finite set of points on X. Show that there exists a meromorphic section of $\mathcal{L}$ which is holomorphic and non-vanishing at all the P_j.

Solution: This is an easy consequence of the Riemann-Roch theorem. Let us choose a point Q which is not one of the P_j. The sheaf $\mathcal{L}(nQ)$ has degree $\deg \mathcal{L} + n$, so for $n \gg 0$, it has a global section by Riemann–Roch.

If we choose n so that $\mathcal{L}(nQ)$ has degree $> 2g - 1$, then for each j, the dimension of $H^0(X, \mathcal{L}(nQ - P_j))$ is one less than the dimension of $H^0(X, \mathcal{L}(nQ))$. So the sections in $H^0(X, \mathcal{L}(nQ))$ that vanish at one or more of the P_j are a finite union of proper subspaces of $H^0(X, \mathcal{L}(nQ))$, and thus we may take a section that is not in any of them. This section is holomorphic away from Q, and has a pole of degree $\leq n$ at Q, so it is in particular a meromorphic section of $\mathcal{L}$; and by construction it does not vanish at any of the P_j.

7. [3 points] Let $f : X \to Y$ be a non-constant morphism of Riemann surfaces. Let $P \in X$ and $Q = f(P) \in Y$. Show that if ω is a differential on Y which is holomorphic and nonvanishing at Q, then the pullback $f^*\omega$ vanishes to order $e_P(f) - 1$ at P.

Hence deduce the Riemann-Hurwitz formula, assuming that the sheaf of holomorphic differentials on a compact Riemann surface of genus g has degree $2g - 2$.

Solution: The first part of the question is a purely local statement, so without loss of generality we may assume that X and Y are open subsets of $\mathbb{C}$, and that $\omega = g(z)\,\mathrm{d}z$ for some $g : Y \to \mathbf{C}$ which is holomorphic and nonvanishing at Q. We then have $f^*\omega = g(f(z))f'(z)\mathrm{d}z$, so the order of vanishing of $f^*\omega$ at P is the same as that of $f'(z)$, which is $e_P(f) - 1$.

Now assume X, Y compact and let ω be a meromorphic differential on Y. By the previous question, we may assume that ω is holomorphic and nonvanishing at each of the ramification points of f. It then follows easily that

$$\sum_{P \in f^{-1}(Q)} v_P(f^*\omega) = N v_Q(\omega) + \sum_{P \in f^{-1}(Q)} (e_P(f) - 1).$$

Summing over all $Q \in Y$ gives

$$\left(\sum_{P \in X} v_P(f^*\omega)\right) = N \cdot \left(\sum_{Q \in Y} v_Q(\omega)\right) + \sum_{P \in X} (e_P(f) - 1).$$

Since ω and $f^*\omega$ are nonzero meromorphic differentials on Y and X respectively, we have $\left(\sum_{Q \in Y} v_Q(\omega)\right) = 2g(Y) - 2$ and similarly for $f^*\omega$, so this is precisely the Riemann–Hurwitz formula.

8. [4 points] Let X be a Riemann surface and ω a meromorphic differential on X. Define the *residue* of ω at a point $P \in X$. Show that if X is compact, then $\sum_{P \in X} \mathrm{Res}_P(\omega) = 0$. (Hint: Use Stokes' theorem.)

Solution: The definition of the residue is the same as in classical complex analysis: just choose a coordinate chart and evaluate the residue in that chart. The chain rule shows that the residue is independent of the choice of chart.

Now, let $P_1, \ldots, P_r$ be the poles of ω. For each i, let D_i be a small neighbourhood of P_i (small enough to be contained in a single coordinate chart) and let C_i be the path in X given by the boundary of D_i oriented anticlockwise. By the residue theorem applied on coordinate charts covering each D_i, we have

$$\sum_i \oint_{C_i} \omega = 2\pi i \sum_j \mathrm{Res}_{P_j} \omega.$$

But $\bigcup C_i$ is the boundary of the complement Y of the D_i in X, with orientation reversed; and applying Stokes' theorem to Y we obtain

$$\sum_i \oint_{C_i} \omega = -\int_{\partial Y} \omega = -\iint_Y d\omega.$$

The 2-form $d\omega$ is zero, by the Cauchy–Riemann equations, so this integral is 0.

9. [4 points] Let Γ be a level, and let $\rho : \mathcal{H} \to \mathcal{H}$ be a homeomorphism such that $\rho(z) \in \Gamma z$ for every $z \in \mathcal{H}$. Show that $\rho(z) = \gamma z$ for a unique $\gamma \in \bar{\Gamma}$ (the image of Γ in $\mathrm{PSL}_2\,\mathbb{Z}$).

Suppose $\bar{\Gamma}$ has no elements of finite order. Show that the fundamental group of $Y(\Gamma)$ (in the sense of algebraic topology) is isomorphic to $\bar{\Gamma}$.

Solution: For the first statement: let $\mathcal{H}'$ be the complement of the elliptic Γ-orbits in $\mathcal{H}$. Then ρ must preserve $\mathcal{H}'$, and for each $z \in \mathcal{H}'$, there is a *unique* $\gamma_z \in \bar{\Gamma}$ such that $\rho(z) = \gamma_z z$.

This defines a map $\mathcal{H}' \to \bar{\Gamma}$ and, since ρ is continuous and Γ acts properly discontinuously on $\mathcal{H}$, this map must be locally constant. The space $\mathcal{H}'$ is obviously connected, so it is constant, and there is a unique $\gamma \in \bar{\Gamma}$ such that $\rho(z) = \gamma z$ for all $z \in \mathcal{H}'$. By continuity this holds for the points in $\mathcal{H} \setminus \mathcal{H}'$ also.

Now, a standard construction in algebraic topology shows that for any locally path-connected topological space Y, there is a unique (up to homeomorphism) *universal cover* of Y, which is a simply connected topological space $\tilde{Y}$ with a continuous map $\tilde{Y} \to Y$ which is a covering (i.e. locally a homeomorphism); and $\pi_1(Y)$ is then isomorphic to the group of deck transformations of $\tilde{Y}$ (homeomorphisms from $\tilde{Y}$ to itself preserving the map to Y).

If $\bar{\Gamma}$ has no elements of finite order, then $\mathcal{H} \to Y(\Gamma)$ is a covering. Since $\mathcal{H}$ is simply-connected, $\mathcal{H}$ must be the universal cover of $Y(\Gamma)$. The first part of the question shows that the group of deck transformations of $\mathcal{H}$ is just $\bar{\Gamma}$, so we are done.

The following questions are optional and wil not be assessed.

10. Show that for any Γ there exists an R such that the graded ring $\bigoplus_{k \geq 0} M_k(\Gamma)$ is generated by forms of weight $\leq R$. What is the best bound you can find for R?

11. Does there exist a proper finite-index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ with only one cusp?

12. Show that there are infinitely many levels such that $X(\Gamma)$ has genus 1. (Hint: Consider subgroups of $\Gamma_0(11)$.)