TCC Modular Forms and Representations of GL_2: Assignment #1

David Loeffler, d.a.loeffler@warwick.ac.uk

18th October 2018

This is the first of 3 problem sheets for this course. Questions *not* marked $*$ are assessed, and students taking this course for credit should submit their solutions to me (by email, or via my pigeonhole for Warwick students) by **noon on Monday 5th November**. Solutions will be distributed that afternoon, so late submissions will not be accepted. The total mark available for the assessed questions is 20.

Questions marked with one or more $*$'s are included for your own interest, and will not be given a numerical mark, but if you would like some (brief) feedback on your answers you are welcome to submit them to me anyway. The number of stars is intended as a rough indication of difficulty.

Throughout the sheet, F denotes a nonarchimedean local field with ring of integers $\mathcal{O}$ and residue field of cardinality q; ϖ denotes an arbitrary uniformiser of $\mathcal{O}$; and $|x|$, for $x \in F$, denotes the absolute value normalised by $|\varpi| = 1/q$.

---

1. [1 point] Show that in a topological group, every open subgroup is also closed.
2. [1 point] Show that if G is locally profinite and compact, then every open neighbourhood of 1_G contains a normal open compact subgroup.
3. [3 points] Let $G = \mathbf{Z}_p^\times$, for p prime, and let $V = \{\text{locally constant functions } G \to \mathbf{C}\}$.
   (a) Show that V is smooth and admissible.
   (b) Show that V has infinitely many (distinct) irreducible subrepresentations.
   (c) Show that the abstract dual V^* is not smooth.
   (d) [*] Show that the smooth dual $V^\vee$ is (non-canonically) isomorphic to V.
4. [2 points] Let G be a locally profinite group. Show that the natural inclusion of $\underline{\mathrm{Smo}}_G$ in $\underline{\mathrm{Rep}}_G$ has a right adjoint (and describe the adjoint functor).
5. (a) [1 point] Show that every compact open subgroup of $\mathrm{GL}_2(F)$ is conjugate to a subgroup of $\mathrm{GL}_2(\mathcal{O})$.
   (b) [**] Show that this is not true for SL_2 in place of GL_2.
6. [2 points] Show that $\mathrm{GL}_n(F)$ is unimodular for all $n \geqslant 1$.
7. Let G be locally profinite, and let μ be a left Haar measure on G. For $f \in C_c^\infty(G)$ and $g \in G$, write $g \cdot f$ for the function $x \mapsto f(xg)$.
   (a) [1 point] Show that the linear functional $\lambda : C_c^\infty(G) \to \mathbf{C}$ defined by
   $$\lambda(f) = \int_{x \in G} \delta_G(x)^{-1} f(x) \, \mathrm{d}\mu(x)$$
   satisfies $\lambda(g \cdot f) = \lambda(f)$ for all $f \in C_c^\infty(G)$ and $g \in G$.
   (b) [2 points] Show that if λ' is a linear map $C_c^\infty(G) \to \mathbf{C}$ such that $\lambda'(g \cdot f) = \lambda'(f)$ for all $f \in C_c^\infty(G)$ and $g \in G$, then there is a constant $\alpha \in \mathbf{C}$ such that $\lambda' = \alpha\lambda$.

8. (a) [*] Show that any open normal subgroup of $\mathrm{GL}_2(F)$ must contain $\mathrm{SL}_2(F)$.
   (b) [*] Hence show that a finite-dimensional irreducible smooth representation of $\mathrm{GL}_2(F)$ must be of the form $\chi \circ \det$, for some smooth $\chi : F^\times \to \mathbf{C}^\times$.

9. A smooth representation of a locally profinite group G is said to be *unitarizable* if there is a pairing $\langle -, - \rangle : V \to \mathbf{C}$ such that the following conditions hold:
   - V is linear in the second variable and conjugate-linear in the first;
   - we have $\langle gx, gy \rangle = \langle x, y \rangle$ for all $x, y \in V$ and $g \in G$;
   - $\langle y, x \rangle = \overline{\langle x, y \rangle}$;
   - $\langle x, x \rangle > 0$ if $x \neq 0$.

   (a) [*] Show that if V is smooth, admissible, and unitarizable, any G-invariant subspace $W \subseteq V$ has a G-invariant complement.
   (b) [1 point] Show that if $H \leqslant G$ is closed and $W \in \underline{\mathrm{Smo}}_G$ is unitarizable, then the normalised induction $V = \text{c-Ind}_H^G \left(W \otimes (\delta_H^{-1} \delta_G)^{1/2} \right)$ is unitarizable.
   (c) [2 points] Show that if the representation $I(\chi, \psi)$ of $\mathrm{GL}_2(F)$ is unitarizable, then the character χ/ψ must be either unitary, or real-valued. (Hint: Consider the representation $I(\overline{\chi}, \overline{\psi})$.)
   (d) [**] Show that the Steinberg representation of $\mathrm{GL}_2(F)$ is unitarizable.

10. Prove the two parts of the first lemma from §2.3, concerning representations of the additive group $N \cong (F, +)$:
   (a) [2 points] If $V \in \underline{\mathrm{Smo}}_N$, then the kernel of $V \mapsto V_N$ coincides with the space
   $$\left\{ v \in V : \int_{N_0} n \cdot v \, \mathrm{d}\mu_N(n) = 0 \quad \text{for some open compact } N_0 \subset N \text{ (depending on } v) \right\}$$
   (b) [2 points] The functor $V \mapsto V_N$ is exact on $\underline{\mathrm{Smo}}_N$.

11. [*] Let χ, ψ be smooth characters of $F^\times$ with $\chi \neq \psi$. We saw in lectures that $\mathrm{Hom}_G(I(\chi, \psi), I(\psi, \chi))$ is 1-dimensional. In this exercise we'll construct an explicit homomorphism between the two (in a special case).
   (a) Show that if the inequality $|\chi(\varpi)/\psi(\varpi)| < 1$ holds for one uniformiser ϖ, this holds for every uniformiser ϖ.
   (b) Suppose the inequality of (a) holds, and let $f \in I(\chi, \psi)$. Show that the integral
   $$\tilde{f}(g) = \int_N f\left(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) ng \right) \, \mathrm{d}\mu_N(n)$$
   converges absolutely. You may find it helpful to note the matrix identity
   $$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -x^{-1} & 1 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}.$$
   (c) Show that in the situation of (a), the map $f \mapsto \tilde{f}$ is a non-zero element of $\mathrm{Hom}_G(I(\chi, \psi), I(\psi, \chi))$.
   (d) [**] What is happening if $\chi/\psi = |\cdot|$?

12. [**] Let G be locally profinite, V an irreducible smooth representation of G, and $H \leqslant G$ an open normal subgroup such that $G/H \cong (\mathbf{Z}/2)^n$ for some n.
   (a) Show that $V|_H$ is a direct sum of finitely many irreducible H-subrepresentations, and the number of these factors divides 2^n. (Hint: Use induction to reduce to the case $n = 1$, and consider the subspace $g \cdot W$ where W is an irreducible H-subrepresentation and $g \in G - H$.)
   (b) Show that $V|_H$ is irreducible if and only if there is no nontrivial character $\chi : G/H \to \mathbf{C}^\times$ such that $V \cong V \otimes \chi$. (Hint: Show that $\mathrm{Ind}_H^G(V|_H) \cong \bigoplus_\chi V \otimes \chi$.)
   (c) Describe the decomposition into irreducible $\mathrm{SL}_2(F)$-representations of every irreducible, non-supercuspidal representation of $\mathrm{GL}_2(F)$. (Hint: $\mathrm{GL}_2(F)/(F^\times \cdot \mathrm{SL}_2(F))$ is finite.)
   (d) [***] Can you find an irreducible representation of $\mathrm{GL}_2(F)$ which decomposes into more than two $\mathrm{SL}_2(F)$ subrepresentations?