TCC Modular Forms and Representations of GL_2: Assignment #2

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This is the second of 3 problem sheets for this course, covering material from lectures 3, 4 and 5.

Questions *not* marked $*$ are assessed, out of a total of 20, and students taking this course for credit should submit their solutions to me (by email, or via my pigeonhole for Warwick students) by **noon on Friday 7th December**. Late submissions will not be accepted.

In questions 1–6, F is a nonarchimedean local field, and $\mathcal{O}$, ϖ, q etc have their usual meanings. From 7 onwards, F is a number field.

Questions marked with one or more $*$'s are included for your own interest, and will not be given a numerical mark, but if you would like some (brief) feedback on your answers you are welcome to submit them to me anyway. The number of stars is intended as a rough indication of difficulty.

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1. [2 points] Let G be locally profinite and $V \in \underline{\mathrm{Smo}}_G$. Show that if $F_1, F_2 \in \mathcal{H}(G)$ then $(F_1 \star F_2) \star v = F_1 \star (F_2 \star v)$. Hence show that $\mathcal{H}(G)$ is a ring.

2. Let G locally profinite, $K \leqslant G$ open compact. We say that $V \in \underline{\mathrm{Smo}}_G$ is K**-spherical** if V^K generates V as a G-representation.
   (a) [1 point] Show that if V is irreducible and K-spherical, then V^K is a simple $\mathcal{H}(G, K)$-module.
   (b) [*] Is $V \mapsto V^K$ an equivalence of categories between K-spherical representations of G and $\mathcal{H}(G, K)$-modules? Give a proof or counterexample as appropriate.

3. [2 points] In the notation of §3.2 of the lectures, prove the identity
$$T \star T = \left[K \left(\begin{smallmatrix} \varpi^2 & 0 \\ 0 & 1 \end{smallmatrix}\right) K\right] + (q+1)S.$$

4. Let χ, ψ be unramified characters of $F^\times$, for F a nonarchimedean local field, and I the Iwahori subgroup of $\mathrm{GL}_2(F)$ (cf. §3.4).
   (a) [2 points] Compute the matrix of $U = [I \left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right) I]$ on $I(\chi, \psi)^I$ in the basis (f_1, f_2), where $f_1(1) = 1, f_1(w) = 0$ and $f_2(w) = 1, f_2(1) = 0$.
   (b) [1 point] Hence show that U is not diagonalisable if $\chi = \psi$.
   (c) [1 point] Show that the I-invariants of the Steinberg representation are 1-dimensional. How does U act on $(\mathrm{St})^I$?

5. Let V be an irreducible infinite-dimensional representation of $\mathrm{GL}_2(F)$, and θ a smooth character of $(F, +)$ which is trivial on $\mathcal{O}$ but not on $\varpi^{-1}\mathcal{O}$.
   (a) [2 points] Justify the claim made in lectures that $v \in V$ is invariant under $\begin{pmatrix} \mathcal{O}^\times & \mathcal{O} \\ 0 & 1 \end{pmatrix}$ if and only if its Kirillov function ϕ_v is supported on $\mathcal{O}$ and constant on cosets of $\mathcal{O}^\times$.
   (b) [1 point] Show that $n \geqslant 1$, $v \in V^{U_n}$, and $v' = [U_n \left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right) U_n] \cdot v$, then for all $x \in \mathcal{O}$ we have
$$\phi_{v'}(a) = q\phi_v(\varpi a).$$

(c) [*] Show that for $v \in V$, the integral $I_s(v) = \int_{x \in F^\times} \phi_v(x)|x|^s \, dx$ converges for $\Re(s) \gg 0$, and has meromorphic continuation to all $s \in \mathbf{C}$ as a rational function of q^{-s}.

6. Let V be an irreducible infinite-dimensional representation of $\mathrm{GL}_2(F)$, and $\chi : F^\times \to \mathbf{C}^\times$ a smooth character.
   (a) [*] Show that $c(V \otimes \chi) \leqslant c(V) + 2c(\chi)$. (Hint: if v is the new vector of V, consider a sum of translates of v by elements of N.)
   (b) [**] Show that if $c(\chi) > c(V)$ then $c(V \otimes \chi) = 2c(\chi)$.

7. Let $N \geqslant 1$ and let χ be a homomorphism $(\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$ (a *Dirichlet character modulo* N).
   (a) [1 point] Show that there exists a unique smooth character $\underline{\chi} : \mathbf{Q}^\times_{>0} \setminus \mathbf{A}_f^\times \to \mathbf{C}^\times$ such that for almost all primes ℓ, the restriction of $\underline{\chi}$ to $\mathbf{Q}_\ell^\times$ is unramified and maps a uniformiser to $\chi(\ell)$.
   (b) [1 point] Show that the restriction of $\underline{\chi}$ to $\hat{\mathbf{Z}}^\times$ is given by the composition
$$\hat{\mathbf{Z}}^\times \twoheadrightarrow (\mathbf{Z}/N\mathbf{Z})^\times \xrightarrow{\chi^{-1}} \mathbf{C}^\times.$$
   (c) [*] If $p \mid N$ is prime, what is $\underline{\chi}(P)$, where P denotes the idèle which is p at the place p and 1 at all other places?

8. [2 points] Let F be a number field. If v is a (finite) prime of F, we denote by F_v the completion of F at v, and $\mathcal{O}_v$ the ring of integers of F_v. Show that for any given prime v of F, we may find an element γ of $\mathrm{SL}_2(F)$ such that
   - the image of γ in $\mathrm{SL}_2(F_v)$ lies in the double coset $\mathrm{SL}_2(\mathcal{O}_v) \begin{pmatrix} \varpi_v & \\ & \varpi_v^{-1} \end{pmatrix} \mathrm{SL}_2(\mathcal{O}_v)$;
   - the image of γ in $\mathrm{SL}_2(F_w)$ lies in $\mathrm{SL}_2(\mathcal{O}_w)$ for all primes $w \neq v$.

   Hence show that $\mathrm{SL}_2(F)$ is dense in $\mathrm{SL}_2(\mathbf{A}_{F,f})$.

   [Hint: the above double coset in $\mathrm{SL}_2(F_v)$ *also contains* $\begin{pmatrix} 1 & \varpi_v^{-1} \\ 0 & 1 \end{pmatrix}$*.]*

9. Let $N \geqslant 1$, and let $U = \{g \in \mathrm{GL}_2(\hat{\mathbf{Z}}) : g = \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \bmod N\}$ and $U' = \{g \in \mathrm{GL}_2(\hat{\mathbf{Z}}) : g = \left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right) \bmod N\}$.
   (a) [1 point] Show that both $Y(U)$ and $Y(U')$ are canonically isomorphic to the classical modular curve $Y_1(N) = \Gamma_1(N) \backslash \mathcal{H}$.
   (b) [1 point] Show that, for ℓ a prime, right-translation by $\begin{pmatrix} \varpi_\ell & 0 \\ 0 & \varpi_\ell \end{pmatrix} \in \mathrm{GL}_2(\mathbf{A}_f)$ acts as the diamond operator $\langle \ell \rangle$ on $Y(U)$, and as $\langle \ell \rangle^{-1}$ on $Y(U')$.

10. [2 points] Let $M_{k,t}$ be the $\mathrm{GL}_2(\mathbf{A}_f)$-representation of modular forms, as in Chapter 6 of the lectures. For $f \in M_{k,t}$, and $s \in \mathbf{R}$, consider the function $f_s : \mathrm{GL}_2(\mathbf{A}_f) \times \mathcal{H} \to \mathbf{C}$ defined by
$$f_s(g, \tau) = f(g, \tau) \| \det g \|^s.$$
    Here $\|x\| = \prod_\ell |x_\ell|$ is the normalised absolute value on $\mathbf{A}_f^\times$. Show that $f_s \in M_{k,t+s}$.

11. [*] Let $\mathcal{H}_\pm = \mathbf{C} - \mathbf{R}$ be the union of the upper and lower half-planes. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbf{R})$ and $f : \mathcal{H}_\pm \to \mathbf{C}$, write
$$(f \mid_{k,t} \gamma)(\tau) = |ad - bc|^t (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right).$$
    (a) Show that every $f \in M_{k,t}$ has a unique extension to a function on $\tilde{f} : \mathrm{GL}_2(\mathbf{A}_f) \times \mathcal{H}_\pm$ satisfying $\tilde{f}(\gamma g, -) = f(g, -) \mid_{k,t} \gamma^{-1}$ for all $\gamma \in \mathrm{GL}_2(\mathbf{Q})$.
    (b) Show that if $\tilde{f}$ is as in (a), then the function $F : \mathrm{GL}_2(\mathbf{A}) \to \mathbf{C}$ defined by $F(g) = \left(\tilde{f}(g_{\mathrm{fin}}, -) \mid_{k,t} g_\infty\right)(i)$ is invariant under left-translation by $\mathrm{GL}_2(\mathbf{Q})$, and satisfies
$$F(gu) = e^{ik\theta} F(g) \quad \text{for all} \quad u = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}_2(\mathbf{R}).$$