TCC Modular Forms and Representations of GL_2: Assignment #3

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This is the last of 3 problem sheets for this course, covering material from lectures 6, 7 and 8.

Questions *not* marked $*$ are assessed, out of a total of 20, and students taking this course for credit should submit their solutions to me (by email, or via my pigeonhole for Warwick students) by **noon on Tuesday 8th January 2019**. Late submissions will not be accepted.

Questions marked with one or more $*$'s are included for your own interest, and will not be given a numerical mark, but if you would like some (brief) feedback on your answers you are welcome to submit them to me anyway. The number of stars is intended as a rough indication of difficulty.

If $\Gamma \leqslant \mathrm{SL}_2(\mathbf{Z})$ and L is a subfield of $\mathbf{C}$, we define $M_k(\Gamma, L)$ to be the L-subspace of $M_k(\Gamma)$ consisting of forms with q-expansion coefficients in L, and similarly $S_k(\Gamma, L)$. For $f \in M_k(\Gamma)$ and $\sigma \in \mathrm{Aut}(\mathbf{C})$, we let f^σ be the formal q-expansion $\sum \sigma(a_n) q^n$, where $f = \sum a_n q^n$.

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1. [1 point] Prove the formula relating the global Kirillov function to q-expansions, $a_n(f(\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right), -)) = n^t \phi_f(nx)$.

2. [1 point] Let Π be a cuspidal automorphic representation of weight (k, t), and $f \in S_k(\Gamma_1(N))$ its normalised new vector. Show that if f transforms under the diamond operators $\langle d \rangle$ via the character $\varepsilon : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$, then the central character of the automorphic representation Π is the character $\| \cdot \|^{2t-k} \underline{\chi}$, where $\underline{\chi}$ is the adelic character attached to χ (as in Q7 of Sheet 1).

3. [*] Let χ be a quadratic Dirichlet character, and Π a cuspidal automorphic representation such that $\Pi = \Pi \otimes \chi$ *[NB: such examples do exist]*. Let $\chi' \neq \chi$ be another quadratic Dirichlet character. Show that the representation $\Pi' = \Pi \otimes \chi'$ satisfies $\Pi_\ell \cong \Pi'_\ell$ for a set of primes ℓ of density $\geqslant \frac{3}{4}$.

4. [2 points] Show (without using Shimura's rationality theorems) that if $f \in M_{k,t}$ then the function $f^*(g, \tau) = \overline{f(\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) g, -\bar{\tau})}$ is also in $M_{k,t}$, and that $(g \cdot f)^* = g \cdot f^*$.

5. [2 points] Let $f \in S_{k,t}(\mathbf{Q})$, for some $k, t \in \mathbf{Z}$, so that the Kirillov function ϕ_f of f takes values in $\mathbf{Q}_\infty$ and satisfies $\sigma(\phi_f(x)) = \phi_f(\chi(\sigma) x)$ for all $\sigma \in \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$. Show that the same is true of the Kirillov function of $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) f$, for any $a \in \mathbf{A}_f^\times, b \in \mathbf{A}_f$. *[You may* ***not*** *use Shimura's theorem that $S_{k,t}(\mathbf{Q})$ is $\mathrm{GL}_2(\mathbf{A}_f)$-stable.]*

6. [3 points] Let $N \geqslant 1$. Define
$$S'_k(\Gamma_1(N), \mathbf{Q}) = \left\{ f \in S_k(\Gamma_1(N), \mathbf{Q}(\zeta_N)) : f^\sigma = \langle \chi(\sigma) \rangle f \ \forall \sigma \in \mathrm{Gal}(\mathbf{Q}(\zeta_N)/\mathbf{Q}) \right\}.$$
   (a) Show that $S'_k(\Gamma_1(N), \mathbf{Q})$ spans $S_k(\Gamma_1(N))$ over $\mathbf{C}$.

   (b) Show that for any integer t the Atkin–Lehner operator W_N, defined by $W_N(f) = f|_{k,t} \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$, is a bijection
$$S_k(\Gamma_1(N), \mathbf{Q}) \cong S'_k(\Gamma_1(N), \mathbf{Q}).$$

   [Hint: Consider the group $\{\gamma \in \mathrm{GL}_2(\hat{\mathbf{Z}}) : \gamma = \left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right) \bmod N\}$.]

7. [4 points] Show that $X_0(16)$ has 6 cusps, of which 4 are defined over $\mathbf{Q}$. What is the field of definition of the remaining two?

8. [*] Let F be a nonarchimedean local field, and π_1, π_2 irreducible infinite-dimensional representations of $\mathrm{GL}_2(F)$. Let χ, ψ be any two characters of $F^\times$ such that $\chi\psi$ is the product of the central characters of the π_i. Show that there is a non-zero homomorphism of $\mathrm{GL}_2(F)$-representations $\pi_1 \otimes \pi_2 \to I(\chi, \psi)$. *[Hint: Consider first the case where at least one of the π_i is supercuspidal.]*

9. Recall the functions $f_\Phi(g, s)$ and $\tilde{f}_\Phi(g, s)$ defined in Jacquet's local Rankin–Selberg theory. *[The parameter s was omitted from the notation in the lecture, but we include it here.]*
   (a) [1 point] Show that if $\mathrm{Re}(s)$ is sufficiently large that $|q^{-2s}\omega(\varpi)| < 1$, then the integral defining $f_\Phi(g, s)$ converges for all g and Φ.
   (b) [1 point] Show that whenever $f_\Phi(g, s)$ is defined, we have $f_\Phi(-, s) \in I\left(|\cdot|^{s-\frac{1}{2}}, |\cdot|^{\frac{1}{2}-s}\omega^{-1}\right)$.
   (c) [*] Let $s_0 \in \mathbf{C}$. Show that the following are equivalent:
   - there exists some $\Phi \in C_c^\infty(F^2)$ and $g \in \mathrm{GL}_2(F)$ such that $f_\Phi(g, s)$ has a pole at $s = s_0$;
   - the representation $I\left(|\cdot|^{s_0-\frac{1}{2}}, |\cdot|^{\frac{1}{2}-s_0}\omega^{-1}\right)$ is reducible with a 1-dimensional subrepresentation.

   Show that if these conditions are satisfied, then the limit

   $$\lim_{s \to s_0} (s - s_0) \cdot f_\Phi(g, s)$$

   exists for all g and Φ, and as a function of g it lies in the 1-dimensional subrepresentation of $I(\dots)$.
   (d) [*] Use (c) to show that if at least one of π_1 and π_2 is supercuspidal, then $L(\pi_1 \times \pi_2, s)$ is identically 1 unless π_1 is isomorphic to a twist of π_2.

10. [2 points] Let F be a nonarchimedean local field. Let θ be a character $F \to \mathbf{C}^\times$ trivial on $\mathcal{O}$ but not on $\varpi^{-1}\mathcal{O}$, and let μ denote the Haar measure on F such that $\mu(\mathcal{O}) = 1$.
    (a) For $\phi \in C_c^\infty(F)$, define $\hat{\phi}$ by

    $$\hat{\phi}(x) = \int_F \phi(u)\theta(xu)\, \mathrm{d}\mu(u).$$

    Show that $\hat{\phi} \in C_c^\infty(F)$, and $\hat{\hat{\phi}}(x) = \phi(-x)$.
    (b) For $\Phi \in C_c^\infty(F^2)$, define $\hat{\Phi}$ by

    $$\hat{\Phi}(x, y) = \iint_{F \times F} \Phi(u, v)\theta(xv - yu)\, \mathrm{d}\mu(u)\mathrm{d}\mu(v).$$

    Show that $\hat{\hat{\Phi}} = \Phi$. *[Hint: $C_c^\infty(F^2)$ is spanned by functions of the form $\Phi(x, y) = \phi_1(x)\phi_2(y)$.]*

11. [3 points] Let $k \geqslant 0$ be an integer, $s \in \mathbf{C}$ with $\mathrm{Re}(s) > 1$, and $\Phi \in C_c^\infty(\mathbf{A}_f^2)$. Show that the Eisenstein series $E_\Phi^k(g, \tau, s)$ and $\tilde{E}_\Phi^k(g, \tau, s)$ transform like elements of $M_{k,k/2}$ under left translation by $\mathrm{GL}_2^+(\mathbf{Q})$. (You may assume that the sums concerned are absolutely convergent.)