TCC Modular Forms and Representations of GL_2: Assignment #2 (Solutions)

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1. [2 points] Let G be locally profinite and $V \in \underline{\mathrm{Smo}}_G$. Show that if $F_1, F_2 \in \mathcal{H}(G)$ then $(F_1 \star F_2) \star v = F_1 \star (F_2 \star v)$. Hence show that $\mathcal{H}(G)$ is a ring.

Solution: We compute

$$\begin{aligned}
(F_1 \star F_2) \star v &= \int_{g \in G} (F_1 \star F_2)(g)(g \cdot v)\, \mathrm{d}\mu(g) && \text{(def of star product)}\\
&= \int_{g \in G} \left(\int_{h \in G} F_1(h) F_2(h^{-1}g)\, \mathrm{d}\mu(h) \right) (g \cdot v)\, \mathrm{d}\mu(g) && \text{(def of } F_1 \star F_2\text{)}\\
&= \int_{h \in G} F_1(h) \left(\int_{g \in G} F_2(h^{-1}g)\, (g \cdot v) \mathrm{d}\mu(g) \right) \mathrm{d}\mu(h) && \text{(Fubini)}\\
&= \int_{h \in G} F_1(h) \left(\int_{g' \in G} F_2(g')\, (hg' \cdot v) \mathrm{d}\mu(g') \right) \mathrm{d}\mu(h) && \text{(translation-invariance)}\\
&= \int_{h \in G} F_1(h) h \cdot \left(\int_{g' \in G} F_2(g')\, (g' \cdot v) \mathrm{d}\mu(g') \right) \mathrm{d}\mu(h) && \text{(linearity)}\\
&= \int_{h \in G} F_1(h)\, (h \cdot (F_2 \star v))\, \mathrm{d}\mu(h) = F_1 \star (F_2 \star v). \quad \square
\end{aligned}$$

(Here the appeal to Fubini's theorem is overkill – the integral is just a finite sum, so the interchange of order of integration is obvious.)

Taking V to be $\mathcal{H}(G)$ itself with its left-regular action of G, we deduce that the star product is associative. Since all the other ring axioms are obvious, this shows that $\mathcal{H}$ is a ring.

2. Let G locally profinite, $K \leqslant G$ open compact. We say that $V \in \underline{\mathrm{Smo}}_G$ is K-**spherical** if V^K generates V as a G-representation.

   (a) [1 point] Show that if V is irreducible and K-spherical, then V^K is a simple $\mathcal{H}(G, K)$-module.

   Solution: It suffices to show that for any $w, v \in V^K$ with $w \neq 0$, the vector v lies in the $\mathcal{H}(G,K)$-span of w. Since V is irreducible, the G-translates of w span V, hence we can write v as a finite sum $v = \sum_{i=1}^{r} a_i g_i w$.
   For each $i \in \{1, \dots, r\}$, write the double coset KgK as a finite disjoint union $\bigsqcup_{j=1}^{n_i} K g_i h_{ij}$ for

$h_{ij} \in K$. Then we have

$$\begin{aligned}\sum_i \frac{a_i}{n_i}[Kg_iK]w &= \sum_i \frac{a_i}{n_i}\sum_j g_i h_{ij} w \\ &= \sum_i \frac{a_i}{n_i}\sum_j g_i w && \text{(as } w \in V^K) \\ &= \sum_i a_i g_i w = v.\end{aligned}$$

Thus $v \in \mathcal{H}(G,K)\cdot w$ as required.

3. [2 points] In the notation of §3.2 of the lectures, prove the identity

$$T \star T = \left[K \begin{pmatrix} \varpi^2 & 0 \\ 0 & 1 \end{pmatrix} K\right] + (q+1)S.$$

Solution: We know that $G = \bigsqcup_{\substack{a,b\in\mathbf{Z}\\ a\geqslant b}} K \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix} K$; and the support of $T\star T$ is clearly contained in the set of matrices lying in $M_{2\times 2}(\mathcal{O})$ and having determinant in $\varpi\mathcal{O}^\times$, so we must have

$$T\star T = c_1\left[K \begin{pmatrix} \varpi^2 & 0 \\ 0 & 1 \end{pmatrix} K\right] + c_2\left[K \begin{pmatrix} \varpi & 0 \\ 0 & \varpi \end{pmatrix} K\right]$$

for some constants c_1 and c_2.
Evaluating c_2 is easier: as shown in lectures we have

$$c_i = \tfrac{1}{\mu(K)}\mu\left(K \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix} K \cap \gamma_i K \begin{pmatrix} \varpi^{-1} & 0 \\ 0 & 1 \end{pmatrix} K\right)$$

where $\gamma_1 = \begin{pmatrix} \varpi^2 & 0 \\ 0 & 1 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} \varpi & 0 \\ 0 & \varpi \end{pmatrix}$. Since γ_2 is central (and K contains the permutation matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$) we have $\gamma_2 K \begin{pmatrix} \varpi^{-1} & 0 \\ 0 & 1 \end{pmatrix} K = K\gamma_2 \begin{pmatrix} \varpi^{-1} & 0 \\ 0 & 1 \end{pmatrix} K = K \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix} K$, so $\gamma_2 = \mu(K \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix} K)/\mu(K) = q+1$ (using the set of coset representatives given in lectures).
Some of you gave similar arguments for c_1, but it is easier to look at how everything acts on the trivial representation: $T\star T$ acts as multiplication by $(q+1)^2$, and $\left[K \begin{pmatrix} \varpi^2 & 0 \\ 0 & 1 \end{pmatrix} K\right]$ as multiplication by $q(q+1)$, so we deduce

$$q^2+2q+1 = (q^2+q)c_1 + (q+1) \Longrightarrow c_1 = 1.$$

An alternative, rather slick solution by Lambert A'Campo (Imperial) was to compare how both sides acted on $I(\chi,\psi)^K$, where χ,ψ are arbitrary unramified characters. If $\chi(\varpi)=\alpha, \psi(\varpi)=\beta$, then one finds that $K \begin{pmatrix} \varpi^2 & 0 \\ 0 & 1 \end{pmatrix} K$ acts as $q\alpha^2+q\beta^2+(q-1)\alpha\beta$, so we must have

$$q(\alpha+\beta)^2 = c_1\left(q\alpha^2+q\beta^2+(q-1)\alpha\beta\right) + c_2\alpha\beta$$

for all $\alpha,\beta \in \mathbf{C}^\times$, from which the result follows immediately.

4. Let χ,ψ be unramified characters of $F^\times$, for F a nonarchimedean local field, and I the Iwahori subgroup of $\mathrm{GL}_2(F)$ (cf. §3.4).
   (a) [2 points] Compute the matrix of $U = [I \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix} I]$ on $I(\chi,\psi)^I$ in the basis (f_1,f_2), where $f_1(1) = 1, f_1(w)=0$ and $f_2(w)=1, f_2(1)=0$.

(b) [1 point] Hence show that U is not diagonalisable if $\chi = \psi$.

Solution: Let $f \in I(\chi,\psi)^I$, and let $g = \left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We have $IgI = \bigsqcup_{a \in \mathcal{O}/\varpi\mathcal{O}} \left(\begin{smallmatrix} \varpi & a \\ 0 & 1 \end{smallmatrix}\right) I$, so

$$(U \cdot f)(x) = \sum_a (\left(\begin{smallmatrix} \varpi & a \\ 0 & 1 \end{smallmatrix}\right) \cdot f)(x) = \sum_a f\left(x \left(\begin{smallmatrix} \varpi & a \\ 0 & 1 \end{smallmatrix}\right)\right).$$

Thus

$$(U \cdot f)(1) = q \cdot \left(|\varpi|^{1/2}\chi(\varpi)\right) \cdot f(1) = q^{1/2}\alpha f(1).$$

On the other hand

$$(U \cdot f)(w) = \sum_a f\left(\left(\begin{smallmatrix} 0 & 1 \\ \varpi & a \end{smallmatrix}\right)\right).$$

The term for $a = 0$ this is just $q^{1/2}\beta f(w)$, whereas for $a \neq 0$ it is

$$f\left(\begin{pmatrix} \varpi & 1/a \\ 0 & 1 \end{pmatrix} \left(\begin{smallmatrix} -1/a & 0 \\ \varpi & a \end{smallmatrix}\right)\right) = q^{-1/2}\alpha f(1),$$

so $(U \cdot f)(w) = (q^{1/2} - q^{-1/2})\alpha f(1) + q^{1/2}\beta f(w)$. So in the basis (f_1, f_2) we have the matrix gives the matrix

$$\begin{pmatrix} q^{1/2}\alpha & 0 \\ (q^{1/2} - q^{-1/2})\alpha & q^{1/2}\beta \end{pmatrix}.$$

In particular, if $\alpha = \beta$ then this matrix has minimal polynomial $(X - \alpha)^2$ and is therefore not diagonalisable.

(c) [1 point] Show that the I-invariants of the Steinberg representation are 1-dimensional. How does U act on $(\mathrm{St})^I$?

Solution: Recall that the Steinberg representation is defined as the kernel of the map

$$I(|\cdot|^{1/2}, |\cdot|^{-1/2}) \to \mathbf{1}_G$$

(the trivial representation of G), given by integration over the quotient $B\backslash G$. Since I is compact, passing to I-invariants is an exact functor, so we have an exact sequence

$$0 \to \mathrm{St}^I \to I(|\cdot|^{1/2}, |\cdot|^{-1/2})^I \to (\mathbf{1}_G)^I \to 0.$$

In the notation of part (a) we have $\alpha = |\varpi|^{1/2} = q^{-1/2}, \beta = q^{1/2}$ *[this way round!]*; so we see that $I(|\cdot|^{1/2}, |\cdot|^{-1/2})^I$ is 2-dimensional, with U acting with eigenvalues 1 and q. On the other hand, $(\mathbf{1}_G)^I$ is clearly 1-dimensional and the action of U is given by summing q coset representatives each of which acts trivially; so U acts on $(\mathbf{1}_G)^I$ as multiplication by q. So we can conclude that St^I is 1-dimensional and U acts on it as the identity.

[Nobody got this question fully correct, so you should all go over your work carefully and make sure you understand where you went wrong. A common mistake was to guess that the map $I(|\cdot|^{1/2}, |\cdot|^{-1/2})^I \to \mathbf{1}_G$ was given by $f \mapsto f(1) + f(w)$ – it is in fact $f \mapsto f(1) + qf(w)$, but you don't need to know that.

One can also argue using the alternative description of St *as the quotient of $I(|\cdot|^{-1/2}, |\cdot|^{1/2}) = C^\infty(B\backslash G)$ by the subrepresentation of constant functions. For this approach, one needs to take $\alpha = q^{1/2}, \beta = q^{-1/2}$, and recognise that the subrepresentation we are* ***quotienting out by*** *is the span of $f_1 + f_2$, which is the $U = q$ eigenspace; so the other eigenspace — the span of f_2, on which U acts as $q^{1/2}\beta = 1$ — surjects onto St^I.]*

5. Let V be an irreducible infinite-dimensional representation of $\mathrm{GL}_2(F)$, and θ a smooth character of $(F, +)$ which is trivial on $\mathcal{O}$ but not on $\varpi^{-1}\mathcal{O}$.

(a) [2 points] Justify the claim made in lectures that $v \in V$ is invariant under $\begin{pmatrix} \mathcal{O}^\times & \mathcal{O} \\ 0 & 1 \end{pmatrix}$ if and only if its Kirillov function ϕ_v is supported on $\mathcal{O}$ and constant on cosets of $\mathcal{O}^\times$.

> **Solution:** We saw in lectures that $\phi_{\left(\begin{smallmatrix} a & b \\ 0 & 1\end{smallmatrix}\right)v}(x) = \theta(bx)\phi_v(ax)$. Since $v \mapsto \phi_v$ is injective, it follows that v is invariant under $\left(\begin{smallmatrix} \mathcal{O}^\times & \mathcal{O} \\ 0 & 1\end{smallmatrix}\right)$ if and only if $\phi_v(x) = \phi_v(ax)$ for all $a \in \mathcal{O}^\times, x \in F^\times$ and $\theta(bx)\phi_v(x) = \phi_v(x)$ for all $b \in \mathcal{O}, x \in F^\times$. The first condition is exactly that ϕ_v be constant on cosets of $\mathcal{O}^\times$, so we must show that the second is equivalent to having support in $\mathcal{O}$.
>
> On one hand, suppose ϕ is supported in $\mathcal{O}$. If $x \notin \mathcal{O}$, then the relation $\phi(x) = \theta(bx)\phi(x)$ is obvious since both sides are 0. If $x \in \mathcal{O}$, then $bx \in \mathcal{O}$ for any $b \in \mathcal{O}$ and hence $\theta(bx) = 1$; so if ϕ has support in $\mathcal{O}$, then we have $\phi(x) = \theta(bx)\phi(x)$ for all $b \in \mathcal{O}$ and $x \in F^\times$.
>
> Conversely, suppose $\phi(x) = \theta(bx)\phi(x)$ holds for all $x \in F^\times$ and $b \in \mathcal{O}$. We are given that there exists some $y \in \varpi^{-1}\mathcal{O}$ with $\theta(y) \neq 1$. For any $x \notin \mathcal{O}$, we have $b = x^{-1}y \in \mathcal{O}$; and hence $\phi(x) = \theta(bx)\phi(x) = \theta(y)\phi(x)$ which forces $\phi(x) = 0$. Hence ϕ is supported in $\mathcal{O}$.
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> *[Note that it is **not** true that $\theta(y) \neq 1$ for every $y \notin \mathcal{O}$, as some of you believed; this implication holds if $F = \mathbf{Q}_\ell$, but not for more general local fields.]*

(b) [1 point] Show that $n \geqslant 1$, $v \in V^{U_n}$, and $v' = [U_n \left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1\end{smallmatrix}\right) U_n] \cdot v$, then for all $x \in \mathcal{O}$ we have

$$\phi_{v'}(a) = q\phi_v(\varpi a).$$

> **Solution:** We compute that if $g = \left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1\end{smallmatrix}\right)$ then $U_n g U_n = \bigsqcup_{a \in \mathcal{O}/\varpi} \left(\begin{smallmatrix} \varpi & a \\ 0 & 1\end{smallmatrix}\right) U_n$, and hence
>
> $$\phi_{[U_n g U_n]\cdot v}(x) = \sum_a \theta(ax)\phi_v(\varpi x).$$
>
> If $x \in \mathcal{O}$ then the θ terms are all 1 and hence we obtain $q\phi_v(\varpi x)$.

7. Let $N \geqslant 1$ and let χ be a homomorphism $(\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$ (a *Dirichlet character modulo N*).

   (a) [1 point] Show that there exists a unique smooth character $\underline{\chi} : \mathbf{Q}^\times_{>0} \setminus \mathbf{A}_f^\times \to \mathbf{C}^\times$ such that for almost all primes ℓ, the restriction of $\underline{\chi}$ to $\mathbf{Q}_\ell^\times$ is unramified and maps a uniformiser to $\chi(\ell)$.

   (b) [1 point] Show that the restriction of $\underline{\chi}$ to $\hat{\mathbf{Z}}^\times$ is given by the composition

$$\hat{\mathbf{Z}}^\times \twoheadrightarrow (\mathbf{Z}/N\mathbf{Z})^\times \xrightarrow{\chi^{-1}} \mathbf{C}^\times.$$

> **Solution:** First we show existence. We have $\mathbf{A}_f^\times = \hat{\mathbf{Z}}^\times \times \mathbf{Q}^\times_{>0}$, and this is even an isomorphism of topological groups if $\mathbf{Q}^\times_{>0}$ is given the discrete topology (and $\hat{\mathbf{Z}}^\times$ its usual profinite topology). Hence restriction to $\hat{\mathbf{Z}}^\times$ gives a bijection between smooth characters of $\mathbf{A}_f^\times$ trivial on $\mathbf{Q}^\times_{>0}$, and smooth characters of $\hat{\mathbf{Z}}^\times$.
>
> We define $\underline{\chi}$ to be the unique character whose restriction to $\hat{\mathbf{Z}}^\times$ is the inflation of χ. For any prime $\ell \nmid N$, and any uniformizer ϖ_ℓ at ℓ, we have
>
> $$\underline{\chi}(\varpi_\ell) = \underline{\chi}(\ell^{-1}\varpi_\ell) = \chi(\ell^{-1} \bmod N)^{-1} = \chi(\ell),$$
>
> since $\ell^{-1}\varpi_\ell$ is in $\hat{\mathbf{Z}}^\times$ and maps to $\ell^{-1} \bmod N$ in the quotient $(\mathbf{Z}/N\mathbf{Z})^\times$. In particular $\underline{\chi}(\varpi_\ell)$ is independent of the choice of ϖ_ℓ, so $\underline{\chi}|_{\mathbf{Q}_\ell^\times}$ is unramified, and it has the specified value on the uniformizer. This shows the existence part of (a) and the character constructed clearly also satisfies (b).

It remains to show uniqueness. If η is any smooth character of $\mathbf{A}_f^\times$ trivial on $\mathbf{Q}_{>0}^\times$ and satisfying the stated conditions, then (by smoothness) there must be some M such that $\eta|_{\hat{\mathbf{Z}}^\times}$ factors through $(\mathbf{Z}/M\mathbf{Z})^\times$. Without loss of generality we may assume $N \mid M$. By Dirichlet's theorem, every class in $(\mathbf{Z}/M\mathbf{Z})^\times$ contains infinitely many primes; hence the character of $(\mathbf{Z}/M\mathbf{Z})^\times$ obtained from η must in fact agree with the map $(\mathbf{Z}/M\mathbf{Z})^\times \twoheadrightarrow (\mathbf{Z}/N\mathbf{Z})^\times \xrightarrow{\chi^{-1}} \mathbf{C}^\times$.

8. [2 points] Let F be a number field. If v is a (finite) prime of F, we denote by F_v the completion of F at v, and $\mathcal{O}_v$ the ring of integers of F_v. Show that for any given prime v of F, we may find an element γ of $\mathrm{SL}_2(F)$ such that

   - the image of γ in $\mathrm{SL}_2(F_v)$ lies in the double coset $\mathrm{SL}_2(\mathcal{O}_v)\begin{pmatrix}\varpi_v & \\ & \varpi_v^{-1}\end{pmatrix}\mathrm{SL}_2(\mathcal{O}_v)$;
   - the image of γ in $\mathrm{SL}_2(F_w)$ lies in $\mathrm{SL}_2(\mathcal{O}_w)$ for all primes $w \neq v$.

   Hence show that $\mathrm{SL}_2(F)$ is dense in $\mathrm{SL}_2(\mathbf{A}_{F,f})$.

   [Hint: the above double coset in $\mathrm{SL}_2(F_v)$ also contains $\begin{pmatrix}1 & \varpi_v^{-1}\\ 0 & 1\end{pmatrix}$.]

Solution: Let ℓ be the rational prime below v. From the density of F in $\mathbf{A}_{F,f}$, we can find an element $x \in F$ which has valuation -1 at v and $\geqslant 0$ at all other primes. Then $\gamma = \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$ works; it is clearly integral away from v by construction, and it must be in the above double coset, because γ does not have matrix entries in $\mathcal{O}_v$ but $\varpi_v\gamma$ does.

Let C be the closure of $\mathrm{SL}_2(F)$ in $\mathrm{SL}_2(\mathbf{A}_{F,f})$. It is clear that C contains $\mathrm{SL}_2(\widehat{\mathcal{O}}_F)$ by a result from lectures, so in particular C is a union of double $\mathrm{SL}_2(\widehat{\mathcal{O}}_F)$-cosets. We have shown that $\begin{pmatrix}\varpi_v & \\ & \varpi_v^{-1}\end{pmatrix} \in C$ for every prime v, and since C is a group, it contains all the powers of this element. From the Cartan decomposition, C contains $\mathrm{SL}_2(F)$ for every F; so it is the whole of $\mathrm{SL}_2(\mathbf{A}_{F,f})$.

9. Let $N \geqslant 1$, and let $U = \{g \in \mathrm{GL}_2(\hat{\mathbf{Z}}) : g = \begin{pmatrix}* & *\\ 0 & 1\end{pmatrix} \bmod N\}$ and $U' = \{g \in \mathrm{GL}_2(\hat{\mathbf{Z}}) : g = \begin{pmatrix}1 & *\\ 0 & *\end{pmatrix} \bmod N\}$.

   (a) [1 point] Show that both $Y(U)$ and $Y(U')$ are canonically isomorphic to the classical modular curve $Y_1(N) = \Gamma_1(N)\backslash\mathcal{H}$.

Solution: We showed in lectures that if U is any open compact in $\mathrm{GL}_2(\mathbf{A}_f)$ and $g_1, \dots, g_r \in \mathrm{GL}_2(\mathbf{A}_f)$ are such that the elts $\{\det g_i\}_{i=1,\dots,r}$ are coset representatives for $\mathbf{A}_f^\times/\mathbf{Q}_{>0}^\times$, then every $x \in Y(U)$ has a representative of the form (g_i, τ), for a unique i and some $\tau \in \mathcal{H}$ unique modulo $\mathrm{GL}_2^+(\mathbf{Q}) \cap g_i U g_i^{-1}$.
In this case, we have $\det(U) = \hat{\mathbf{Z}}^\times$, and since $\mathbf{A}_f^\times/\mathbf{Q}_{>0}^\times\hat{\mathbf{Z}}^\times = \{1\}$, we can take $r = 1$ and $g = \mathrm{id}$ to see that $Y(U) = \Gamma\backslash\mathcal{H}$ where $\Gamma = U \cap \mathrm{GL}_2^+(\mathbf{Q})$, which is clearly $\Gamma_1(N)$. The argument for U' is identical.

   (b) [1 point] Show that, for ℓ a prime not dividing N, right-translation by $\begin{pmatrix}\varpi_\ell & 0\\ 0 & \varpi_\ell\end{pmatrix} \in \mathrm{GL}_2(\mathbf{A}_f)$ acts as the diamond operator $\langle \ell \rangle$ on $Y(U)$, and as $\langle \ell \rangle^{-1}$ on $Y(U')$.

Solution: If we choose any $a, b \in \mathbf{Z}$ such that $a\ell - bN = 1$, then

$$\begin{pmatrix}a & b\\ N & \ell\end{pmatrix}\begin{pmatrix}\ell^{-1} & 0\\ 0 & \ell^{-1}\end{pmatrix}\begin{pmatrix}\varpi_\ell & 0\\ 0 & \varpi_\ell\end{pmatrix} \in U.$$

As points of $Y(U)$ we have

$$\begin{aligned}(1,\tau)\cdot\begin{pmatrix}\varpi_\ell & 0\\ 0 & \varpi_\ell\end{pmatrix} &= \left(\begin{pmatrix}\varpi_\ell & 0\\ 0 & \varpi_\ell\end{pmatrix},\tau\right)\\ &= \left(\begin{pmatrix}\varpi_\ell & 0\\ 0 & \varpi_\ell\end{pmatrix}\cdot\left[\begin{pmatrix}a & b\\ N & \ell\end{pmatrix}\begin{pmatrix}\ell^{-1} & 0\\ 0 & \ell^{-1}\end{pmatrix}\begin{pmatrix}\varpi_\ell & 0\\ 0 & \varpi_\ell\end{pmatrix}\right]^{-1},\tau\right)\\ &= \left(\left[\begin{pmatrix}a & b\\ N & \ell\end{pmatrix}\begin{pmatrix}\ell^{-1} & 0\\ 0 & \ell^{-1}\end{pmatrix}\right]^{-1},\tau\right)\\ &= \left(1,\left[\begin{pmatrix}a & b\\ N & \ell\end{pmatrix}\begin{pmatrix}\ell^{-1} & 0\\ 0 & \ell^{-1}\end{pmatrix}\right]\cdot\tau\right)\\ &= (1,\langle\ell\rangle\cdot\tau).\end{aligned}$$

If we use U' in place of U, then we need to replace $\left(\begin{smallmatrix}a & b\\ N & \ell\end{smallmatrix}\right)$ with $\left(\begin{smallmatrix}\ell & b\\ N & a\end{smallmatrix}\right)$. The condition $a\ell - bN = 1$ implies that $a = \ell^{-1} \bmod N$, so this matrix represents $\langle\ell^{-1}\rangle$.

[The condition $\ell \nmid N$ was accidentally omitted from the question, but if $\ell \mid N$ the operator $\langle\ell\rangle$ is not defined.]

10. [2 points] Let $M_{k,t}$ be the $\mathrm{GL}_2(\mathbf{A}_f)$-representation of modular forms, as in Chapter 6 of the lectures. For $f \in M_{k,t}$, and $s \in \mathbf{R}$, consider the function $f_s : \mathrm{GL}_2(\mathbf{A}_f) \times \mathcal{H} \to \mathbf{C}$ defined by

$$f_s(g,\tau) = f(g,\tau)\|\det g\|^s.$$

Here $\|x\| = \prod_\ell |x_\ell|$ is the normalised absolute value on $\mathbf{A}_f^\times$. Show that $f_s \in M_{k,t+s}$.

Solution: We need to check the following:

(i) $f_s(g,-)$ is holomorphic and bounded at the cusps for any $g \in \mathrm{GL}_2(\mathbf{A}_f)$;

(ii) f_s is stable under right-translation by some open subgroup of $\mathrm{GL}_2(\mathbf{A}_f)$;

(iii) $f_s(\gamma g,-) = f_s(g,-)|_{k,t+s}\gamma^{-1}$ for $\gamma \in \mathrm{GL}_2^+(\mathbf{Q})$.

Part (i) is obvious since $f_s(g,-)$ is a scalar multiple of $f(g,-)$.

For part (ii), let U be any open compact fixing f. Then the image of U under $x \mapsto \|\det x\|$ is an open compact subgroup of $\mathbf{R}_{>0}$, hence it's trivial; so $\|\det u\| = 1$ for all $u \in U$. Thus U fixes f_s.

For part (iii), we have $\|x\| = 1/x$ for $x \in \mathbf{Q}_{>0}$, so

$$\begin{aligned}f_s(\gamma g,-) &= \|\det \gamma g\|^s f(\gamma g,-)\\ &= \|\det \gamma g\|^s f(g,-)|_{k,t}\gamma^{-1}\\ &= (\det\gamma^{-1})^s f_s(g,-)|_{k,t}\gamma^{-1}\\ &= f_s(g,-)|_{k,t+s}\gamma^{-1}.\end{aligned}$$

[For (ii), it suffices to argue – as several of you did – that $U' = \{u \in U : \|\det u\| = 1\}$ is open, e.g. because it contains $U \cap \mathrm{GL}_2(\hat{\mathbf{Z}})$. However, the above argument shows that we always have $U' = U$.]