A trace formula for Hecke operators for modular groups

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Thesis
Submitted to the University of Warwick
for the degree of
Master of Science

Supervised by David Loeffler

THE UNIVERSITY OF WARWICK

Mathematics Institute

September 2012

Acknowledgements

I would like to express my deepest gratitude to my supervisor David Loeffler for suggesting this thesis and supporting me throughout the year with regular meetings and on the spot answers to almost all of my mails. I would also like to thank him for his brilliant introduction to the theory of modular forms given as a lecture at the University of Warwick during the academic year 2011/12, without which I might have missed this fascinating area of mathematics.

Further, thanks go to the Mathematics Department of the University of Warwick for offering a perfect work environment. In particular, I would like to thank Carole Fisher for always being there for us MSc students. In addition, my thanks go to the "Studienstiftung des Deutschen Volkes" for their financial support during the year.

Last but not least I would like to thank my parents for always supporting me spending a year abroad. I also thank Joakim Skogholt and Kien Nguyen for all the time spent together at the Department working hard and playing "Skat" during the breaks. Moreover, I would like to thank Daniel Reker and Stefan Schmid for their valuable suggestions to this work, and finally, I would like to express my gratitude to David Wegmann and Patrick Tolksdorf for their fantastic last-minute support.

The beginner should not be discouraged
if he finds that he does not have the prerequisites
for reading the prerequisites.

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Paul Halmos

Contents

1 Introduction

This thesis yields an introduction to the *Eichler-Selberg trace formula*, which is a formula for the trace of Hecke operators acting on spaces of cusp forms. For the reader unfamiliar with the area we quickly comment on the mentioned terms: Roughly speaking, a cusp form is an holomorphic function on the upper half-plane $\mathcal{H}$ which behaves "nicely" on the closure of $\mathcal{H}$ in the Riemann sphere, and which is invariant under a certain weight k action of some matrix group Γ. For a given weight k and a given group Γ, the set of cusp forms is a finite dimensional vetor space, and Hecke operators are particular linear operators acting on this space. (We give a more detailed introduction to the theory in Section 2.1.)

The eigenvalues of Hecke operators are of interest as they describe Fourier coefficients of modular forms, and these coefficients are important for applications to number theory. For example the number of ways of representing an integer as a sum of four squares is encoded in the Fourier coefficients of a certain modular form. (See Section 1.2 in [DS05] for details on this matter.) Using traces of different Hecke operators it is possible to recover their eigenvalues. (We refer to pages 266, 267 in [Miy06] for details.) This motivates the study of trace formulae.

The first formulae were given by A. Selberg and M. Eichler, who studied trace formulae simultaneously. In 1956 Selberg stated a trace formula in [Sel56] without proof for the well-known T_n operators for the full modular group, and in 1957 Eichler proved a trace formula for such T_n operators for the full modular group in the case that n is squarefree in [Eic57]. He also states a trace formula for n not being squarefree. However, Eichler had already been studying trace formulae during the past years (see [Eic55] and [Eic56]). Since then a lot of different authors have contributed to the area. Therefore the term *Eichler-Selberg trace formula* denotes a whole class of trace formulae being due to the original formulae by Selberg and Eichler. In particular, we mention the paper [Hij74] published by H. Hijikata in 1974, in which he proves a trace formula for T_n operators acting on spaces of cusp forms of level $\Gamma_0(N)$ with N being coprime to n.

The present work is mainly based on the first half of Chapter 6 of T. Miyake's book on modular forms ([Miy06]). We start by recalling the basics of the theory of modular forms in Chapter 2. Afterwards we introduce two concepts which will be fundamental in the course of the thesis:

- The theory of *reproducing kernel Hilbert spaces*, and
- the classification of elements in $\mathrm{GL}_2(\mathbb{R})$ into *scalar*, *elliptic*, *parabolic* and *hyperbolic* elements.

The former concept is based on the beginning of the article [Aro50] by N. Aronszajn, and the latter relies on Section 1.3 of [Miy06]. We end Chapter 2 by giving a brief introduction to algebraic number theory based on [ST02], which will be necessary to understand the mentioned trace formula by Hijikata in Chapter 5.

Chapter 3 and Chapter 4 are completely based on Miyake's book. More precisely, we deal with Section 6.1 to 6.3 of [Miy06] in Chapter 3 where a first trace formula is developed by applying the theory of reproducing kernel Hilbert spaces to some function spaces related to spaces of cusp forms. Subsequently, we simplify this trace formula in Chapter 4 which covers Section 6.4 of [Miy06].

These two chapters provide the core of this thesis. Since trace formulae have been studied intensely for years it has not been our goal to extend the theory as this would go beyond the scope of this thesis. Instead we aim to give an easily accessible introduction. The corresponding sections in Miyake's book are often slightly vague, missing technical details and structure. It has been our intention to improve on these points, which essentially meant providing proofs for statements trivial to the author (such as Proposition 4.1.9 and Proposition 4.1.11) and filling in details for existing proofs (such as Lemma 3.4.4, Theorem 3.4.5 and Theorem 4.1.13). Apart from that we claim the following to be original work:

- On the pages 222 to 225 in [Miy06] Miyake uses Fourier analysis to develop a precise formula for the reproducing kernel K_k of $H_k^2(\mathcal{H})$. We use a shorter and more elementary method at this point. (This is the second half of our Section 3.2.)
- In the first part of Section 6.4. in [Miy06] one carefully interchanges summation and integration as a step in the derivation of the trace formula. We point out that one has to fix a fundamental domain first as for example the integrals in equation (6.4.7) in [Miy06] will in general not be well-defined. This is an issue Miyake ignores, though it turns out to be purely formal.

In Chapter 5 we present Hijikata's trace formula though we cannot give a proof as the gap between our final trace formula given in Section 4.3 and Hijikata's formula is still too big. Instead we explain the different terms appearing in Hijikata's formula with the help of two examples. Finally, we quickly summarise our results and give an outlook for further studies in Chapter 6.

2 Fundamental concepts

In the present chapter we introduce some concepts that will be fundamental for this thesis. Most of the results we present will not be proved, though we give references for further reading. Section 2.1 and Section 2.2 are based on Miyake's book [Miy06], Section 2.3 follows [Aro50] and Section 2.4 is mainly due to Stewart's and Tall's book [ST02].

2.1 Short introduction to modular forms

We start by quickly recalling the basic notation of modular forms and Hecke operators used in this thesis. We mainly follow [Miy06], though we work in a less general setting which sometimes simplifies things. We also mention [DS05] as a good introduction to the theory of modular forms and Hecke operators.

2.1.1 Some group actions

We denote the upper half-plane in $\mathbb{C}$ by $\mathcal{H}$. It is well known that the group $\mathrm{GL}_2(\mathbb{C})$ acts on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ via $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z = \frac{az+b}{cz+d}$ for $z \in \mathbb{C}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \infty = \frac{a}{c}$. In both cases we interpret the right-hand side as ∞ if the corresponding denominator vanishes. In particular, one can check that $\mathrm{GL}_2^+(\mathbb{R})$ acts on $\mathcal{H}$. This gives rise to an action of $\mathrm{GL}_2^+(\mathbb{R})$ on the space of functions $f \colon \mathcal{H} \to \mathbb{C}$ via

$$(f|_k\alpha)(z) = \det(\alpha)^{k-1} j(\alpha, z)^{-k} f(\alpha z)$$

where k is an arbitrary integer and $j(\alpha, z) = cz + d$ for $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We call this the **weight k action**. A function $f \colon \mathcal{H} \to \mathbb{C}$ is called **Γ-invariant of weight k** for some integer $k \in \mathbb{Z}$ if $f|_k\gamma = f$ for all $\gamma \in \Gamma$. If the context determines the weight k, we call such f simply Γ-invariant.

We collect some properties of the function j which can be checked easily: For α, β in $\mathrm{GL}_2(\mathbb{C})$ and arbitrary $z \in \mathbb{C}$ we have

$$j(\alpha\beta, z) = j(\alpha, \beta z) j(\beta, z) \quad \text{and} \quad j(\alpha^{-1}, z) = j(\alpha, \alpha^{-1}z)^{-1}$$

as in equation (1.1.5), (1.1.6) on page 1,2 in [Miy06]. (For the second equation we formally require $j(\alpha^{-1}, z) \neq 0$.) Further, we have for $\alpha, \beta \in \mathrm{GL}_2(\mathbb{R})$ and $z \in \mathcal{H}$ that

$$\mathrm{Im}(\alpha z) = \det(\alpha) \, \mathrm{Im}(z) \, |j(\alpha, z)|^{-2}$$

as in equation (1.1.7) on page 3 in [Miy06]. We will use all of these properties without further notice from now on.

2.1.2 Modular groups

Consider the group $\mathrm{SL}_2(\mathbb{Z})$ and its finite index subgroups. The former is called the **full modular group**, and the latter **modular groups**. Moreover, we define for $N \in \mathbb{N}$ the **modular group of level** N by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \colon c = 0 \bmod N \right\}.$$

We will work with general modular groups up to Chapter 5, where we specialise to modular groups of level N.

Let Γ be a modular group. Following Section 1.6 of [Miy06] we call $F \subseteq \mathcal{H}$ a **fundamental domain of** Γ if the following conditions hold:

(i) $\mathcal{H} = \bigcup_{\gamma \in \Gamma} \gamma F$,

(ii) $F = \overline{U}$ where U is the set of interior points of F, and

(iii) $\lambda U \cap U = \varnothing$ for all $\gamma \in \Gamma \backslash \{\pm 1\}$.

Note that we do not require F to be connected. Theorem 4.1.2 on page 97 in [Miy06] proves that the set

$$D := \{z \in \mathcal{H} \colon |\operatorname{Re}(z)| \leqslant 1/2 \text{ and } |z| \geqslant 1\}$$

is a fundamental domain for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. Let $g_1, \ldots, g_l$ be coset representatives for the quotient $\Gamma \backslash \mathcal{H}$, and put $F = \bigcup_{j=1}^{l} g_j D$. Then F is a fundamental domain for Γ as one can check. In particular, this shows that every modular group has a fundamental domain. Note that for a fixed integer k a function $f \colon \mathcal{H} \to \mathbb{C}$ that is Γ-invariant of weight k is completely determined by its values on a fundamental domain for Γ.

One can easily check that the full modular group acts transitively on $\mathbb{Q} \cup \{\infty\}$. For a modular group Γ we define the **set of cusps** of Γ as the set of Γ-orbits in $\mathbb{Q} \cup \{\infty\}$, and denote it by $C(\Gamma)$. Since Γ is of finite index in $\mathrm{SL}_2(\mathbb{Z})$ the set of cusps is finite. It turns out that the set of cusps of Γ contains exactly the elements missing for the compactification of the quotient $\Gamma \backslash \mathcal{H}$. More precisely, one can show that the quotient $\Gamma \backslash (\mathcal{H} \cup \mathbb{Q} \cup \{\infty\})$ is a compact Riemann surface. (This is quite involved. For details we refer to Section 1.7, Section 1.8, Theorem 1.9.1 and Theorem 4.1.2 in [Miy06].) The topology we use for the Riemann surface is the one induced by the topology on $\mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$ introduced in Section 1.7 of [Miy06], which has the normal open sets in $\mathcal{H}$ and sets of the form $\sigma(\{z \in \mathcal{H} \colon \operatorname{Im}(z) > \delta\})$ with $\sigma \in \mathrm{SL}_2(\mathbb{Z})$, $\delta > 0$ as a basis.

2.1.3 Modular forms

Next we want to introduce spaces of modular forms. Let Γ be a modular group and let $f \colon \mathcal{H} \to \mathbb{C}$ be Γ-invariant of weight k. We call f a **modular form of weight** k

and level Γ if f is holomorphic on $\mathcal{H}$ and well-defined as a function on the quotient $\Gamma\backslash(\mathcal{H} \cup \mathbb{Q} \cup \{\infty\})$ mapping to $\mathbb{C}$. So roughly spoken, modular forms are holomorphic functions on $\mathcal{H}$ that are Γ-invariant and behave "nicely" at the cusps. We denote the space of modular forms of weight k and level Γ by $M_k(\Gamma)$. Further, we call a modular form f a **cusp form** if it vanishes at all cusps, and we denote the space of cusp forms of weight k and level Γ by $S_k(\Gamma)$. (For a more detailed definition of these spaces and their corresponding forms we refer to Section 2.1 of [Miy06].)

Clearly $M_k(\Gamma)$ is a vector space over $\mathbb{C}$, and $S_k(\Gamma)$ is a linear subspace. Moreover, one can show that these spaces are finite dimensional for every integer k and every modular group Γ. (We refer to Theorem 2.5.2 on page 60, 61 in [Miy06] for a proof.) For $k \geqslant 3$ the simplest example of a modular form is the **Eisenstein series**

$$G_{k,\Gamma,\infty}(z) = \sum_{\gamma \in \Gamma_\infty^+ \backslash \Gamma} j(\gamma, z)^{-k}, \quad z \in \mathcal{H},$$

where $\Gamma_\infty^+ = \Gamma \cap \{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\}$. One can check that the sum defining $G_{k,\Gamma,\infty}$ converges absolutely and uniformly on compact subsets of $\mathcal{H}$, and that $G_{k,\Gamma,\infty}$ is a modular form of weight k and level Γ. A detailed discussion of these functions is given in Section 2.6 of [Miy06]. In the case of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ we also refer to Lemma 4.1.6 on page 100 in [Miy06], since E_k as given in the corresponding section equals $G_{k,\mathrm{SL}_2(\mathbb{Z}),\infty}$ up to a scalar multiple.

2.1.4 The Petersson inner product

Identifying the upper half-plane $\mathcal{H}$ with the upper half-plane in $\mathbb{R}^2$ we define the measure $d\nu(z)$ on $\mathcal{H}$ by $y^{-2}d(x, y)$ where $z = x + iy$, as in (1.4.2) on page 11 in [Miy06]. We are interested in this measure since it is invariant under the action of $\mathrm{GL}_2^+(\mathbb{R})$. Furthermore, it is straightforward to check:

Lemma 2.1.1. *Let $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$, $U \subseteq \mathcal{H}$ open and $f : U \to \mathbb{C}$ such that $\int_U |f(z)|d\nu(z)$ exists. Then*

$$\int_U f(z)d\nu(z) = \int_{\alpha^{-1}U} f(\alpha z)d\nu(z).$$

Let Γ be a modular group. The previous lemma shows that

$$\int_{\Gamma\backslash\mathcal{H}} F(z)d\nu(z)$$

is well-defined if the function F is Γ-invariant of weight 0 and $\int_F |F(z)|d\nu(z)$ is finite for some fundamental domain F of Γ. Hence it makes sense to determine the measure of the quotient $\Gamma\backslash\mathcal{H}$. A direct calculation shows that $\nu(\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}) = \pi/3$. One may use this to show:

Lemma 2.1.2. *Let Γ be a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. The quotient $\Gamma\backslash\mathcal{H}$ has finite measure with respect to ν. More precisely,*

$$\nu(\Gamma\backslash\mathcal{H}) = d_\Gamma \cdot \pi/3$$

where $d_\Gamma = [\mathrm{SL}_2(\mathbb{Z})/\{\pm 1\} : \Gamma/(\{\pm 1\} \cap \Gamma)]$.

Moreover, we may define for $f, g \in M_k(\Gamma)$

$$\langle f, g \rangle_\Gamma = \int_{\Gamma\backslash\mathcal{H}} f(z)\overline{g(z)}\,\mathrm{Im}(z)^k d\nu(z).$$

One can check that the integral exists if at least one of f and g vanish at every cusp of Γ. (Compare page 44 of [Miy06].) In particular, $\langle f, g \rangle_\Gamma$ is well-defined for every $f, g \in S_k(\Gamma)$. We call $\langle \cdot, \cdot \rangle_\Gamma$ the **Petersson inner product**. One can check that it indeed defines an inner product on the space of cusp forms $S_k(\Gamma)$.

2.1.5 Hecke operators

Let Γ be a modular group. For an element $g \in \mathrm{GL}_2^+(\mathbb{Q})$ we let $\Gamma g \Gamma$ denote the double coset $\{\gamma_1 g \gamma_2 \colon \gamma_1, \gamma_2 \in \Gamma\}$. One can show that for any such $g \in \mathrm{GL}_2^+(\mathbb{Q})$ the intersection $\Gamma \cap (g^{-1}\Gamma g)$ has finite index in Γ and $g^{-1}\Gamma g$. We say Γ and $g^{-1}\Gamma g$ are commensurable. Hence there are $\alpha_1, \ldots, \alpha_r$ such that $\Gamma g \Gamma = \bigsqcup_{j=1}^r \Gamma \alpha_j$ by Lemma 2.7.1 on page 69 in [Miy06]. (Note that the set $\tilde{\Gamma}$ given in the lemma equals $\mathrm{GL}_2^+(\mathbb{Q})$ in our case by the above observation.) Let $\mathcal{R}(\Gamma)$ be the $\mathbb{C}$-vector space with basis the symbols $[\Gamma g \Gamma]$ for each $g \in \Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})/\Gamma$. Following Section 2.7 of [Miy06] one can show that $\mathcal{R}(\Gamma)$ equipped with a suitable multiplication is also a ring, so an algebra over $\mathbb{C}$. It is called the **Hecke algebra of** Γ.

For elements in $\mathcal{R}(\Gamma)$ we define an action on the space of modular forms $M_k(\Gamma)$ via

$$f|_k[\Gamma g \Gamma] = \sum_{j=1}^r f|_k \alpha_j$$

where $\alpha_1, \ldots, \alpha_r \in \mathrm{GL}_2^+(\mathbb{Q})$ such that $\Gamma g \Gamma = \bigsqcup_{j=1}^r \Gamma \alpha_j$. Theorem 2.8.1 on page 74, 75 in [Miy06] proves the following:

(1) The above definition is independent of the choice of representatives $\alpha_1, \ldots, \alpha_r$, and thus well-defined.

(2) We have $f|_k[\Gamma g \Gamma] \in M_k(\Gamma)$ if $f \in M_k(\Gamma)$, and $f|_k[\Gamma g \Gamma] \in S_k(\Gamma)$ if $f \in S_k(\Gamma)$.

(3) $M_k(\Gamma)$ and $S_k(\Gamma)$ are right modules over $\mathcal{R}(\Gamma)$.

We will mainly consider elements of $\mathcal{R}(\Gamma)$ as linear operators acting on the spaces $M_k(\Gamma)$ and $S_k(\Gamma)$. In an abuse of notation we denote such operators by $T = \Gamma g \Gamma$, and write $T(f)$ for $f|_k[\Gamma g \Gamma]$. These operators are the so called **Hecke operators**.

2.2 Classification of elements in $\mathrm{GL}_2^+(\mathbb{R})$

In this section we introduce the classification of elements in $\mathrm{GL}_2^+(\mathbb{R})$ following Section 1.3 and 1.5 of [Miy06]. The classification is essential for the understanding of the trace formula presented in this thesis. It will be used first in Section 4.2.

Let $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$. We call α **scalar** if it is of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ for some $a \in \mathbb{R}^\times$. For some subset M of $\mathrm{GL}_2^+(\mathbb{R})$ we write $Z(M)$ for the set of scalar elements in M, even though coventionally $Z(G)$ denotes the centre of a group G. However, a direct calculation proves that the two notations agree for example if there are $h_1, h_2 \in \mathbb{Z}\backslash\{0\}$ such that $\left(\begin{smallmatrix} 1 & h_1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ h_2 & 1 \end{smallmatrix}\right)$ are elements of M. This is in particular the case for M being a modular group as one can check. We also note that $Z(\Gamma) = \Gamma \cap \{\pm 1\}$ if Γ is a modular group.

For non-scalar α we say α is **elliptic**, **parabolic** or **hyperbolic**, if

$$\mathrm{Tr}(\alpha)^2 < 4\det(\alpha), \quad \mathrm{Tr}(\alpha)^2 = 4\det(\alpha) \quad \text{or} \quad \mathrm{Tr}(\alpha)^2 > 4\det(\alpha),$$

respectively. Since eigenvalues of α are given by $\lambda_{1,2} = \frac{\mathrm{Tr}(\alpha)}{2} \pm \frac{1}{2}\sqrt{\mathrm{Tr}(\alpha)^2 - 4\det(\alpha)}$, we see for non-scalar α:

- The element α is elliptic if and only if the eigenvalues of α are complex conjugates with non-zero imaginary part.
- The element α is parabolic if and only if α has only one eigenvalue which has algebraic multiplicity two and geometric multiplicity one. This eigenvalue is rational if $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$.
- The element α is hyperbolic if and only if α has two distinct real eigenvalues. These eigenvalues lie at worst in a quadratic extension of $\mathbb{Q}$ if $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$.

We note that an element α is elliptic, parabolic or hyperbolic if and only if all conjugates of α are so, since trace and determinant are stable under conjugation.

In the following we consider fixed points of $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$ acting on the Riemann sphere $\mathbb{C} \cup \{\infty\}$. If α is scalar, it acts trivially on $\mathbb{C} \cup \{\infty\}$, and thus fixes every point. Suppose α is non-scalar. If $c = 0$ then ∞ is a fixed point of α and $\mathrm{Tr}(\alpha)^2 - 4\det(\alpha) = (a-d)^2$. Thus α cannot be elliptic, α is parabolic if and only if $a = d$, and α is hyperbolic if and only if $a \neq d$. Moreover, since $\alpha x = x$ for $x \in \mathbb{C}$ is equivalent to $(d-a)x = b$, we see that

- α is parabolic if and only if ∞ is the unique fixed point of α, and
- α is hyperbolic if and only if the only fixed points of α are ∞ and $\frac{b}{d-a}$.

Now suppose that $c \neq 0$. Then $\alpha x = x$ is equivalent to $x = \frac{a-d}{2c} \pm \frac{1}{2c}\sqrt{\mathrm{Tr}(\alpha)^2 - 4\det(\alpha)}$. Hence we have that

- α is elliptic if and only if the fixed points of α are complex conjugates with non-zero imaginary part,
- α is parabolic if and only if $\frac{a-d}{2c}$ is the only fixed point of α, and
- α is hyperbolic if and only if the fixed points of α are two distinct real values. If $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ then these fixed points are either both rational, or both non-rational. In the latter case they lie in an quadratic extension of $\mathbb{Q}$.

Combing these results we can characterise elements in $\mathrm{GL}_2^+(\mathbb{R})$ by means of their fixed points on the Riemann sphere.

Corollary 2.2.1. *Let $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$.*

- *The element α is elliptic if and only if there is $z \in \mathcal{H}$ such that z and $\overline{z}$ are the unique fixed points of α.*
- *The element α is parabolic if and only if α has a unique fixed point in $\mathbb{R} \cup \{\infty\}$. If $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ then this fixed point is in $\mathbb{Q} \cup \{\infty\}$.*
- *The element α is hyperbolic if and only if α has exactly two distinct fixed points on $\mathbb{R} \cup \{\infty\}$. If $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, either both fixed points lie in $\mathbb{Q} \cup \{\infty\}$, or they are both irrational.*

For a subgroup U of $\mathrm{GL}_2^+(\mathbb{R})$ we denote the **stabilizer** of $z \in \mathbb{C} \cup \{\infty\}$ in U by

$$U_z = \{\beta \in U : \beta z = z\}.$$

Lemma 2.2.2. *(1) For $z \in \mathcal{H}$ we have*

$$(\mathrm{GL}_2^+(\mathbb{R}))_z = \sigma \left\{\lambda \cdot \Phi : \lambda \in \mathbb{R}^\times, \Phi \in \mathrm{SO}_2(\mathbb{R})\right\} \sigma^{-1}$$

where $\sigma \in \mathrm{SL}_2(\mathbb{R})$ with $\sigma i = z$, and $\mathrm{SO}_2(\mathbb{R})$ denotes the special orthogonal group. In particular, we may write $\alpha = \sqrt{\det(\alpha)}\,\sigma\Phi\sigma^{-1}$ for any elliptic $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$ with fixed point z, where Φ is an element of $\mathrm{SO}_2(\mathbb{R})$.

(2) For $x \in \mathbb{R} \cup \{\infty\}$ we have

$$\left\{\alpha \in (\mathrm{GL}_2^+(\mathbb{R}))_x : \alpha \text{ parabolic or scalar}\right\} = \sigma \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a \in \mathbb{R}^\times, b \in \mathbb{R} \right\} \sigma^{-1}$$

where $\sigma \in \mathrm{SL}_2(\mathbb{R})$ with $\sigma\infty = x$.

(3) For distinct $x_1, x_2 \in \mathbb{R} \cup \{\infty\}$ we have

$$(\mathrm{GL}_2^+(\mathbb{R}))_{x_1} \cap (\mathrm{GL}_2^+(\mathbb{R}))_{x_2} = \sigma \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a, d \in \mathbb{R}^\times, ad > 0 \right\} \sigma^{-1}$$

where $\sigma \in \mathrm{SL}_2(\mathbb{R})$ with $\sigma\infty = x_1$ and $\sigma 0 = x_2$.

We refer to Lemma 1.3.2 on page 8 in [Miy06] for a proof.

Lemma 2.2.3. *For $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$ we define the* ***centralizer*** *of α by*

$$Z(\alpha) = \{\beta \in \mathrm{GL}_2(\mathbb{R}) : \alpha\beta = \beta\alpha\}.$$

(1) If α is elliptic with fixed point $z \in \mathcal{H}$ then $Z(\alpha) = (\mathrm{GL}_2^+(\mathbb{R}))_z$.

(2) If α is parabolic with fixed point $x \in \mathbb{R} \cup \{\infty\}$ then

$$Z(\alpha) = \{\beta \in (\mathrm{GL}_2^+(\mathbb{R}))_x \colon \beta \text{ parabolic or scalar}\}.$$

(3) If α is hyperbolic with distinct fixed points $x_1, x_2 \in \mathbb{R} \cup \{\infty\}$ then

$$Z(\alpha) \cap \mathrm{GL}_2^+(\mathbb{R}) = (\mathrm{GL}_2^+(\mathbb{R}))_{x_1} \cap (\mathrm{GL}_2^+(\mathbb{R}))_{x_2}$$

and $[Z(\alpha) : Z(\alpha) \cap \mathrm{GL}_2^+(\mathbb{R})] = 2$.

Again, we omit the proof, and refer to Lemma 1.3.3 on page 9 in [Miy06]. In Section 4.1 we will define $\Gamma(\alpha) = \{\gamma \in \Gamma \colon \gamma\alpha = \alpha\gamma\}$ for some finite index subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ and some $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Note that we have $\Gamma(\alpha) = \Gamma \cap Z(\alpha)$ by definition. We use this equality and the previous lemma to describe $\Gamma(\alpha)$:

Corollary 2.2.4. *Let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ and Γ be a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.*

(1) If α is elliptic with fixed point $z \in \mathcal{H}$ then $\Gamma(\alpha) = \Gamma_z$.

(2) If α is parabolic with fixed point $x \in \mathbb{Q} \cup \{\infty\}$ then $\Gamma(\alpha) = \Gamma_x$.

(3) If α is hyperbolic with fixed points $x_1, x_2 \in \mathbb{Q} \cup \{\infty\}$ then $\Gamma(\alpha) = Z(\Gamma)$.

(4) If α is hyperbolic with fixed points $x_1, x_2 \in \mathbb{R} \backslash \mathbb{Q}$ then $\Gamma(\alpha) = \Gamma_{x_1} \cap \Gamma_{x_2}$.

Proof. For α elliptic with fixed point $z \in \mathcal{H}$ we see $\Gamma(\alpha) = \Gamma \cap Z(\alpha) = \Gamma_z$ using part (1) of Lemma 2.2.3, which proves (1). Similarly, (4) follows directly from part (3) of Lemma 2.2.3. For (2) and (3) we claim that $\mathrm{SL}_2(\mathbb{Z})$ does not contain any hyperbolic elements with fixed points in $\mathbb{Q} \cup \{\infty\}$. Assume that γ is such an element with distinct fixed points $x, x' \in \mathbb{Q} \cup \{\infty\}$. Let $\sigma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\sigma\infty = x$, then $\sigma^{-1}\gamma\sigma$ fixes ∞, and is therefore of the form $\left(\begin{smallmatrix} \pm 1 & * \\ 0 & \pm 1 \end{smallmatrix}\right)$. But elements of this form are parabolic, so γ itself has to be parabolic, which is a contradiction and thus proves the claim.

To show (2) let α be parabolic with fixed point $x \in \mathbb{Q} \cup \{\infty\}$. By part (2) of Lemma 2.2.3 we have that $\Gamma(\alpha)$ consists precisely of parabolic elements in Γ fixing x and scalar elements in Γ. Therefore $\Gamma(\alpha)$ is a subset of Γ_x, and the only elements that might be in Γ_x but not in $\Gamma(\alpha)$, are hyperbolic elements in Γ fixing x. We proved that such elements do not exist.

To see (3) note that we have for α hyperbolic $\Gamma(\alpha) = \Gamma_x \cap \Gamma_{x'}$ by part (3) of Lemma 2.2.3. The right-hand side consists of hyperbolic elements in Γ fixing x and x' and all scalar elements in Γ. We proved that there are no hyperbolic elements with fixed points in $\mathbb{Q} \cup \{\infty\}$. □

We close this section with two more lemma. For the corresponding proofs see part (2) of Lemma 1.3.5 on page 10 and Theorem 1.5.4 on page 18, 19 in [Miy06].

Lemma 2.2.5. *If two distinct elements of $\mathrm{GL}_2^+(\mathbb{R})$ are either both elliptic or both parabolic, and if they are conjugate by a matrix in $\mathrm{GL}_2(\mathbb{R})$ of negative determinant, then they are not conjugate in $\mathrm{GL}_2^+(\mathbb{R})$.*

Note that the corresponding lemma in [Miy06] is stated only for parabolic elements, but the given proof works exactly the same for elliptic elements since we have for any elliptic $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$ that $Z(\alpha) \subseteq \mathrm{GL}_2^+(\mathbb{R})$ by part (1) of Lemma 2.2.3.

Lemma 2.2.6. *Let Γ be a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.*

(1) For $z \in \mathcal{H}$, the stabilizer group Γ_z is finite.

(2) For $x \in \mathbb{Q} \cup \{\infty\}$, the quotient $\Gamma_x/Z(\Gamma)$ is isomorphic to $\mathbb{Z}$. Moreover, we have

$$\left(\sigma^{-1}\Gamma_x\sigma\right) \cdot \{\pm 1\} = \left\{\pm \begin{pmatrix} 1 & hm \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z}\right\}$$

where $\sigma \in \mathrm{SL}_2(\mathbb{R})$ with $\sigma\infty = x$ and h is the width of the cusp $[x]$ for Γ.

(3) For distinct $x_1, x_2 \in \mathbb{R} \cup \{\infty\}$ with $\Gamma_{x_1} \cap \Gamma_{x_2} \neq Z(\Gamma)$, the quotient $(\Gamma_{x_1} \cap \Gamma_{x_2})/Z(\Gamma)$ is isomorphic to $\mathbb{Z}$. Moreover, there is $u > 0$ such that for $\sigma \in \mathrm{SL}_2(\mathbb{R})$ with $\sigma\infty = x_1$ and $\sigma 0 = x_2$ we have

$$\left(\sigma^{-1}(\Gamma_{x_1} \cap \Gamma_{x_2})\sigma\right) \cdot \{\pm 1\} = \left\{\pm \begin{pmatrix} u^m & 0 \\ 0 & u^{-m} \end{pmatrix} : m \in \mathbb{Z}\right\}.$$

2.3 Introduction to reproducing kernel Hilbert spaces

In this section we will give a short introduction to the theory of reproducing kernel Hilbert spaces based on the beginning of [Aro50]. The concept will be fundamental for the third chapter of this thesis.

Definition 2.3.1. Let X be an arbitrary set and let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space consisting of complex valued functions on X. A function $K\colon X \times X \to \mathbb{C}$ is called the **reproducing kernel** of H if

(1) the function $K(\cdot, x)$ is an element of H for each fixed $x \in X$, and

(2) for every function $f \in H$ and every $x \in X$ we have $f(x) = \langle f, K(\cdot, x)\rangle$.

Property (2) is called the **reproducing property** of the kernel K. If such a function K exists then H is called a **reproducing kernel Hilbert space**.

Following Section 1.2 of [Aro50] we will now prove some basic properties of reproducing kernel Hilbert spaces. Throughout the section we assume $(H, \langle\cdot,\cdot\rangle)$ to be a Hilbert space with $H \subseteq \{f\colon X \to \mathbb{C}\}$ where X is an arbitrary set. Further, we will sometimes write *kernel* instead of *reproducing kernel.*

Proposition 2.3.2 (Uniqueness)**.** *If H is a reproducing kernel Hilbert space, then its kernel K is unique.*

Proof. Suppose $K' \colon X \times X \to \mathbb{C}$ is another reproducing kernel of H. Then we see for any $x \in X$ using the reproducing property of K and K' that

$$\begin{aligned}\|K(\cdot,x) - K'(\cdot,x)\|^2 &= \langle K(\cdot,x) - K'(\cdot,x), K(\cdot,x) - K'(\cdot,x)\rangle \\ &= \langle K(\cdot,x) - K'(\cdot,x), K(\cdot,x)\rangle - \langle K(\cdot,x) - K'(\cdot,x), K'(\cdot,x)\rangle \\ &= (K(x,x) - K'(x,x)) - (K(x,x) - K'(x,x)) \\ &= 0.\end{aligned}$$

Here $\|\cdot\|$ denotes the norm of H induced by $\langle\cdot,\cdot\rangle$. Hence $K = K'$ as claimed. □

Proposition 2.3.3 (Existence)**.** *The Hilbert space H has a reproducing kernel K if and only if the evaluation functional $E_x \colon H \to \mathbb{C}$, $f \mapsto f(x)$ is continuous for every $x \in X$.*

Proof. Suppose that K is the kernel of H, and fix $x \in X$. Then

$$|E_x(f)| = |f(x)| = |\langle f, K(\cdot,x)\rangle| \leqslant \|f\|\sqrt{\langle K(\cdot,x), K(\cdot,x)\rangle} = \sqrt{K(x,x)}\|f\|$$

by the Cauchy-Schwarz inequality and the reproducing property of K, so E_x is continuous. Conversely suppose that E_x is continuous for every $x \in X$. Then every E_x is an element of the dual space of H since it is clearly linear. By the Riesz representation theorem (see Theorem 3.4 on page 13 in [Con97]) we find for every such E_x a unique $g_x \in H$ such that $E_x = \langle\cdot, g_x\rangle$. Put $K(y,x) = g_x(y)$. Then $K(\cdot,x) = g_x \in H$ for all $x \in X$ and

$$f(x) = E_x(f) = \langle f, g_x\rangle = \langle f, K(\cdot,x)\rangle.$$

Thus K is the reproducing kernel of H. □

Corollary 2.3.4. *If H is a reproducing kernel Hilbert space, then its kernel K has the following properties:*

(i) We have $K(x,x) \in [0,\infty)$ for all $x \in X$, and $K(x,x) = 0$ if and only if $f(x) = 0$ for all $f \in H$.

(ii) The kernel K is conjugate symmetric, that is $K(x,y) = \overline{K(y,x)}$ for all $x,y \in X$.

Proof. Using the reproducing property of K we see $K(x,y) = \langle K(\cdot,y), K(\cdot,x)\rangle$. Hence the first part of property (i) follows from $\langle\cdot,\cdot\rangle$ being a scalar product and thus positive definite, and (ii) follows since $\langle\cdot,\cdot\rangle$ is also conjugate symmetric.

It remains to show that $K(x,x) = 0$ if and only if $f(x) = 0$ for all $f \in H$. Fix $x \in X$. Suppose that $0 = K(x,x) = \langle K(\cdot,x), K(\cdot,x)\rangle$. Then $K(\cdot,x) = 0$ since the scalar product is positive definite, and thus $f(x) = \langle f, K(\cdot,x)\rangle = 0$. Conversly suppose that $0 = f(x)$ for all $f \in H$. Then $K(x,x) = 0$ follows trivially since $K(\cdot,x) \in H$. □

Proposition 2.3.5. *Let H be a reproducing kernel Hilbert space with kernel K. If H is a subspace of a larger Hilbert space J, then*

$$\pi_K \colon J \to H, \ (\pi_K f)(x) = \langle f, K(\cdot,x)\rangle$$

is a well-defined operator which projects J onto H.

Proof. Let $f \in J$. Then there is a unique element $g \in H$ such that $f - g \in H^\perp$ where $H^\perp$ denotes the orthogonal complement of H in J. (We refer to Section 1.2 of [Con97] for details on this matter.) Hence

$$(\pi_K f)(x) = \langle g, K(\cdot, x)\rangle + \langle f - g, K(\cdot, x)\rangle = \langle g, K(\cdot, x)\rangle = g(x)$$

since $K(\cdot, x) \in H$ and $f - g \in H^\perp$. In particular, we have $\pi_K f = f$ for all $f \in H$. □

Proposition 2.3.6. *Let H be a reproducing kernel Hilbert space with kernel K and let $\{e_j\}_{j\in J}$ be an orthonormal basis of H. Then*

$$K(x, y) = \sum_{j\in J} e_j(x)\overline{e_j(y)}.$$

Proof. Since $\{e_j\}_{j\in J}$ is an orthonormal basis we have $f = \sum_{j\in J} \langle f, e_j\rangle e_j$ for every $f \in H$. (This is part of Theorem 4.13 on page 16 in [Con97].) Fix $y \in X$. Then

$$K(\cdot, y) = \sum_{j\in J} \langle K(\cdot, y), e_j\rangle e_j$$

and thus $\langle K(\cdot, y), e_j\rangle = \overline{\langle e_j, K(\cdot, y)\rangle} = \overline{e_j(y)}$ gives the claimed statement. □

2.4 Some algebraic number theory

Finally we recall some basic notation and facts about algebraic number theory. In particular, we are interested in quadratic fields over $\mathbb{Q}$ and their orders, which will appear in Chapter 5 of this thesis. The present section is based on [ST02], which we follow closely. Throughout we require a ring to have a multiplicative identity element, and a homomorphism of rings needs to map the unity of its domain to the unity of its target.

2.4.1 Number fields and their rings of integers

We call a complex number $\alpha \in \mathbb{C}$ an **algebraic number** if it is algebraic over $\mathbb{Q}$, so if there is a polynomial $p \in \mathbb{Q}[t]$, $p \neq 0$, such that $p(\alpha) = 0$. Note that this is equivalent to p having coefficients in $\mathbb{Z}$ as we can clear denominators. Define $\mathbb{A}$ to be the set of all algebraic numbers in $\mathbb{C}$. One can show that $\mathbb{A}$ is a subfield of $\mathbb{C}$. (Compare Theorem 2.1 on page 36 in [ST02].) Note that the field $\mathbb{A}$ is an infinite field extension of $\mathbb{Q}$. We define a subfield K of $\mathbb{C}$ to be a **number field** if it is a finite extension of $\mathbb{Q}$. Note that any element in K is algebraic, so we have $K \subseteq \mathbb{A}$. Moreover, K is separable over $\mathbb{Q}$ since $\mathbb{Q}$ and thus K have characteristic 0. Hence we have $K = \mathbb{Q}(\theta)$ for some $\theta \in K$ by the primitive element theorem. (See for example Theorem 4.6 on page 243 in [Lan02].)

We call a complex number $\theta \in \mathbb{C}$ an **algebraic integer** if there is a monic polynomial $p \in \mathbb{Z}[t]$, $p \neq 0$, such that $p(\theta) = 0$, so

$$\theta^n + a_{n-1}\theta^{n-1} + \ldots + a_1\theta + a_0 = 0$$

for some $a_0, \ldots, a_{n-1} \in \mathbb{Z}$. Define $\mathbb{B}$ to be the set of all algebraic integers in $\mathbb{C}$. Then $\mathbb{B}$ is a subring of $\mathbb{A}$, and we have $\mathbb{B} \cap \mathbb{Q} = \mathbb{Z}$ by Theorem 2.9 on page 43 and Lemma 2.14 on page 45 in [ST02]. Finally, we define for any number field K the **ring of integers** of K by $\mathcal{O}_K = K \cap \mathbb{B}$. Note that $\mathcal{O}_K$ is indeed a subring of K and $\mathbb{Z} \subseteq \mathcal{O}_K$. One can check that for any $\alpha \in K$ there is $N \in \mathbb{Z}$, $N \neq 0$, such that $N\alpha \in \mathcal{O}_K$. Hence we have $\mathcal{O}_K \otimes \mathbb{Q} = K$. Further, this shows that we can always find $\theta \in \mathcal{O}_K$ such that $K = \mathbb{Q}(\theta)$.

2.4.2 The discriminant of a number field

Let $K = \mathbb{Q}(\theta)$ be a number field. We want to define the discriminant of K using homomorphisms of the form $\sigma \colon K \to \mathbb{C}$. Recall that a homomorphism of fields is always injective, so such σ is an embedding. Further one can easily check that any such σ fixes $\mathbb{Q}$. Hence we are looking for possible extensions of the trivial embedding $\mathbb{Q} \hookrightarrow \mathbb{C}$ to the number field $K = \mathbb{Q}(\theta)$. By Proposition 2.7 on page 233 in [Lan02] there are exactly n embeddings $\sigma_i \colon K \hookrightarrow \mathbb{C}$ where n is the number of distinct roots of the minimal polynomial of θ, which agrees with the degree of K since K is separable over $\mathbb{Q}$ as mentioned earlier. Further, one can check that the elements $\theta_i := \sigma_i(\theta)$ are precisely the distinct zeros of the minimal polynomial of θ over $\mathbb{Q}$.

Let now $\{\alpha_1, \ldots, \alpha_n\}$ be a basis of K as a vector space over $\mathbb{Q}$. Then we define the **discriminant** of this basis to be

$$\Delta[\alpha_1, \ldots, \alpha_n] = \left[\det\left((\sigma_i(\alpha_j))_{1 \leqslant i,j \leqslant n}\right)\right]^2.$$

If $\{\beta_1, \ldots, \beta_n\}$ is another basis of K over $\mathbb{Q}$, then we can write $\beta_k = \sum_{j=1}^n c_{j,k}\alpha_j$ for some $c_{j,k} \in \mathbb{Q}$, $k = 1, \ldots, n$, and hence

$$\Delta[\beta_1, \ldots, \beta_n] = \left[\det\left((c_{j,k})_{1 \leqslant j,k \leqslant n}\right)\right]^2 \Delta[\alpha_1, \ldots, \alpha_n]. \tag{2.4.1}$$

The obvious choice of a basis for K over $\mathbb{Q}$ is $\{1, \theta, \ldots, \theta^{n-1}\}$. To see that these elements are indeed linearly independent we only have to note that the minimal polynomial of θ has degree n. One can check that

$$\Delta\left[1, \theta, \ldots, \theta^{n-1}\right] = \prod_{1 \leqslant i < j \leqslant n} (\theta_i - \theta_j)^2.$$

Here the right-hand side is rational and non-zero, and thus the discriminant of any basis of K over $\mathbb{Q}$ is so, since the determinant of $(c_{j,k})_{1 \leqslant j,k \leqslant n}$ is rational and non-zero as well.

Next we note that $\mathcal{O}_K$, the ring of integers of K, is an abelian group under addition, and thus a $\mathbb{Z}$-module. More precisely, Theorem 2.16 on page 46 in [ST02] proves that $\mathcal{O}_K$ is a free abelian group of rank n where n is the degree of K, and thus the $\mathbb{Z}$-module $\mathcal{O}_K$ always has a basis. We call this basis an **integral basis** of K, which is reasonable since any $\mathbb{Z}$-basis of $\mathcal{O}_K$ is also a $\mathbb{Q}$-basis for K as $\mathcal{O}_K \otimes \mathbb{Q} = K$.

Let $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1, \ldots, \beta_n\}$ be two integral bases of K. Since they are bases of the $\mathbb{Z}$-module $\mathcal{O}_K$ we can write $\beta_k = \sum_{j=1}^n c_{j,k}\alpha_j$ for some $c_{j,k} \in \mathbb{Z}$, $k = 1, \ldots, n$, and conversly $\alpha_k = \sum_{j=1}^n \tilde{c}_{j,k}\beta_j$ for some $\tilde{c}_{j,k} \in \mathbb{Z}$, $k = 1, \ldots, n$. Since the matrix $(c_{j,k})_{1 \leqslant j,k \leqslant n}$ is the inverse of $(\tilde{c}_{j,k})_{1 \leqslant j,k \leqslant n}$, and both matrices have integer entries, they both have to have determinant ± 1. Therefore the discriminant of an integral basis is indepenent of the choice of integral basis by equation (2.4.1), and we can define the **discriminant of the number field** K, denoted by Δ_K, as the discriminant of any integral basis of K.

2.4.3 Quadratic fields

We call a number field K a **quadratic field** if it is of degree 2 over $\mathbb{Q}$. As noted at the end of Subsection 2.4.1 we can write $K = \mathbb{Q}(\theta)$ for some $\theta \in \mathcal{O}_K$. Let $m_\theta = t^2 + at + b$, $a, b \in \mathbb{Q}$, be the minimal polynomial of θ. One can check that a and b have to be integers since θ is an algebraic integer. (Compare Lemma 2.13 on page 45 in [ST02].) Further, we can write

$$\theta = \frac{-a \pm \sqrt{a^2 - 4b}}{2}.$$

Since $a \in \mathbb{Z}$ we have $K = \mathbb{Q}(\theta) = \mathbb{Q}(\sqrt{a^2 - 4b}\,)$. This is a quadratic extension of $\mathbb{Q}$ if and only if $a^2 - 4b$ does not have a rational square root, which is the case if and only if there is no integer $r \in \mathbb{N}_0$ such that $a^2 - 4b = r^2$. Hence we can write $a^2 - 4b = r^2 d$ for a unique pair of integers $r \in \mathbb{N}$ and $d \in \mathbb{Z} \backslash \{0, 1\}$ with d being squarefree, and thus $K = \mathbb{Q}(r\sqrt{d}\,) = \mathbb{Q}(\sqrt{d}\,)$. Conversly any field of the form $\mathbb{Q}(\sqrt{d}\,)$ for some squarefree integer d not equal to 0 or 1 is obviously a quadratic field, so we have shown:

Proposition 2.4.1. *The quadratic fields are precisely the fields of the form* $\mathbb{Q}(\sqrt{d}\,)$ *with* d *being a squarefree integer and* $d \neq 0, 1$.

If such squarefree d is positive, we call $\mathbb{Q}(\sqrt{d}\,)$ a **real quadratic field**, and if d is negative we call $\mathbb{Q}(\sqrt{d}\,)$ an **imaginary quadratic field**. Next we determine the ring of integers of a quadratic field.

Proposition 2.4.2. *Let* $d \neq 0, 1$ *be a squarefree integer and* K *the quadratic field* $\mathbb{Q}(\sqrt{d}\,)$. *Then* $\mathcal{O}_K = \mathbb{Z}[\sqrt{d}\,]$ *if* $d = 2, 3$ *mod* 4, *and* $\mathcal{O}_K = \mathbb{Z}[(1 + \sqrt{d}\,)/2]$ *if* $d = 1$ *mod* 4.

For a proof see Theorem 3.2 on page 62 in [ST02]. Note that $d = 0 \bmod 4$ is not possible since we assume d to be squarefree. We use this result to compute the discriminant of a quadratic field.

Theorem 2.4.3. *Let* $d \neq 0, 1$ *be a squarefree integer and* $K = \mathbb{Q}(\sqrt{d}\,)$. *Then* $\Delta_K = 4d$ *if* $d = 2, 3$ *mod* 4, *and* $\Delta_K = d$ *if* $d = 1$ *mod* 4.

Proof. Note that the two distinct embeddings $\sigma_1, \sigma_2 \colon K \hookrightarrow \mathbb{C}$ are the identity and the conjugation, so for $a, b \in \mathbb{Q}$ we have

$$\sigma_1(a + b\sqrt{d}\,) = a + b\sqrt{d} \quad \text{and} \quad \sigma_2(a + b\sqrt{d}\,) = a - b\sqrt{d}.$$

Suppose that $d = 2, 3 \bmod 4$. By the previous proposition an integral basis of K is given by $\{1, \sqrt{d}\,\}$. Hence we can compute

$$\Delta_K = \Delta[1, \sqrt{d}\,] = \left[\det\left(\begin{pmatrix} \sigma_1(1) & \sigma_1(\sqrt{d}\,) \\ \sigma_2(1) & \sigma_2(\sqrt{d}\,) \end{pmatrix}\right)\right]^2 = \left[\det\left(\begin{pmatrix} 1 & \sqrt{d} \\ 1 & -\sqrt{d} \end{pmatrix}\right)\right]^2 = 4d.$$

Similarly we can compute $\Delta_K = d$ if $d = 1 \bmod 4$ using that an integral basis of K is given by $\{1, (1+\sqrt{d})/2\}$ in this case. □

Further we want to describe the set of units of a ring of integers for some quadratic field. Recall that the set of units of a ring R forms a group under multiplication. We denote this group by $U(R)$.

Proposition 2.4.4. *Let $K = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field. Then*

$$U(\mathcal{O}_K) = \begin{cases} \{\pm 1, \pm i\}, & \text{if } d = -1, \\ \{\pm 1, \pm e^{2\pi i/3}, \pm e^{4\pi i/3}\}, & \text{if } d = -3, \\ \{\pm 1\}, & \text{otherwise.} \end{cases}$$

We refer to Proposition 4.2 on page 77 in [ST02] for a proof. The general case of groups of units in quadratic fields is more complicated and known as the *Dirichlet Units Theorem*. It is dealt with in the Appendix B of [ST02], and in Section 3.3 of [Ono90]. We will not need the general case in this thesis.

2.4.4 Ideal class group and class number

Let K be a number field as before. In this subsection we use the term *ideal* to denote a non-zero ideal, and we put $\mathcal{O}_K^\times := \mathcal{O}_K \backslash \{0\}$.

We call an $\mathcal{O}_K$-submodule $\mathfrak{a}$ of K a **fractional ideal** of $\mathcal{O}_K$ if there is some $c \in \mathcal{O}_K^\times$ such that $c\mathfrak{a} \subseteq \mathcal{O}_K$. Note that $c\mathfrak{a}$ is still an $\mathcal{O}_K$-submodule of K. More precisely, it is an $\mathcal{O}_K$-submodule of $\mathcal{O}_K$, and thus an ideal of $\mathcal{O}_K$. Hence the fractional ideals of $\mathcal{O}_K$ have the form $c^{-1}\mathfrak{b}$ with $c \in \mathcal{O}_K^\times$ and $\mathfrak{b}$ being an ideal of O_K. In particular, every ideal of $\mathcal{O}_K$ is a fractional ideal. Conversly, a fractional ideal $\mathfrak{a}$ is an ideal of $\mathcal{O}_K$ if and only if $\mathfrak{a} \subseteq \mathcal{O}_K$.

Therefore we have generalised the concept of ideals of $\mathcal{O}_K$ to fractional ideals. The advantage of this generalisation is given by Theorem 5.5 on page 107 in [ST02]:

Theorem 2.4.5. *The set of non-zero fractional ideals of $\mathcal{O}_K$ is an abelian group under multiplication with identity $\mathcal{O}_K$.*

We omit the proof and denote the group of non-zero fractional ideals by $\mathcal{F}$. Further, we note that the set of ideals of $\mathcal{O}_K$ itself is only a commutative semigroup, but not a group as we are missing inverses in general.

Next we define a fractional ideal $\mathfrak{a}$ to be **principal** if it comes from a principal ideal in $\mathcal{O}_K$, so if it is of the form $c^{-1}\mathfrak{b}$ with $c \in \mathcal{O}_K^\times$ and $\mathfrak{b}$ being a principal ideal of $\mathcal{O}_K$. Let $\mathcal{P}$ be the subset of $\mathcal{F}$ consisting of all principal fractional ideals. One can check that $\mathcal{P}$ is a subgroup of $\mathcal{F}$. Therefore we may define the **ideal class-group** of $\mathcal{O}_K$ as the quotient group

$$H := \mathcal{F}/\mathcal{P}.$$

Further, we define the **class-number** $h(\mathcal{O}_K)$ as the order of the group H.

Theorem 2.4.6. *The ideal class-group of a number field is a finite abelian group.*

Hence the class-number is always a well-defined natural number. A proof of the statement is given in [ST02]. The theorem itself can be found on page 157, but the corresponding proof uses some more advanced techniques we haven't developed here, like Minkowski's theorem. For a more elementary proof we refere to Section 2.10 on page 74 to 76 in [Ono90] where a slightly different approach is used to introduce the class-number:

For two ideals $\mathfrak{b}$, $\mathfrak{b}'$ of $\mathcal{O}_K$ we define $\mathfrak{b} \sim \mathfrak{b}'$ if there are elements $c, c' \in \mathcal{O}_K^\times$ such that $\langle c \rangle \mathfrak{b} = \langle c' \rangle \mathfrak{b}'$. This gives an equivalence relation on the set of ideals of $\mathcal{O}_K$ whose equivalence classes are called **ideal classes**, denoted by $[\mathfrak{b}]$ for some ideal $\mathfrak{b}$. Let $\mathfrak{J}$ be the set of ideal classes, then we can give $\mathfrak{J}$ a group structure by defining $[\mathfrak{b}] \cdot [\mathfrak{b}'] = [\mathfrak{b}\mathfrak{b}']$. (Compare Theorem 2.14 on page 77 in [Ono90].)

It turns out that this group is isomorphic to the ideal class-group as defined above, and thus the number of ideal classes equals the class-number of K. We will explain this correspondence in the following:

Define two fractional ideals $\mathfrak{a}$ and $\mathfrak{a}'$ to be equivalent, denoted by $\mathfrak{a} \approx \mathfrak{a}'$, if they represent the same coset in H, and denote such a coset by $[[\mathfrak{a}]]$. Write $\mathfrak{a} = c^{-1}\mathfrak{b}$ with $c \in \mathcal{O}_K^\times$ and $\mathfrak{b}$ being an ideal in $\mathcal{O}_K$. Then $\mathfrak{b} = c\mathfrak{a} = \langle c \rangle \mathfrak{a}$, and thus $\mathfrak{a} \approx \mathfrak{b}$, so every coset contains at least one proper ideal of $\mathcal{O}_K$. Hence we can choose a set of representatives $\{\mathfrak{b}_1, \ldots, \mathfrak{b}_h\}$ for the quotient $\mathcal{F}/\mathcal{P}$ where all $\mathfrak{b}_j$ are proper ideals of $\mathcal{O}_K$.

Now let $\mathfrak{b}$, $\mathfrak{b}'$ be ideals of $\mathcal{O}_K$ that are equivalent as fractional ideals, so $\mathfrak{b} \approx \mathfrak{b}'$. We claim that this implies $\mathfrak{b}$ and $\mathfrak{b}'$ to be equivalent as ideals in $\mathcal{O}_K$ as well. To see this let $\mathfrak{p} \in \mathcal{P}$ such that $\mathfrak{b} = \mathfrak{p}\mathfrak{b}'$ and write $\mathfrak{p} = c^{-1}\langle c' \rangle$ with $c, c' \in \mathcal{O}_K^\times$. Then

$$\langle c \rangle \mathfrak{b} = c\mathfrak{b} = \langle c' \rangle \mathfrak{b}',$$

and thus $\mathfrak{b} \sim \mathfrak{b}'$ as claimed. Conversely, $\mathfrak{b} \sim \mathfrak{b}'$ clearly implies $\mathfrak{b} \approx \mathfrak{b}'$ since all principal ideals of $\mathcal{O}_K$ are elements of $\mathcal{P}$.

This shows that $[\mathfrak{b}] = [\mathfrak{b}']$ if and only if $[[\mathfrak{b}]] = [[\mathfrak{b}']]$ for ideals $\mathfrak{b}$, $\mathfrak{b}'$ of $\mathcal{O}_K$, and thus the set $\{\mathfrak{b}_1, \ldots, \mathfrak{b}_h\}$ is also a set of representatives for the set of ideal classes $\mathfrak{J}$. This proves that the ideal class group H and $\mathfrak{J}$ are indeed isomorphic since the group operation is in both groups given by multiplication of ideals.

2.4.5 Orders of number fields

Next we introduce orders of number fields. As these are not treated in [ST02] we have to use a different reference at this point, namely [Ste08].

Let K be a number field. We call a subring $\mathcal{O}$ of K an **order** in K, if $\mathcal{O}$ is a $\mathbb{Z}$-module which is free of rank $n = [K : \mathbb{Q}]$. Thus a $\mathbb{Z}$-basis of an order $\mathcal{O}$ is always a $\mathbb{Q}$-basis of the corresponding number field, so $\mathcal{O} \otimes \mathbb{Q} = K$. Further, $\mathcal{O}_K$ is clearly an order since we remarked in Subsection 2.4.2 that $\mathcal{O}_K$ is a free abelian group of rank n. We call an order **maximal** if it is maximal with respect to inclusion. By Theorem 2.2 on page 213 of [Ste08] a subring $\mathcal{O}$ of K is an order if and only if $\mathcal{O}$ is of finite index in the ring of integers $\mathcal{O}_K$. Thus $\mathcal{O}_K$ is the unique maximal order of K.

Recall that we used fractional ideals of $\mathcal{O}_K$ to introduce the ideal class-group $\mathcal{H}$ of $\mathcal{O}_K$. Section 4 of [Ste08] shows how to generalise this concept to arbitrary orders in K:

Let $\mathcal{O}$ be an order in K. We may extend the definition of fractional ideals of $\mathcal{O}_K$ to fractional ideals of $\mathcal{O}$, but in contrast to the former ones, non-zero fractional ideals of $\mathcal{O}$ do in general not have a natural inverse. So to generalise Theorem 2.4.5 we only consider invertible fractional ideals $\mathcal{F}(\mathcal{O})$ of $\mathcal{O}$. Stevenhagen shows on page 216 that these indeed form an abelian group under multiplication with identity $\mathcal{O}$. Since in particular all principal fractional ideals $\mathcal{P}(\mathcal{O})$ of $\mathcal{O}$ are invertible we may proceed as before, namely define the **ideal class-group** of $\mathcal{O}$ as the quotient $\mathcal{F}(\mathcal{O})/\mathcal{P}(\mathcal{O})$ and the **class-number** $h(\mathcal{O})$ as the order of this quotient group. Finally Corollary 10.6 on page 238 of [Ste08] generalises Theorem 2.4.6 from the previous subsection, so $h(\mathcal{O})$ is a well-defined natural number.

We quote and explain a formula which lets us compute the class-number of an order in an imaginary quadratic field directly. This will be very helpful in the course of Chaper 5. The formula is given in part (1) of Theorem 6.7.2 on page 257 in [Miy06]. We omit the proof.

Theorem 2.4.7. *Let $K = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field, so d a squarefree and negative integer, and let $\mathcal{O}$ be an arbitrary order in K with $[\mathcal{O}_K : \mathcal{O}] = n$. Then the class number of $\mathcal{O}$ is given by*

$$h(\mathcal{O}) \;=\; \frac{n \cdot h(\mathcal{O}_K)}{[U(\mathcal{O}_K) : U(\mathcal{O})]} \prod_{\substack{p \text{ prime} \\ p \mid n}} \left(1 - \frac{L(d,p)}{p}\right)$$

where $L(d,p)$ denotes the Legendre-Symbol.

The Legendre-Symbol $L(d,p)$ is introduced in the appendix of [ST02], more precisely, on page 283. It is defined for an odd prime p and an integer d not divisible by p via

$$L(d,p) = \begin{cases} 1, & \text{if there is } m \in \mathbb{Z} \text{ such that } m^2 = d \bmod p, \\ -1, & \text{if there is no such } m \in \mathbb{Z}. \end{cases}$$

If $L(d,p) = 1$ then d is called a quadratic residue modulo p. To calculate $L(d,p)$ we can use Proposition A.15 on page 284 of [ST02] which states $L(d,p) = d^{(p-1)/2} \bmod p$ for odd primes p and integers d not divisible by p. One may naturally extend the Legendre-Symbol to integers d divisible by p via $L(d,p) = 0$. Further, Kronecker defined $L(d,p)$ for $p = 2$ via

$$L(d,2) = \begin{cases} 1, & \text{if } d = \pm 1 \bmod 8, \\ 0, & \text{if } d \text{ is even}, \\ -1, & \text{if } d = \pm 3 \bmod 8. \end{cases}$$

(See for example [MV07] for details on this matter. In particular, the case $p = 2$ is dealt with on page 296.)

Next we quickly consider the quantity $[U(\mathcal{O}_K) : U(\mathcal{O})]$ appearing in the formula. Let d be a squarefree negative integer and let $\mathcal{O}$ be an order in the imaginary quadratic field

$K = \mathbb{Q}(\sqrt{d}\,)$ with $[\mathcal{O}_K : \mathcal{O}] = n$. By Proposition 2.4.4 we clearly have $U(\mathcal{O}) = \{\pm 1\}$ if d is neither -1 nor -3. Let $d = -1$. If $\pm i \in \mathcal{O}$ then $\mathcal{O}_K = \mathbb{Q}(\pm i) \subseteq \mathcal{O}$, so $\mathcal{O} = \mathcal{O}_K$. Hence we have $\pm i \in \mathcal{O}$ if and only if $n = 1$. Now let $d = -3$. A similar argument shows that one of $\pm e^{2\pi i/3}$, $\pm e^{4\pi i/3}$ is in $\mathcal{O}$ if and only if $\mathcal{O} = \mathcal{O}_K$. Therefore we get

$$|U(\mathcal{O})| = \begin{cases} 4, & \text{if } d = -1 \text{ and } n = 1, \\ 6, & \text{if } d = -3 \text{ and } n = 1, \\ 2, & \text{otherwise,} \end{cases}$$

and thus

$$[U(\mathcal{O}_K) : U(\mathcal{O})] = \begin{cases} 2, & \text{if } d = -1 \text{ and } n > 1, \\ 3, & \text{if } d = -3 \text{ and } n > 1, \\ 1, & \text{otherwise.} \end{cases}$$

Finally, we consider discriminants of orders in imaginary quadratic fields. Let d be a squarefree negative integer and let $\mathcal{O}$ be an order in the imaginary quadratic field $K = \mathbb{Q}(\sqrt{d}\,)$ with $[\mathcal{O}_K : \mathcal{O}] = n$ as before. By definition we have $1 \in \mathcal{O}$, so $\mathbb{Z} \subseteq \mathcal{O}$, and $\mathcal{O} \subseteq \mathcal{O}_K$. Using Proposition 2.4.2 we get that

$$\mathcal{O} = \begin{cases} \mathbb{Z} + n\sqrt{d}\,\mathbb{Z}, & \text{if } d = 2, 3 \bmod 4, \\ \mathbb{Z} + n(1 + \sqrt{d}\,)/2 \cdot \mathbb{Z}, & \text{if } d = 1 \bmod 4. \end{cases}$$

We may now define the discriminant of an order as the discriminant of the $\mathbb{Z}$-basis of the order. This is well defined, since such a basis is always a $\mathbb{Q}$-basis for K. Hence we can compute as in the proof of Theorem 2.4.3

$$\Delta(\mathcal{O}) := \Delta[1, n\sqrt{d}\,] = 4n^2 d \qquad \text{if } d = 2, 3 \bmod 4,$$

and

$$\Delta(\mathcal{O}) := \Delta\left[1, n(1 + \sqrt{d}\,)/2\right] = n^2 d \qquad \text{if } d = 1 \bmod 4.$$

2.4.6 Table of class numbers of imaginary quadratic fields

We finish this section by quoting a small part of the table given on page 180 in Section 10.4 of [ST02]. It contains a list of class-numbers $h(\mathcal{O}_K)$ for some imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-d}\,)$. By Theorem 2.4.7 this list also enables us to compute class-numbers of arbitrary orders in these fields. We will use the list in Chapter 5 while calculating examples.

d	1	2	3	5	6	7	10	11	13	14	15	17	19	21	22	23	$\cdots$
$h(\mathcal{O}_K)$	1	1	1	2	2	1	2	1	2	4	2	4	1	4	2	3	$\cdots$

Table 2.1: class-numbers of imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-d}\,)$

3 The reproducing kernel of $S_k(\Gamma)$

At the end of this chapter we will be able to express the trace of a Hecke operator for some modular group in terms of a kernel function of some reproducing kernel Hilbert space, namely $H_k^2(\mathcal{H})$. We begin by introducing this space and some related function spaces in the first section. In the second section we will determine its kernel K_k, which plays a central role in this and the next chapter. Section three shows that the space of cusp forms $S_k(\Gamma)$ for some modular group Γ is itself a reproducing kernel Hilbert space. We use the kernel K_k to write down an expression for the kernel of $S_k(\Gamma)$, which finally leads to a first trace formula.

This chapter is based on the first part of Chapter 6 in [Miy06]. More precisely, we deal with Section 6.1 and 6.3 in detail, quote an important result from Section 6.2, and finish with Theorem 6.4.2.

3.1 Some function spaces on $\mathcal{H}$

Throughout the following sections we assume k to be a fixed non-negative integer.

Definition 3.1.1. For $p \in [1, \infty)$ and $f\colon \mathcal{H} \to \mathbb{C}$ we define

$$\|f\|_{k,p} = \left(\int_{\mathcal{H}} \left| f(z) \operatorname{Im}(z)^{k/2} \right|^p d\nu(z) \right)^{1/p}$$

and

$$\|f\|_{k,\infty} = \operatorname{ess\,sup}_{z \in \mathcal{H}} |f(z) \operatorname{Im}(z)^{k/2}|.$$

Moreover, we define $L_k^p(\mathcal{H})$ to be the space of measurable functions $f\colon \mathcal{H} \to \mathbb{C}$ such that $\|f\|_{k,p} < \infty$ where we identify $f, g \in L_k^p(\mathcal{H})$ with each other if $\|f - g\|_{k,p} = 0$. Further, we define $H_k^p(\mathcal{H})$ to be the subspace consisting of all holomorphic functions in $L_k^p(\mathcal{H})$.

It can be easily checked that $L_k^p(\mathcal{H})$ is a normed space with respect to $\| \cdot \|_{k,p}$ for any $p \in [1, \infty]$. In the case $p = 2$ we can define

$$\langle f, g \rangle_k = \int_{\mathcal{H}} f(z) \overline{g(z)} \operatorname{Im}(z)^k d\nu(z)$$

for $f, g \in L_k^2(\mathcal{H})$. Again it can be easily checked that this defines an inner product on $L_k^2(\mathcal{H})$ which induces the norm $\| \cdot \|_{k,2}$, so $L_k^2(\mathcal{H})$ is an inner product space. As $H_k^p(\mathcal{H})$ is a linear subspace of $L_k^p(\mathcal{H})$ it is also a normed space and in the case of $p = 2$ an inner product space.

Proposition 3.1.2. *The space $L^p_k(\mathcal{H})$ is a Banach space for any $p \in [1, \infty]$. In particular, $L^2_k(\mathcal{H})$ is a Hilbert space.*

Proof. By the above observations it sufficies to show that $L^p_k(\mathcal{H})$ is complete. This is clear for $k = 0$ as $L^p_0(\mathcal{H})$ is the usual L^p-space of functions on $\mathcal{H}$ with respect to the measure $d\nu(z)$. But $L^p_k(\mathcal{H})$ and $L^p_0(\mathcal{H})$ are isomorphic as normed spaces via the map $f(z) \mapsto f(z)\operatorname{Im}(z)^{k/2}$, so $L^p_k(\mathcal{H})$ is complete for any integer k. □

We will see that $H^p_k(\mathcal{H})$ is a closed subspace of $L^p_k(\mathcal{H})$ for any $p \in [1, \infty]$, and thus also complete, so a Banach space and if $p = 2$ a Hilbert space. To prove this we need some basic complex analysis:

Lemma 3.1.3. *Let $p \in [1, \infty)$, $z_0 \in \mathcal{H}$ and $\varepsilon > 0$ such that $B := \overline{B_{3\varepsilon}(z_0)}$ is contained in $\mathcal{H}$. Moreover, let $f\colon \mathcal{H} \to \mathbb{C}$ be holomorphic. Then there is $C > 0$ depending on p, z_0 and ε but not on f such that*

$$\sup_{z \in \overline{B_\varepsilon(z_0)}} |f(z)| \leqslant C \left(\int_B \left| f(z) \operatorname{Im}(z)^{k/2} \right|^p d\nu(z) \right)^{1/p}.$$

For a proof we refer to Theorem 2.6.1 on page 61 in [Miy06].

Corollary 3.1.4. *Let $p \in [1, \infty)$ and $f\colon \mathcal{H} \to \mathbb{C}$ be holomorphic. For every $z_0 \in U$ there is $C_{z_0} > 0$ such that $|f(z_0)| \leqslant C_{z_0} \|f\|_{k,p}$.*

Proof. Let $z_0 \in U$ and choose $\varepsilon > 0$ such that $\overline{B_{3\varepsilon}(z_0)} \subseteq \mathcal{H}$. Then by Lemma 3.1.3 there is $C > 0$ depending on z_0 but not on f such that $|f(z_0)| \leqslant C\|f\|_{k,p}$. □

Corollary 3.1.5. *Let $p \in [1, \infty)$ and let $(f_n)_n \subseteq H^p_k(\mathcal{H})$ be a Cauchy sequence with respect to $\|\cdot\|_{k,p}$. Then there is $f\colon \mathcal{H} \to \mathbb{C}$ holomorphic such that $f_n \to f$ uniformly on any compact subset of $\mathcal{H}$.*

Again we omit the proof as the statement is proven in detail in [Miy06], compare Corollary 2.6.4 on page 63. The next proposition is Theorem 6.1.1 on page 220 in [Miy06]. We give a proof since the one Miyake presents misses some details.

Proposition 3.1.6. *For any $p \in [1, \infty]$ the space $H^p_k(\mathcal{H})$ is a Banach space. In particular, $H^2_k(\mathcal{H})$ is a Hilbert space.*

Proof. We already know that $H^p_k(\mathcal{H})$ is a linear subspace of the Banach space $L^p_k(\mathcal{H})$. Therefore it sufficies to show that $H^p_k(\mathcal{H})$ is closed in $L^p_k(\mathcal{H})$ with respect to $\|\cdot\|_{k,p}$.

Let $(f_n)_n$ be a Cauchy sequence in $H^p_k(\mathcal{H})$, then $(f_n)_n$ is also a Cauchy sequence in $L^p_k(\mathcal{H})$ and thus has a limit $f \in L^p_k(\mathcal{H})$. First suppose that $p \in [1, \infty)$. Then there is $h\colon \mathcal{H} \to \mathbb{C}$ holomorphic such that $f_n \to h$ uniformly on any compact subset of $\mathcal{H}$ by Corollary 3.1.5. We have to show that $f = h$ almost everywhere.

By Theorem 5.2 on page 138 ($p = 1$) and Theorem 5.2 on page 210 ($1 < p < \infty$) in [Lan93] a sequence $(g_n)_n$ of some general L^p space that converges to some $g \in L^p$ with respect to the corresponding norm $\|\cdot\|_p$, has a subsequence which converges almost everywhere to g. Put $g_n(z) := f_n(z)\,\mathrm{Im}(z)^{k/2}$ and $g(z) := f(z)\,\mathrm{Im}(z)^{k/2}$. Then $g_n \to g$ in $L_0^p(\mathcal{H})$ which is a general L^p space, so there is a subsequence $(g_{n_l})_l$ such that $g_{n_l} \to g$ almost everywhere. Hence we also have $f_{n_l} \to f$ almost everywhere, and therefore $f = h$ almost everywhere since $f_n \to h$ pointwise.

Now let $p = \infty$. Note that $\|g\|_{k,\infty} = \sup_{z\in\mathcal{H}} |g(z)\,\mathrm{Im}(z)^{k/2}|$ for any continuous function $g \in L_k^\infty(\mathcal{H})$. Put $g_n(z) := f_n(z)\,\mathrm{Im}(z)^{k/2}$ as before, then g_n is continuous since f_n is, and

$$\|g_n\|_{\infty,\mathcal{H}} := \sup_{z\in\mathcal{H}} |g_n(z)| = \|f_n\|_{k,\infty}.$$

Thus $(g_n)_n$ is a Cauchy sequence with respect to the uniform norm $\|\cdot\|_{\infty,\mathcal{H}}$ on $\mathcal{H}$ as $(f_n)_n$ is a Cauchy sequence with respect to $\|\cdot\|_{k,\infty}$. Since the space of continuous functions $C(\mathcal{H},\mathbb{C})$ is complete with respect to the uniform norm, there is $g \in C(\mathcal{H},\mathbb{C})$ such that $g_n \to g$ uniformly. Put $h(z) := g(z)\,\mathrm{Im}(z)^{-k/2}$, then $f = h$ in $L_k^\infty(\mathcal{H})$ by construction. Let K be a compact subset of $\mathcal{H}$. Then

$$\sup_{z\in K} |f_n(z) - h(z)| \leqslant \sup_{z\in K} \mathrm{Im}(z)^{k/2} \cdot \|g_n - g\|_{\infty,\mathcal{H}}.$$

Here the right-hand side goes to 0 as $n \to \infty$ since the continuous function $z \mapsto \mathrm{Im}(z)^{k/2}$ is bounded on the compact set K and $g_n \to g$ uniformly. Therefore we have shown that $f_n \to h$ uniformly on any compact subset of $\mathcal{H}$, which implies that h is holomorphic. Thus the sequence $(f_n)_n$ has a limit in $\mathcal{H}_k^\infty(\mathcal{H})$ as claimed. □

Theorem 3.1.7. *The space $H_k^2(\mathcal{H})$ is a reproducing kernel Hilbert space.*

Proof. We showed in the previous proposition that $H_k^2(\mathcal{H})$ is a Hilbert space. Fix $z \in \mathcal{H}$ and let $E_z(f) := f(z)$ be the evaluation functional on $H_k^2(\mathcal{H})$. By Corollary 3.1.4 we have $|E_z(f)| = |f(z)| \leqslant C_z\|f\|_{k,2}$ for some $C_z > 0$ depending on z but not on f. Hence E_z is continuous for every $z \in \mathcal{H}$ and therefore $H_k^2(\mathcal{H})$ is a reproducing kernel Hilbert space by Proposition 2.3.3. □

Notation. We denote the kernel of $H_k^2(\mathcal{H})$ by K_k.

Recall that by definition K_k is a function of the form $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that $K_k(\cdot, w)$ is an element of $H_k^2(\mathcal{H})$ for every fixed $w \in \mathcal{H}$, and $f(w) = \langle f, K_k(\cdot, w)\rangle_k$ for every $f \in H_k^2(\mathcal{H})$, $w \in \mathcal{H}$. Using part (ii) of Corollary 2.3.4 we see

$$f(w) = \int_{\mathcal{H}} f(z) K_k(w,z)\,\mathrm{Im}(z)^k d\nu(z). \qquad (3.1.1)$$

for any such $f \in H_k^2(\mathcal{H})$, $w \in \mathcal{H}$.

3.2 Computation of the kernel of $H_k^2(\mathcal{H})$

We will now develop a precise formula for K_k. The following proposition starts characterising the kernel and will be very useful in the course of this thesis, too.

Proposition 3.2.1. *For any $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$ we have*

$$K_k(\alpha z, \alpha w) \;=\; \det(\alpha)^{-k}\; j(\alpha, z)^k\; \overline{j(\alpha, w)}^k\; K_k(z, w), \quad z, w \in \mathcal{H}.$$

To prove this proposition we use a simple lemma:

Lemma 3.2.2. *Let $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$ and $f \in L_k^2(\mathcal{H})$. Then*

$$\| f|_k\alpha \|_{k,2} = \det(\alpha)^{k/2-1} \|f\|_{k,2}.$$

In particular, we have $f \in L_k^2(\mathcal{H})$ if and only if $f|_k\alpha \in L_k^2(\mathcal{H})$, and similarly $f \in H_k^2(\mathcal{H})$ if and only if $f|_k\alpha \in H_k^2(\mathcal{H})$.

Proof. Let $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$ and $f \in L_k^2(\mathcal{H})$. Since elements in $\mathrm{GL}_2^+(\mathbb{R})$ act as automorphisms on $\mathcal{H}$ we have $\alpha\mathcal{H} = \mathcal{H}$. Thus we see using Lemma 2.1.1

$$\begin{aligned}\int_{\mathcal{H}} |f(z)|^2 \,\mathrm{Im}(z)^k d\nu(z) &= \int_{\alpha^{-1}\mathcal{H}} |f(\alpha z)|^2 \,\mathrm{Im}(\alpha z)^k d\nu(z) \\ &= \det(\alpha)^{2-k} \int_{\mathcal{H}} |(f|_k\alpha)(z)|^2 \,\mathrm{Im}(z)^k d\nu(z).\end{aligned}$$

Hence $\|f\|_{k,2} = \det(\alpha)^{1-k/2} \| f|_k\alpha \|_{k,2}$. For the second part of the lemma we only have to note that f is holomorphic on $\mathcal{H}$ if and only if $f|_k\alpha$ is. □

Proof of Proposition 3.2.1. Let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ and define

$$K_k^{(\alpha)}(z, w) := \det(\alpha)^k\, j(\alpha, z)^{-k}\, \overline{j(\alpha, w)}^{-k}\, K_k(\alpha z, \alpha w).$$

Using Proposition 2.1.1 one can easily check that

$$\left\langle f, K_k^{(\alpha)}(\cdot, w) \right\rangle_k = \det(\alpha)^{k-1} j(\alpha, w)^{-k} \left\langle f|_k\alpha^{-1}, K_k(\cdot, \alpha w) \right\rangle_k$$

for any $f \in H_k^2(\mathcal{H})$ and any $w \in \mathcal{H}$. By Lemma 3.2.2 we know that $f|_k\alpha^{-1}$ is an element of $H_k^2(\mathcal{H})$. Therefore we can use the reproducing property of the kernel K_k which yields

$$\left\langle f|_k\alpha^{-1}, K_k(\cdot, \alpha w) \right\rangle_k = \left(f|_k\alpha^{-1}\right)(\alpha w) = \det(\alpha)^{1-k} j(\alpha, w)^k f(w).$$

Hence $\langle f, K_k^{(\alpha)}(\cdot, w)\rangle_k = f(w)$ for all $f \in H_k^2(\mathcal{H})$, $w \in \mathcal{H}$. Moreover, we have

$$K_k^{(\alpha)}(z, w) = \det(\alpha)\, \overline{j(\alpha, w)}^{-k} \left(K_k(\cdot, \alpha w)|_k\alpha\right)(z).$$

Since K_k is a reproducing kernel, $K_k(\cdot, \alpha w)$ is an element of $H_k^2(\mathcal{H})$ for fixed $w \in \mathcal{H}$. Thus $K_k(\cdot, \alpha w)|_k\alpha$ is an element of $H_k^2(\mathcal{H})$ by Lemma 3.2.2, and therefore $K_k^{(\alpha)}(\cdot, w)$ itself is an element of $H_k^2(\mathcal{H})$. Hence $K_k^{(\alpha)}$ is a reproducing kernel of $H_k^2(\mathcal{H})$, and thus by uniqueness of the kernel (Proposition 2.3.2) we have $K_k^{(\alpha)} = K_k$. □

Using Proposition 3.2.1 with $\alpha_1 = \left(\begin{smallmatrix} \sqrt{a} & 0 \\ 0 & 1/\sqrt{a} \end{smallmatrix}\right)$ for some $a > 0$, and $\alpha_2 = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ for some $b \in \mathbb{R}$ we directly get the following corollary, which describes the kernel further:

Corollary 3.2.3. *We have for any $a > 0$*

$$K_k(az, aw) = a^{-k} K_k(z, w), \quad z, w \in \mathcal{H},$$

and for any $b \in \mathbb{R}$

$$K_k(z + b, w + b) = K_k(z, w), \quad z, w \in \mathcal{H}.$$

The second part of this corollary is equation (6.1.7) on page 221 in [Miy06]. The subsequent discussion on the same page of [Miy06] proves the following proposition, which is Theorem 6.1.2 on page 222 in [Miy06].

Proposition 3.2.4. *The reproducing kernel of $H_k^2(\mathcal{H})$ is given by*

$$K_k(z, w) = c_k \left(\frac{z - \overline{w}}{2i} \right)^{-k}$$

for some constant $c_k \in [0, \infty)$.

We present Miyake's proof as the statement is essential for this section.

Proof. Define the set $\Omega = \{(z, w) \in \mathbb{C}^2 \colon z \in \mathcal{H}, \overline{z - w} \in \mathcal{H}\}$ and the function

$$h \colon \Omega \to \mathbb{C}, \ (z, w) \mapsto K_k(z, \overline{z - w}).$$

One can check that if $\varphi(z)$ is holomorphic on an open domain U then $\overline{\varphi(\overline{z})}$ is holomorphic on $\overline{U}$. Hence $h(z, w)$ is holomorphic in w since K_k is holomorphic in the first argument and $h(z, w) = \overline{K_k(\overline{z} - \overline{w}, z)}$ as the kernel K_k is conjugate symmetric. Next we fix $w \in \mathcal{H}$ and consider $h(\cdot, w)$ as a composition of the functions $i \colon z \mapsto (z, z)$ and $H \colon (z_1, z_2) \mapsto K_k(z_1, \overline{z_2 - w})$. The function H is holomorphic in both arguments by the above considerations, and the components i_1, i_2 are trivially holomorphic, too.

We are now going to use complex analysis of several variables to show that h is holomorphic in z. First we note that i is holomorphic as a function from $\mathbb{C}$ to $\mathbb{C}^2$ by part (5) of Proposition 1.2.2 on page 8 of [Sch05]. Further, H is partially holomorphic in the sense of Definition 1.2.21 on page 13 of [Sch05]. This implies H to be holomorphic by Hartogs' theorem which is a deep result of the theory of complex analysis of several variables. It is remarked in 1.2.28 on page 17 of [Sch05] and proven in Section 2.4 of [Kra01]. Therefore we have that the composition $h(\cdot, w) = H \circ i$ is holomorphic in the usual sense by part (4) of Proposition 1.2.2 of [Sch05].

So we have shown that h is holomorphic in both arguments. Moreover, we have by part (2) of Corollary 3.2.3 that

$$h(z + b, w) = K_k(z + b, \overline{z - w} + b) = K_k(z, \overline{z - w}) = h(z, w)$$

for any $(z, w) \in \Omega$, $b \in \mathbb{R}$. We claim this implies $h(z, w) = h(z', w)$ for any z, z', w with $(z, w), (z', w) \in \Omega$. To see this consider the map $\Phi(\tau) = h(z + \tau, w) - h(z, w)$ for $\tau \in \mathbb{C}$

and fixed $(z, w) \in \Omega$. Then Φ is holomorphic on a neighbourhood of the real line as h is holomorphic in the first argument, and Φ vanishes on the real line by the above observation. Hence Φ vanishes everywhere. Consequently h is locally constant in z and thus also globally as h is holomorphic in z. Therefore we can define

$$l\colon \mathcal{H} \to \mathbb{C},\ w \mapsto h(z_\omega, w) = K_k\left(z_\omega, \overline{z_\omega - w}\right)$$

where z_ω is any element of the upper half-plane such that $(z_\omega, w) \in \Omega$. (One can easily see that there is such an element z_ω for every $w \in \mathcal{H}$.) The map l is holomorphic as h is in w, and

$$K_k(z, w) = K_k\left(z, \overline{z - (z - \overline{w})}\right) = l\left(z - \overline{w}\right)$$

for any $z, w \in \mathcal{H}$. Next we use part (1) of Corollary 3.2.3 which yields

$$l(2aw) = l\left(aw - \overline{a(-\overline{w})}\right) = K_k\left(aw, a(-\overline{w})\right) = a^{-k} K_k\left(w, -\overline{w}\right) = a^{-k} l(2w)$$

for any $a > 0$. In particular, taking $w = i/2$ we get $l(iy) = y^{-k} l(i)$ for all $y > 0$. Finally, we define

$$L\colon \mathcal{H} \to \mathbb{C},\ z \mapsto \left(\frac{z}{i}\right)^{-k} l(i).$$

Then L is obviously holomorphic on $\mathcal{H}$ and $L(iy) = l(iy)$ for all $y > 0$. So L and l agree on the imaginary axis, and thus everywhere on $\mathcal{H}$ as they are both holomorphic. Therefore we get

$$K_k(z, w) = l\left(z - \overline{w}\right) = L\left(z - \overline{w}\right) = \left(\frac{z - \overline{w}}{i}\right)^{-k} l(i)$$

for any $z, w \in \mathcal{H}$. Put $c_k = 2^{-k} l(i)$ then we see $K_k(z, w) = c_k((z - \overline{w})/2i)^{-k}$ as claimed. Moreover, we have

$$l(i) = K_k\left(i/2, \overline{i/2 - i}\right) = K_k\left(i/2, i/2\right) \in [0, \infty)$$

by part (1) of Corollary 2.3.4, and thus also $c_k \in [0, \infty)$. Therefore we are done. □

Next we want to compute the constant c_k. Up to now we have been following [Miy06] very closely. Though we filled in the details for some arguments, we kept the given structure. The next part will differ from the book. On the pages 222 to 225 Miyake uses Fourier analysis to calculate the constant c_k. We will use a much simpler and shorter argument, which does not describe the space $H_k^2(\mathcal{H})$ as nicely as Miyake's work does (compare Theorem 6.1.6 and Corollary 6.1.7 on page 224), but gives the desired result nevertheless.

The idea is to use the reproducing property of the kernel K_k (equation (3.1.1)) with some explicit function $f \in H_k^2(\mathcal{H})$. This reduces the problem of computing c_k to

(1) finding an element of $H_k^2(\mathcal{H})$, and

(2) evaluating the corresponding integral.

Define

$$f_0 \colon \mathcal{H} \to \mathbb{C}, \; z \mapsto (z+i)^{-k}.$$

We claim that f_0 is an element of $H_k^2(\mathcal{H})$ for any integer $k \geqslant 2$. (Note that this is not a limitation as Miyake proves in Corollary 6.1.7 that $H_k^2(\mathcal{H}) = \{0\}$ for any integer $k \leqslant 1$.) We have

$$\begin{aligned}\|f_0\|_{k,2}^2 &= \int_{\mathbb{R}\times(0,\infty)} |(x+iy)+i|^{-2k}\, y^k \frac{d(x,y)}{y^2} \\ &= \int_{\mathbb{R}\times(1,\infty)} \left(x^2+y^2\right)^{-k} (y-1)^{k-2} d(x,y).\end{aligned}$$

Clearly $(y-1)^{k-2} \leqslant y^{k-2}$ for $y \in (1,\infty)$, $k \geqslant 2$. Hence

$$\begin{aligned}\|f_0\|_{k,2}^2 &\leqslant \int_{\mathbb{R}\times(1,\infty)} \left(x^2+y^2\right)^{-k} y^{k-2} d(x,y) \\ &\leqslant \int_{(1,\infty)\times(0,\pi)} r^{-2k}(r\sin(\varphi))^{k-2} r\, d(r,\varphi) \\ &= \int_0^{\pi} \sin(\varphi)^{k-2} d\varphi \int_1^{\infty} r^{-(k+1)} dr.\end{aligned}$$

The last expression is obviously finite for $k \geqslant 2$, so $f_0 \in H_k^2(\mathcal{H})$ as claimed. Thus we may use equation (3.1.1) with $f = f_0$ and $w = i$. We have

$$\begin{aligned}(2i)^{-k} = f_0(i) &= \int_{\mathcal{H}} f_0(z) K_k(i,z) \operatorname{Im}(z)^k d\nu(z) \\ &= \int_{\mathcal{H}} (z+i)^{-k} (2i)^k c_k (i-\overline{z})^{-k} \operatorname{Im}(z)^k d\nu(z). \qquad (3.2.1)\end{aligned}$$

One can check that $(z+i)^{-k}(i-\overline{z})^{-k} = (-1)^k \left(x^2+(1+y)^2\right)^{-k}$ where $z = x+iy$. Hence equation (3.2.1) gives

$$c_k = 4^{-k} \left[\int_{\mathbb{R}\times(0,\infty)} \left(x^2+(1+y)^2\right)^{-k} y^{k-2} d(x,y) \right]^{-1}. \qquad (3.2.2)$$

It remains to compute this integral. Substituting $s = x/(1+y)$ for x yields

$$\int_{\mathbb{R}\times(0,\infty)} \left(x^2+(1+y)^2\right)^{-k} y^{k-2} d(x,y) = \int_{\mathbb{R}} (1+s^2)^{-k} ds \int_0^{\infty} (1+y)^{-2k+1} y^{k-2} dy. \quad (3.2.3)$$

Let $B(a,b)$ denote the so called *beta function* as defined on the bottom of page 20 in [BW10]. By Theorem 2.1.2 on page 21 of the book we have

$$B(a,b) = \int_0^{\infty} u^{a-1}(1+u)^{-(a+b)} du = \frac{\Gamma(a)\cdot\Gamma(b)}{\Gamma(a+b)}$$

for all $a, b \in \mathbb{C}$ with positive real part. Here $\Gamma(z)$ denotes the *gamma function* as defined on page 19 of the mentioned book. It has the well-known property $\Gamma(n) = (n-1)!$ for all positive integers n. For a further discussion of these two functions we refer to Chapter 2 of [BW10]. In particular, Section 2.1 covers the mentioned results. However, using the stated identities, the second integral on the right-hand side of equation (3.2.3) solves to

$$\int_0^\infty (1+y)^{-2k+1} y^{k-2} dy = B(k-1, k) = \frac{(k-2)! \cdot (k-1)!}{(2k-2)!}. \tag{3.2.4}$$

Thus we are left with the first integral on the right-hand side of equation (3.2.3). Define $g(z) = (1+z^2)^{-k}$, then g is a meromorphic function on $\mathbb{C}$ with poles of order k at $\pm i$. One can check that the residue of g at i is given by

$$\operatorname{Res}(g, i) = -2i \cdot 4^{-k} \cdot \frac{(2k-2)!}{(k-1)! \cdot (k-1)!}.$$

Fix $R > 1$. Put $\gamma_0(t) = Re^{\pi i t}$ and $\gamma_1(t) = R(2t-1)$ for $t \in [0,1]$. By the residue theorem we have

$$\int_{\gamma_0+\gamma_1} g(z) dz = 2\pi i \operatorname{Res}(g, i) = \frac{4^{1-k}\pi \cdot (2k-2)!}{(k-1)! \cdot (k-1)!} \tag{3.2.5}$$

Clearly $\int_{\gamma_1} g(z)dz \to \int_{\mathbb{R}} (1+s^2)^{-k} ds$ as $R \to \infty$ which is the integral we want to compute. On the other hand we have

$$\left| \int_{\gamma_0} g(z) dz \right| \leqslant \int_0^1 \left| f\left(Re^{\pi i t}\right) \cdot R\pi i e^{\pi i t} \right| dt = R\pi \int_0^1 \left|1 + R^2 e^{2\pi i t}\right|^{-k} dt.$$

For sufficiently large R we have $|1 + R^2 e^{2\pi i t}| \geqslant R^2/2$, so

$$\left| \int_{\gamma_0} g(z) dz \right| \leqslant 2^k R^{1-2k} \pi \int_0^1 dt.$$

Evidently the right-hand side goes to 0 as $R \to \infty$. Therefore we get taking the limit $R \to \infty$ on both sides of equation (3.2.5)

$$\int_{\mathbb{R}} \left(1 + s^2\right)^{-k} ds = \frac{4^{1-k}\pi \cdot (2k-2)!}{(k-1)! \cdot (k-1)!}.$$

Combining this equality with equation (3.2.4) we are finally able to compute the integral in (3.2.3). We have

$$\int_{\mathbb{R}\times(0,\infty)} \left(x^2 + (1+y)^2\right)^{-k} y^{k-2} d(x,y) = \frac{4^{1-k}\pi}{k-1},$$

and thus $c_k = (k-1)/(4\pi)$ by equation (3.2.2). Therefore we have shown:

Theorem 3.2.5. *For any integer $k \geqslant 2$ the reproducing kernel of $H_k^2(\mathcal{H})$ is given by*

$$K_k(z,w) = \frac{k-1}{4\pi}\left(\frac{z-\overline{w}}{2i}\right)^{-k}.$$

We end this section by studying some more properties of the kernel K_k, which will be needed in Section 3.4 to determine the kernel of $S_k(\Gamma)$.

Lemma 3.2.6. *For any integer $k \geqslant 3$ and any fixed $w \in \mathcal{H}$ the kernel function $K_k(\cdot, w)$ is an element of $H_k^1(\mathcal{H})$.*

Proof. Let $k \geqslant 3$ and fix $w \in \mathcal{H}$. We already know that $K_k(\cdot, w)$ is holomorphic. Put $\delta = \operatorname{Im}(w)$. Then

$$\begin{aligned}
\|K_k(\cdot,w)\|_{k,1} &= \frac{2^k(k-1)}{4\pi}\int_{\mathcal{H}} |z-\overline{w}|^{-k}\operatorname{Im}(z)^{k/2} d\nu(z) \\
&= \frac{2^k(k-1)}{4\pi}\int_{\mathbb{R}\times(\delta,\infty)} |x+iy|^{-k}(y-\delta)^{k/2-2} d(x,y) \\
&= \frac{2^k(k-1)}{4\pi}\int_0^{\pi}\int_{\delta/\sin(\varphi)}^{\infty} r^{-k}(r\sin(\varphi)-\delta)^{k/2-2} r\,dr\,d\varphi.
\end{aligned}$$

Substituting $t = r\sin(\varphi)/\delta - 1$ for fixed $\varphi \in (0,\pi)$ yields

$$\int_{\delta/\sin(\varphi)}^{\infty} r^{-k+1}(r\sin(\varphi)-\delta)^{k/2-2} dr \;=\; \delta^{-k/2}\sin(\varphi)^{k-2}\int_0^{\infty}(1+t)^{-k+1}t^{k/2-2} dt.$$

For the integral on the right we may again use the identity of the beta function as given in Theorem 2.1.2 on page 21 in [BW10]. We have

$$\int_0^{\infty}(1+t)^{-k+1}t^{k/2-2} dt \;=\; B\left(\frac{k}{2}-1, \frac{k}{2}\right),$$

and therefore

$$\|K_k(\cdot,w)\|_{k,1} \;=\; \frac{2^k(k-1)}{4\pi}\operatorname{Im}(w)^{-k/2} B\left(\frac{k}{2}-1,\frac{k}{2}\right)\int_0^{\pi}\sin(\varphi)^{k-2} d\varphi.$$

Hence we have $\|K_k(\cdot,w)\|_{k,1} < \infty$ since all the terms on the right are finite. $\square$

Lemma 3.2.7. *For any integer $k \geqslant 2$ and any fixed $w \in \mathcal{H}$ the kernel function $K_k(\cdot, w)$ is an element of $H_k^{\infty}(\mathcal{H})$.*

Proof. Let $k \geqslant 2$ and fix $w \in \mathcal{H}$. Writing $z = x+iy$ and $w = a+ib$ one can check that

$$\left|K_k(z,w)\operatorname{Im}(z)^{k/2}\right| = \frac{2^k(k-1)}{4\pi}\left(\frac{y}{(x-a)^2+(y+b)^2}\right)^{k/2}.$$

Hence

$$\|K_k(\cdot,w)\|_{k,\infty} = \frac{2^k(k-1)}{4\pi}\left(\sup_{y>0}\frac{y}{(y+b)^2}\right)^{k/2}.$$

Further, one can check that $\sup_{y>0} y(y+b)^{-2} = (4b)^{-1}$, so

$$\|K_k(\cdot,w)\|_{k,\infty} = \frac{k-1}{4\pi}\,\mathrm{Im}(w)^{-k/2} < \infty.$$

Since $K_k(\cdot,w)$ is also holomorphic we are done. $\square$

Combining Lemma 3.2.6 and Lemma 3.2.7 we can prove part (3) of Theorem 6.2.1 on page 226 in [Miy06]:

Corollary 3.2.8. *For any integer $k \geqslant 3$ and any fixed $w \in \mathcal{H}$ the kernel function $K_k(\cdot,w)$ is an element of $H_k^p(\mathcal{H})$ for all $p \in [1,\infty]$.*

Proof. Let $k \geqslant 3$ and fix $w \in \mathcal{H}$. We have $K_k(\cdot,w) \in H_k^1(\mathcal{H}) \cap H_k^\infty(\mathcal{H})$. Let $p \in (1,\infty)$ and put $f(z) = K_k(z,w)\,\mathrm{Im}(z)^{k/2}$. Then

$$\begin{aligned}
\|K_k(\cdot,w)\|_{k,p}^p &= \int_{\mathcal{H}} |f(z)|^p \chi_{\{z:\,|f(z)|\leqslant 1\}}(z)d\nu(z) + \int_{\mathcal{H}} |f(z)|^p \chi_{\{z:\,|f(z)|>1\}}(z)d\nu(z)\\
&\leqslant \int_{\mathcal{H}} |f(z)| \chi_{\{z:\,|f(z)|\leqslant 1\}}(z)d\nu(z) + \|f\|_\infty^p \int_{\mathcal{H}} |f(z)|\cdot \chi_{\{z:\,|f(z)|>1\}}(z)d\nu(z)\\
&\leqslant \|K_k(\cdot,w)\|_{k,1}\left(1 + \|K_k(\cdot,w)\|_{k,\infty}\right).
\end{aligned}$$

So $\|K_k(\cdot,w)\|_{k,p} < \infty$ and thus $K_k(\cdot,w) \in H_k^p(\mathcal{H})$. $\square$

Now recall that by Proposition 2.3.5 the map

$$\pi_k\colon L_k^2(\mathcal{H}) \to H_k^2(\mathcal{H}),\ (\pi_k f)(w) = \langle f, K_k(\cdot,w)\rangle$$

is a well-defined operator which projects $L_k^2(\mathcal{H})$ onto its subspace $H_k^2(\mathcal{H})$. Let $f \in L_k^p(\mathcal{H})$ for any $p \in [1,\infty]$. Using Hölder's inequality (compare Theorem 5.1 on page 209, 210 in [Lan93]) we get

$$\begin{aligned}
|\langle f, K_k(\cdot,w)\rangle| &\leqslant \int_{\mathcal{H}} \left|f(z)\,\mathrm{Im}(z)^{k/2}\right| \cdot \left|\overline{K_k(z,w)}\,\mathrm{Im}(z)^{k/2}\right| d\nu(z)\\
&\leqslant \left(\int_{\mathcal{H}} \left|f(z)\,\mathrm{Im}(z)^{k/2}\right|^p d\nu(z)\right)^{1/p} \left(\int_{\mathcal{H}} \left|K_k(z,w)\,\mathrm{Im}(z)^{k/2}\right|^q d\nu(z)\right)^{1/q}\\
&= \|f\|_{k,p}\|K_k(\cdot,w)\|_{k,q}
\end{aligned}$$

where q is the well-known Hölder conjugate of p. Hence we may extend the operator π_k to any $L_k^p(\mathcal{H})$ space. Moreover, the following is true:

Theorem 3.2.9. *For any integer $k \geqslant 3$ and any $p \in [1,\infty]$ the operator*

$$\pi_k\colon L_k^p(\mathcal{H}) \to H_k^p(\mathcal{H}),\ (\pi_k f)(w) = \langle f, K_k(\cdot,w)\rangle$$

is well-defined and projects $L_k^p(\mathcal{H})$ onto its subspace $H_k^p(\mathcal{H})$.

Note that this is a very strong statement as it implies that the reproducing property of K_k, namely $f(w) = \langle f, K_k(\cdot, w)\rangle$, does not only hold for $f \in H_k^2(\mathcal{H})$, but for $f \in H_k^p(\mathcal{H})$ for all $p \in [1, \infty]$. We omit the corresponding proof as it is very involved and refer to Section 6.2 in [Miy06] instead which deals with it in detail. In particular, the statement is given by Theorem 6.2.2 on page 226.

Further, we note that we we will only need two special cases of this theorem in the course of this theses, namely that π_k maps $L_k^1(\mathcal{H})$ to $H_k^1(\mathcal{H})$ and that π_k acts trivially on $H_k^\infty(\mathcal{H})$. These facts will be essential for the proof of Theorem 3.4.5.

3.3 Interpretation of $S_k(\Gamma)$ as a reproducing kernel Hilbert space

Throughout this and the following sections of the present chapter let Γ be a modular group, so a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and $k \geqslant 3$ an integer. We start by defining spaces of Γ-invariant functions similar to the ones introduced in Section 3.1. These will be denoted by $L_k^p(\Gamma)$ and $H_k^p(\Gamma)$, respectively. As in the case of $H_k^2(\mathcal{H})$ it turns out that $H_k^2(\Gamma)$ is a reproducing kernel Hilbert space, and one can easily check that $S_k(\Gamma)$, the space of cusp forms of weight k and level Γ, is contained in $H_k^2(\Gamma)$. In fact, these spaces agree for $k \geqslant 3$. (This also explains why we only consider integers $k \geqslant 3$ from now on.)

First note that for f being Γ-invariant of weight k the map $z \mapsto |f(z)\,\mathrm{Im}(z)^{k/2}|$ is Γ-invariant of weight 0, so $\left|f(\gamma z)\,\mathrm{Im}(\gamma z)^{k/2}\right| = \left|f(z)\,\mathrm{Im}(z)^{k/2}\right|$ for all $\gamma \in \Gamma$. Thus the following definition is well-defined:

Definition 3.3.1. For $p \in [1, \infty)$ and $f\colon \mathcal{H} \to \mathbb{C}$ satisfying $f|_k\gamma = f$ for all $\gamma \in \Gamma$ we define

$$\|f\|_{\Gamma,p} = \left(\int_{\Gamma\backslash\mathcal{H}} \left|f(z)\,\mathrm{Im}(z)^{k/2}\right|^p d\nu(z)\right)^{1/p}$$

and

$$\|f\|_{\Gamma,\infty} = \operatorname{ess\,sup}_{z\in\Gamma\backslash\mathcal{H}} |f(z)\,\mathrm{Im}(z)^{k/2}|.$$

Moreover, we define $L_k^p(\Gamma)$ to be the space of measurable functions $f\colon \mathcal{H} \to \mathbb{C}$ such that $f|_k\gamma = f$ for all $\gamma \in \Gamma$ and $\|f\|_{\Gamma,p} < \infty$ where we identify $f, g \in L_k^p(\Gamma)$ with each other if $\|f - g\|_{\Gamma,p} = 0$. Further, we define $H_k^p(\Gamma)$ to be the subspace consisting of all holomorphic functions in $L_k^p(\Gamma)$.

Clearly $\|\cdot\|_{k,\infty}$ and $\|\cdot\|_{\Gamma,\infty}$ agree on $L_k^\infty(\Gamma)$ since $z \mapsto |f(z)\,\mathrm{Im}(z)^{k/2}|$ is Γ-invariant of weight 0 as mentioned earlier. In particular, $L_k^\infty(\Gamma)$ is a subspace of $L_k^\infty(\mathcal{H})$. As in Section 3.1 one can easily check that $L_k^p(\Gamma)$ is a normed space with respect to $\|\cdot\|_{\Gamma,p}$ for any $p \in [1, \infty]$, and similarly to $\langle\cdot,\cdot\rangle_k$ we can put

$$\langle f, g\rangle_\Gamma = \int_{\Gamma\backslash\mathcal{H}} f(z)\overline{g(z)}\,\mathrm{Im}(z)^k d\nu(z)$$

for $f, g \in L_k^2(\Gamma)$. This is well-defined since $z \mapsto f(z)\overline{g(z)}\,\mathrm{Im}(z)^k$ is also Γ-invariant of weight 0, and thus $\langle\cdot,\cdot\rangle_\Gamma$ defines an inner product on $L_k^2(\Gamma)$ which clearly corresponds to the norm $\|\cdot\|_{\Gamma,2}$. So $L_k^2(\Gamma)$ is an inner product space. As in Section 3.1, $H_k^p(\Gamma)$ is a normed space and in the case of $p = 2$ an inner product space, since it is a linear subspace of $L_k^p(\Gamma)$.

Proposition 3.3.2. *The space $H_k^p(\Gamma)$ is a Banach space for any $p \in [1, \infty]$. In particular, $H_k^2(\Gamma)$ is a Hilbert space.*

We only sketch the proof:

Proof. Similarly to $L_k^p(\mathcal{H})$ one may define the space $L_k^p(U)$ where we replace the upper half-plane $\mathcal{H}$ by some open subset U of $\mathcal{H}$. Let $\|\cdot\|_{U,p}$ denote the corresponding norm. As in Proposition 3.1.2 one can check that this space is a Banach space.

Let U be the interior of some fundamental domain F of Γ. Then $H_k^p(\Gamma)$ is a subspace of $L_k^p(U)$. Let $(f_n)_n$ be a Cauchy sequence in $H_k^p(\Gamma)$ with respect to $\|\cdot\|_{U,p}$. Using similar arguments as in Proposition 3.1.6 one can check that there is a holomorphic function $f \in L_k^p(U)$ such that $f_n \to f$ with respect to $\|\cdot\|_{U,p}$. Clearly f is also Γ-invariant since all f_n are Γ-invariant and continuous. □

Theorem 3.3.3. *We have*

$$S_k(\Gamma) = H_k^\infty(\Gamma) = H_k^2(\Gamma).$$

Moreover, $S_k(\Gamma)$ and $H_k^2(\Gamma)$ are isomorphic as Hilbert spaces.

The first equality of this theorem is exactly Theorem 2.1.5 on page 42 in [Miy06]. Further, $H_k^\infty(\Gamma) \subseteq H_k^2(\Gamma)$ is obvious since $\Gamma\backslash\mathcal{H}$ has finite measure with respect to ν. Hence it remains to check that $H_k^2(\Gamma)$ is contained in $S_k(\Gamma)$, so that functions in $H_k^2(\Gamma)$ vanish at the cusps of Γ. This is done in Theorem 6.1.3 on page 228, 229 in [Miy06]. We omit these slightly technical proofs here, as they are given in appropriate detail in Miyake's book. Finally, the second part of the theorem is clear since the Petersson inner product of $S_k(\Gamma)$ coincides with the inner product of $H_k^2(\Gamma)$.

The following theorem is almost a direct consequence of the previous identity.

Theorem 3.3.4. *The space $S_k(\Gamma)$ is a reproducing kernel Hilbert space.*

Proof. We already know that $S_k(\Gamma) = H_k^2(\Gamma)$ is a Hilbert space. Fix $z \in \mathcal{H}$ and let $E_z(f) := f(z)$ be the evaluation functional on $S_k(\Gamma)$. One can check that there is always a finite number of fundamental domains $F_1, \ldots, F_n$ such that the corresponding interiors are pairwise disjoint and z lies in the interior of $\bigcup_{j=1}^n F_j$. We call this interior U and may choose $\varepsilon > 0$ such that $B := \overline{B_{3\varepsilon}(z)} \subseteq U$. By Lemma 3.1.3 there is $C > 0$ depending on z but not on f such that

$$\begin{aligned}|E_z(f)|^2 = |f(z)|^2 &\leqslant C^2 \int_U \left|f(z)\operatorname{Im}(z)^{k/2}\right|^2 d\nu(z)\\ &= C^2 \sum_{j=1}^n \int_{F_j} \left|f(z)\operatorname{Im}(z)^{k/2}\right|^2 d\nu(z) = C^2 n \|f\|_{\Gamma,2}^2.\end{aligned}$$

Hence E_z is continuous for every $z \in \mathcal{H}$ and therefore $S_k(\Gamma)$ is a reproducing kernel Hilbert space by Proposition 2.3.3. □

3.4 Computation of the kernel of $S_k(\Gamma)$

Next we want to determine the kernel of the reproducing kernel Hilbert space $S_k(\Gamma)$ where $k \geqslant 3$ is an integer and Γ is a modular group. Instead of characterising it step by step as in Section 3.2, we will write down a guess for the reproducing kernel function, and prove that this function has the desired properties.

We define for $f \in L^1_k(\mathcal{H})$ the function

$$f^\Gamma(z) = \frac{1}{|Z(\Gamma)|} \sum_{\gamma \in \Gamma} (f|_k\gamma)(z), \quad z \in \mathcal{H}.$$

By definition f^Γ is clearly Γ-invariant if it is well-defined, which is not obvious and will be shown in the following proposition.

Proposition 3.4.1. *Let $f \in L^1_k(\mathcal{H})$. The sum defining f^Γ converges absolutely almost everywhere on $\mathcal{H}$ and f^Γ is an element of $L^1_k(\Gamma)$.*

Note that this is part (1) of Theorem 6.3.2 on page 229 in [Miy06]. Since the corresponding proof is not very detailed, we will give a proof here.

Proof. Fix a fundamental domain F of Γ and note that

$$\int_F \left|f^\Gamma(z)\right| \operatorname{Im}(z)^{k/2} d\nu(z) \leqslant \frac{1}{|Z(\Gamma)|} \int_F \sum_{\gamma \in \Gamma} |f(\gamma z)| \operatorname{Im}(\gamma z)^{k/2} d\nu(z).$$

We can interchange summation and integration since the integrand $|f(\gamma z)| \operatorname{Im}(\gamma z)^{k/2}$ is positive. (This follows for example from Corollary 5.13 on page 143 in [Lan93].) So

$$\begin{aligned}\int_F \sum_{\gamma \in \Gamma} |f(\gamma z)| \operatorname{Im}(\gamma z)^{k/2} d\nu(z) &= \sum_{\gamma \in \Gamma} \int_F |f(\gamma z)| \operatorname{Im}(\gamma z)^{k/2} d\nu(z) \\ &= \sum_{\gamma \in \Gamma} \int_{\gamma(F)} |f(z)| \operatorname{Im}(z)^{k/2} d\nu(z) \\ &= |Z(\Gamma)| \int_{\mathcal{H}} |f(z)| \operatorname{Im}(z)^{k/2} d\nu(z).\end{aligned}$$

Therefore $\sum_{\gamma \in \Gamma} |f(\gamma z) j(\gamma, z)^{-k}| \operatorname{Im}(z)^{k/2}$ is ν-integrable over any fundamental domain F of Γ. Hence the sum defining f^Γ is absolutely convergent almost everywhere on any fundamental domain of Γ, so on $\mathcal{H}$. Furthermore, we have shown that

$$\left\|f^\Gamma\right\|_{\Gamma,1} = \int_{\Gamma \backslash \mathcal{H}} \left|f^\Gamma(z)\right| \operatorname{Im}(z)^{k/2} d\nu(z) \leqslant \|f\|_{k,1}.$$

Thus f^Γ is a well-defined element of $L^1_k(\Gamma)$. □

Proposition 3.4.2. *If $f \in H^1_k(\mathcal{H})$ then $f^\Gamma \in H^1_k(\Gamma)$ and the sum defining f^Γ is absolutely convergent everywhere on $\mathcal{H}$.*

This is part (2) of Theorem 6.3.2 on page 229 in [Miy06], and it follows directly from the previous proposition and the following lemma, which is a special case of part (1) of Theorem 2.6.6 on page 64 in [Miy06]. Considering the notation on the bottom of page 63 we chose Λ to be trivial and f to be holomorphic everywhere on $\mathcal{H}$. We omit the proof.

Lemma 3.4.3. *Let $f\colon \mathcal{H} \to \mathbb{C}$ be holomorphic, and let $\{c_1, \ldots, c_r\}$ be the set of cusps of Γ. Put*

$$H' = \mathcal{H} \backslash \left(\bigcup_{i=1}^{r} \bigcup_{\gamma \in \Gamma} \gamma V_i \right)$$

where V_i is any neighbourhood of the cusp c_i. If f satisfies

$$\int_{H'} |f(z)| \operatorname{Im}(z)^{k/2} d\nu(z) < \infty$$

for all such H', we define the ***Poincaré series*** *of f as*

$$F_k(z) = \sum_{\gamma \in \Gamma} (f|_k \gamma)(z).$$

The series defining F_k converges absolutely and uniformly on any compact subset of $\mathcal{H}$. In particular, F_k is Γ-invariant and holomorphic on $\mathcal{H}$.

Now we define what will turn out to be the reproducing kernel of $S_k(\Gamma)$. Recall that K_k is the reproducing kernel of $H_k^2(\mathcal{H})$ as determined in Section 3.2. We have shown in Lemma 3.2.6 that $K_k(\cdot, w) \in H_k^1(\mathcal{H})$ for every fixed $w \in \mathcal{H}$. This allows us to define

$$K_k^\Gamma(z, w) = \frac{1}{|Z(\Gamma)|} \sum_{\gamma \in \Gamma} (K_k(\cdot, w)|_k \gamma)(z), \quad z, w \in \mathcal{H}.$$

By Proposition 3.4.2 the sum defining K_k^Γ is absolutely convergent for every fixed pair of elements $(z, w) \in \mathcal{H} \times \mathcal{H}$, and $K_k^\Gamma(\cdot, w)$ is an element of $H_k^1(\Gamma)$ for every fixed w in $\mathcal{H}$. Moreover, the following statement holds, which will be essential for the proof of Proposition 4.1.9 in the next chapter:

Lemma 3.4.4. *The sum defining K_k^Γ is uniformly convergent on any compact subset of $\mathcal{H} \times \mathcal{H}$.*

This is part (3) of Theorem 6.3.2 in [Miy06]. In the end of the corresponding proof on page 230 Miyake refers to (his) Corollary 2.6.4. We remark that he probably means (his) Theorem 2.6.1 instead. In the following we fill in the details for the proof Miyake gives using Theorem 2.6.1 which corresponds to our Lemma 3.1.3.

Proof. Fix $w \in \mathcal{H}$ and note that

$$\left\| \sum_{\gamma \in \Gamma} |(K_k(\cdot, w)|_k \gamma)(z)| \right\|_{\Gamma, 1} = \int_{\Gamma \backslash \mathcal{H}} \sum_{\gamma \in \Gamma} |K_k(\gamma z, w)| \operatorname{Im}(\gamma z)^{k/2} d\nu(z).$$

We may interchange summation and integration for a fixed fundamental domain F of Γ since $|K_k(\gamma z, w)| \operatorname{Im}(\gamma z)^{k/2}$ is positive. So we see

$$\begin{aligned}\left\|\sum\nolimits_{\gamma\in\Gamma} |(K_k(\cdot,w)|_k\gamma)(z)|\right\|_{\Gamma,1} &= \sum_{\gamma\in\Gamma}\int_F |K_k(\gamma z,w)| \operatorname{Im}(\gamma z)^{k/2} d\nu(z)\\ &= \sum_{\gamma\in\Gamma}\int_{\gamma F} |K_k(z,w)| \operatorname{Im}(z)^{k/2} d\nu(z)\\ &= |Z(\Gamma)| \int_{\mathcal{H}} |K_k(z,w)| \operatorname{Im}(z)^{k/2} d\nu(z).\end{aligned}$$

Write $w = x + iy$ for $w \in \mathcal{H}$ and put $\sigma = y^{-1/2}\left(\begin{smallmatrix} 1 & -x \\ 0 & y\end{smallmatrix}\right)$. Then $\sigma \in \mathrm{SL}_2(\mathbb{R})$ with $\sigma w = i$. Further, we have $j(\sigma, w) = \operatorname{Im}(w)^{1/2}$. We use Proposition 3.2.1 with $\alpha = \sigma$ to get

$$|K_k(z,w)| \operatorname{Im}(z)^{k/2} = \operatorname{Im}(w)^{-k/2} |K_k(\sigma z, i)| \operatorname{Im}(\sigma z)^{k/2}.$$

Therefore we have by Lemma 2.1.1 substituting $z' = \sigma z$ in the integral that

$$\left\|\sum\nolimits_{\gamma\in\Gamma} |(K_k(\cdot,w)|_k\gamma)(z)|\right\|_{\Gamma,1} = |Z(\Gamma)| \operatorname{Im}(w)^{-k/2} \, \|K_k(\cdot,i)\|_{k,1}$$

and thus

$$\left\|K_k^\Gamma(\cdot,w)\right\|_{\Gamma,1} \leqslant \|K_k(\cdot,i)\|_{k,1} \operatorname{Im}(w)^{-k/2} \tag{3.4.1}$$

for any $w \in \mathcal{H}$.

Now let $z_0 \in \mathcal{H}$ be arbitrary. As in the proof of Theorem 3.3.4 we may choose a finite number of fundamental domains $F_1, \ldots, F_n$ for Γ such that the corresponding interiors are pairwise disjoint and z_0 lies in the interior U of $\bigcup_{j=1}^n F_j$. Further, we choose $\varepsilon > 0$ such that $\overline{B_{3\varepsilon}(z_0)} \subseteq U$. By Lemma 3.1.3 there is a constant $C > 0$ depending only on z_0 and ε such that

$$\begin{aligned}\sup_{z\in B_\varepsilon(z_0)} \left|K_k^\Gamma(z,w)\right| &\leqslant C \int_U \left|K_k^\Gamma(z,w)\operatorname{Im}(z)^{k/2}\right| d\nu(z)\\ &= C\sum_{j=1}^n \int_{F_j} \left|K_k^\Gamma(z,w)\operatorname{Im}(z)^{k/2}\right| d\nu(z) = Cn \left\|K_k^\Gamma(\cdot,w)\right\|_{\Gamma,1}.\end{aligned}$$

for all $w \in \mathcal{H}$ since all functions $K_k^\Gamma(\cdot,w)$, $w \in \mathcal{H}$, are holomorphic on $\mathcal{H}$ as remarked earlier. Let $w_0 \in \mathcal{H}$ and put $\delta = \operatorname{Im}(w_0)/2$. Then $\overline{B_\delta(w_0)} \subseteq \mathcal{H}$ and by equation (3.4.1)

$$\sup_{\substack{z\in B_\varepsilon(z_0)\\ w\in B_\delta(w_0)}} \left|K_k^\Gamma(z,w)\right| \leqslant Cn\|K_k(\cdot,i)\|_{k,1} \cdot \sup_{w\in B_\delta(w_0)} \operatorname{Im}(w)^{-k/2} < \infty.$$

Therefore we have shown that for any pair $(z_0, w_0) \in \mathcal{H}\times\mathcal{H}$ there are $\varepsilon > 0$, $\delta > 0$ such that the sum defining K_k^Γ is unformly convergent on $B_\varepsilon(z_0) \times B_\delta(w_0)$. In other words, the sum is locally uniformly convergent, and hence also uniformly convergent on compact subsets of $\mathcal{H}\times\mathcal{H}$. $\square$

We will now prove that K_k^Γ is indeed the reproducing kernel of $S_k(\Gamma)$. Even though the proof is given in [Miy06] fairly detailed (see Theorem 6.3.3 on page 230), we will present it here, too, since the result is central for this thesis.

Theorem 3.4.5. *The reproducing kernel of $S_k(\Gamma)$ is given by*

$$K_k^\Gamma(z,w) = \frac{1}{|Z(\Gamma)|}\sum_{\gamma\in\Gamma}\left(K_k(\cdot,w)|_k\gamma\right)(z).$$

Proof. By definition we have to check that

(1) $K_k^\Gamma(\cdot,w)\in S_k(\Gamma)$ for every fixed $w\in\mathcal{H}$, and

(2) $f(w) = \left\langle f, K_k^\Gamma(\cdot,w)\right\rangle_\Gamma$ for every $f\in S_k(\Gamma)$ and every $w\in\mathcal{H}$.

We start with (1). Fix $w\in\mathcal{H}$. Recall that $K_k^\Gamma(\cdot,w)\in H_k^1(\Gamma)$ by Proposition 3.4.2. In particular, $K_k^\Gamma(\cdot,w)$ is Γ-invariant and holomorphic. Thus it sufficies to show that $K_k^\Gamma(\cdot,w)$ is an element of $L_k^\infty(\mathcal{H})$, as this would imply

$$\|K_k^\Gamma(\cdot,w)\|_{\Gamma,\infty} = \|K_k^\Gamma(\cdot,w)\|_{k,\infty} < \infty,$$

which then gives $K_k^\Gamma(\cdot,w)\in H_k^\infty(\Gamma) = S_k(\Gamma)$ as desired.

To show that $K_k^\Gamma(\cdot,w)\in L_k^\infty(\mathcal{H})$ we need to recall some basic functional analysis. Let X be a Banach space over $\mathbb{C}$. A sequence $(x_n')_n$ in the dual space X' of X is called weakly-* convergent if the sequence $(x_n'(x))_n$ converges in $\mathbb{C}$ for every fixed $x\in X$. Similarly we say that the sequence $(x_n')_n$ is weakly-* convergent to $x'\in X'$ if $x_n'(x)\to x'(x)$ in $\mathbb{C}$ for every $x\in X$. The Banach-Steinhaus Theorem (see for example Theorem 14.6 on page 96/97 in [Con97]) implies that every weakly-* convergent sequence has a weakly-* limit, and one can easily check that this limit is unique.

Now recall that the usual L^∞-space is isomorphic to the dual space of L^1 via the isomorphism

$$\Phi\colon L^\infty \to \left(L^1\right)',\ f\mapsto [g\mapsto \langle f,g\rangle].$$

(For a proof of this see for example Theorem 2.2 on page 188 in [Lan93].) Hence $L_k^\infty(\mathcal{H})$ is isomorphic to the dual space of $L_k^1(\mathcal{H})$ since $L_k^p(\mathcal{H})$ is isomorphic to $L_0^p(\mathcal{H})$ which is the usual L^p-space with respect to the measure ν.

Consider a sequence $(f_n)_n\subseteq L_k^\infty(\mathcal{H})$. Then every f_n corresponds to some x_n' in $(L_k^1(\mathcal{H}))'$ where x_n' is the linear functional given by $g\mapsto\langle f_n,g\rangle_k$, $g\in L_k^1(\mathcal{H})$. Suppose that $(\langle f_n,g\rangle_k)_n$ is convergent as a sequence in $\mathbb{C}$ for every fixed $g\in L_k^1(\mathcal{H})$. Then the sequence $(x_n')_n$ is weakly-* convergent, and thus has a unique limit $x'\in(L_k^1(\mathcal{H}))'$ by the above observations. Corresponding to x' there is a unique element $f\in L_k^\infty(\mathcal{H})$ such that $x'(g) = \langle f,g\rangle_k$ for $g\in L_k^1(\mathcal{H})$, and since x' is the weakly-* limit of $(x_n')_n$ we have that $\langle f_n,g\rangle_k\to\langle f,g\rangle_k$ in $\mathbb{C}$ for every fixed $g\in L_k^1(\mathcal{H})$.

The idea is now to interpret the sum defining $K_k^\Gamma(\cdot,w)$ as the weakly-* limit of its partial sums. Denote the partial sums of $K_k^\Gamma(\cdot,w)$ by f_n. Then each f_n is up to a scalar factor a finite sum of terms of the form $K_k(\cdot,w)|_k\gamma$ for different $\gamma\in\Gamma$. Hence each f_n is in $L_k^\infty(\mathcal{H})$ as $K_k(\cdot, w)$ is by Lemma 3.2.7. Thus $(f_n)_n$ can be identified with a sequence in $(L_k^1(\mathcal{H}))'$. Suppose that this sequence is weakly-* convergent, then it has a unique limit in $(L_k^1(\mathcal{H}))'$, which again can be identified with some element $f \in L_k^\infty(\mathcal{H})$. In the following we will check that $K_k^\Gamma(\cdot, w)$ and f agree almost everywhere on $\mathcal{H}$, meaning that they denote the same element in $L_k^\infty(\mathcal{H})$, so $K_k^\Gamma(\cdot, w) \in L_k^\infty(\mathcal{H})$ as desired. Afterwards we will prove that the sequence of partial sums is indeed weakly-* convergent as assumed earlier.

By the above considerations there is $f \in L_k^\infty(\mathcal{H})$ such that $\langle f_n, g\rangle_k \to \langle f, g\rangle_k$ in $\mathbb{C}$ for every fixed $g \in L_k^1(\mathcal{H})$. We want to show that f and $K_k^\Gamma(\cdot, w)$ agree almost everywhere on $\mathcal{H}$. Suppose this is not the case, then there is a compact set $K \subseteq \mathcal{H}$ such that f and $K_k^\Gamma(\cdot, w)$ do not agree almost everywhere on K. Put $N = \{z \in K \colon f(z) \neq K_k^\Gamma(z, w)\}$, then N is measurable with $0 < \nu(N) \leqslant \nu(K) < \infty$. Recall that the sequence of partial sums f_n converges pointwise to $K_k^\Gamma(\cdot, w)$ on $\mathcal{H}$, and note that each f_n is obviously measurable as it is continuous. Therefore we may use Egorov's Theorem (see for example Theorem 4.4 on page 33 in [SS05]) which tells us that we can find a closed set $A \subseteq N$ such that $\nu(N \backslash A) \leqslant \nu(N)/2$ and f_n converges to $K_k^\Gamma(\cdot, w)$ uniformly on A.

Now let $g \in L_k^1(\mathcal{H})$ be a function with compact support in A, then we see

$$\langle f, g\rangle_{A,k} = \langle f, g\rangle_k = \lim_{n\to\infty} \langle f_n, g\rangle_k = \lim_{n\to\infty} \langle f_n, g\rangle_{A,k} = \left\langle K_k^\Gamma(\cdot, w), g\right\rangle_{A,k}.$$

Here $\langle \cdot, \cdot\rangle_{A,k}$ denotes the restriction of the scalar product introduced in Section 3.1 to A, and the last equality holds since $f_n \to K_k^\Gamma(\cdot, w)$ uniformly on A. Define

$$G(z) = \operatorname{sign}\left[f(z) - K_k^\Gamma(z, w)\right] \cdot \chi_A(z)$$

where χ_A denotes the characteristic function of A which is 1 on A and 0 otherwise. Then $G \in L_k^1(\mathcal{H})$ as $\nu(A)$ is finite, and G has compact support in A. Therefore we get

$$\begin{aligned}\int_A \left|f(z) - K_k^\Gamma(z, w)\right| \operatorname{Im}(z)^{k/2} d\nu(z) &= \int_A \left(f(z) - K_k^\Gamma(z, w)\right) G(z) \operatorname{Im}(z)^{k/2} d\nu(z) \\ &= \left\langle f - K_k^\Gamma(\cdot, w), G\right\rangle_{A,k} \\ &= 0.\end{aligned}$$

Therefore f and $K_k^\Gamma(\cdot, w)$ agree almost everywhere on A. But this is a contradiction since $A \subseteq N$ and $\nu(A) \geqslant \nu(N)/2 > 0$. Hence f and $K_k^\Gamma(\cdot, w)$ agree almost everywhere on $\mathcal{H}$, and thus $K_k^\Gamma(\cdot, w) \in L_k^\infty(\mathcal{H})$ as claimed. It remains to prove that the sequence of partial sums is indeed weakly-* convergent.

The functional in $(L_k^1(\mathcal{H}))'$ associated to some f_n is given by $g \mapsto \langle f_n, g\rangle_k$, $g \in L_k^1(\mathcal{H})$. We want to show that $\langle f_n, g\rangle_k$ is convergent as a sequence in $\mathbb{C}$ for every fixed $g \in L_k^1(\mathcal{H})$. Fix such a g. Then

$$\langle f_n, g\rangle_k = \int_{\mathcal{H}} \left(\frac{1}{|Z(\Gamma)|} \sum_{\gamma \in A_n} \left(K_k(\cdot, w)|_k \gamma\right)(z)\right) \overline{g(z)} \operatorname{Im}(z)^k d\nu(z)$$

where A_n is a finite subset of Γ with $A_n \to \Gamma$. Since the sum is finite we may interchange summation and integration. Fix $\gamma \in \Gamma$. Using Proposition 3.2.1 with $\alpha = \gamma$ we get

$$(K_k(\cdot, w)|_k\gamma)(z) = \overline{j(\gamma^{-1}, w)}^{-k} K_k(z, \gamma^{-1}w),$$

so

$$\int_{\mathcal{H}} (K_k(\cdot, w)|_k\gamma)(z)\, \overline{g(z)}\, \mathrm{Im}(z)^k d\nu(z) = \overline{j(\gamma^{-1}, w)}^{-k} \left\langle K_k(\cdot, \gamma^{-1}w), g\right\rangle_k .$$

Let π_k be the projection operator as defined in Theorem 3.2.9. Then

$$\left\langle K_k(\cdot, \gamma^{-1}w), g\right\rangle_k = \overline{(\pi_k g)(\gamma^{-1}w)},$$

and therefore

$$\begin{aligned}\lim_{n\to\infty} \langle f_n, g\rangle_k &= \lim_{n\to\infty} \left[|Z(\Gamma)|^{-1} \sum_{\gamma\in A_n} \overline{j(\gamma^{-1}, w)}^{-k}\, \overline{(\pi_k g)(\gamma^{-1}w)} \right] \\ &= |Z(\Gamma)|^{-1} \sum_{\gamma\in\Gamma} \overline{((\pi_k g)|_k\gamma^{-1})(w)} \\ &= \overline{|Z(\Gamma)|^{-1} \sum_{\gamma\in\Gamma} ((\pi_k g)|_k\gamma)(w)}.\end{aligned}$$

Hence we have $\lim_{n\to\infty} \langle f_n, g\rangle_k = \overline{(\pi_k g)^\Gamma(w)}$ which is absolutely convergent for every $w \in \mathcal{H}$ by Proposition 3.4.2 and since π_k maps $g \in L^1_k(\mathcal{H})$ to $\pi_k g \in H^1_k(\mathcal{H})$ by Theorem 3.2.9. So the sequence of functionals associated to the partial sums of K^Γ_k is indeed weakly-* convergent as claimed. Thus we are done with (1).

It remains to prove (2), so that $f(w) = \left\langle f, K^\Gamma_k(\cdot, w)\right\rangle_\Gamma$ for every $f \in S_k(\Gamma)$ and every $w \in \mathcal{H}$. Fix such f and w. Recall that we have $\overline{K^\Gamma_k(z, w)} = K^\Gamma_k(w, z)$ by part (ii) of Corollary 2.3.4. So

$$\begin{aligned}\left\langle f, K^\Gamma_k(\cdot, w)\right\rangle_\Gamma &= \int_{\Gamma\backslash\mathcal{H}} f(z) K^\Gamma_k(w, z)\, \mathrm{Im}(z)^k d\nu(z) \\ &= \frac{1}{|Z(\Gamma)|} \int_{\Gamma\backslash\mathcal{H}} \sum_{\gamma\in\Gamma} f(z) K_k(\gamma w, z) j(\gamma, w)^{-k}\, \mathrm{Im}(z)^k d\nu(z). \qquad (3.4.2)\end{aligned}$$

We want to interchange summation and integration. Note that by Proposition 3.2.1 and since $K_k(z, w) = \overline{K_k(w, z)}$

$$\left|K_k(\gamma w, z) j(\gamma, w)^{-k}\right| = \left|K_k(w, \gamma^{-1}z)\overline{j(\gamma^{-1}, z)}^{-k}\right| = |K_k(\gamma^{-1}z, w)||j(\gamma^{-1}, z)|^{-k}.$$

Now fix a fundamental domain F of Γ. Then

$$\begin{aligned}&\sum_{\gamma\in\Gamma} \int_{\Gamma\backslash\mathcal{H}} \left|f(z) K_k(\gamma w, z) j(\gamma, w)^{-k}\, \mathrm{Im}(z)^k\right| d\nu(z) \\ &\qquad \leqslant \sup_{z\in\mathcal{H}} \left|f(z)\, \mathrm{Im}(z)^{k/2}\right| \cdot \sum_{\gamma\in\Gamma} \int_F |K_k(\gamma^{-1}z, w)|\, \mathrm{Im}(\gamma^{-1}z)^{k/2} d\nu(z) \\ &\qquad = \|f\|_{\Gamma,\infty} \cdot |Z(\Gamma)|\, \|K_k(\cdot, w)\|_{k,1} \\ &\qquad < \infty\end{aligned}$$

since $f \in S_k(\Gamma) = H_k^\infty(\Gamma)$ and $K_k(\cdot, w) \in L_k^1(\mathcal{H})$. Thus we may interchange summation and integration in equation (3.4.2), so

$$\begin{aligned}\left\langle f, K_k^\Gamma(\cdot, w)\right\rangle_\Gamma &= \frac{1}{|Z(\Gamma)|}\sum_{\gamma\in\Gamma}\int_F f(z)K_k(\gamma w, z)j(\gamma, w)^{-k}\operatorname{Im}(z)^k d\nu(z)\\ &= \frac{1}{|Z(\Gamma)|}\sum_{\gamma\in\Gamma}\int_{\gamma^{-1}F} f(\gamma z)K_k(\gamma w, \gamma z)j(\gamma, w)^{-k}\operatorname{Im}(\gamma z)^k d\nu(z).\end{aligned}$$

Since $f \in S_k(\Gamma)$ we have $f(\gamma z) = j(\gamma, z)^k f(z)$. Using again Proposition 3.2.1 we get

$$\begin{aligned}\left\langle f, K_k^\Gamma(\cdot, w)\right\rangle_\Gamma &= \frac{1}{|Z(\Gamma)|}\sum_{\gamma\in\Gamma}\int_{\gamma^{-1}F} f(z)j(\gamma, z)^k K_k(w, z)\overline{j(\gamma, z)}^k\operatorname{Im}(\gamma z)^k d\nu(z)\\ &= \frac{1}{|Z(\Gamma)|}\sum_{\gamma\in\Gamma}\int_{\gamma^{-1}F} f(z)K_k(w, z)\operatorname{Im}(z)^k d\nu(z)\\ &= \int_{\mathcal{H}} f(z)\overline{K_k(z, w)}\operatorname{Im}(z)^k d\nu(z).\end{aligned}$$

So $\langle f, K_k^\Gamma(\cdot, w)\rangle_\Gamma = \langle f, K_k(\cdot, w)\rangle = (\pi_k f)(w)$ where π_k still denotes the projection operator defined in Theorem 3.2.9. Note that $f \in S_k(\Gamma) = H_k^\infty(\Gamma) \subseteq H_k^\infty(\mathcal{H})$ as remarked earlier. Therefore we have

$$\left\langle f, K_k^\Gamma(\cdot, w)\right\rangle_\Gamma = (\pi_k f)(w) = f(w)$$

since π_k projects $L_k^\infty(\mathcal{H})$ onto $H_k^\infty(\mathcal{H})$ and thus acts trivially on $H_k^\infty(\mathcal{H})$ itself. So we are done. □

3.5 A first trace formula

Let's recall some basic linear algebra: Let $(V, \langle\cdot,\cdot\rangle)$ be a finite dimensional inner product space, and let $B = \{b_1, \ldots, b_n\}$ be an orthonormal basis of V. Further, let φ be a linear operator on V. Then we can write φ as a matix $A = (a_{ij})$ in terms of the basis B, and the trace of A is given by the sum of the diagonal entries of A, so $\operatorname{Tr}(A) = a_{11} + \ldots + a_{nn}$. One can show that the trace of A does not depend on the choice of basis B, so we can define the trace of the operator φ as the trace of one of its matrix representations.

Keeping notation we have

$$\langle\varphi(b_l), b_l\rangle = \langle a_{1l}b_1 + \ldots + a_{nl}b_n, b_l\rangle = \sum\nolimits_{j=1}^n a_{jl}\langle b_j, b_l\rangle = a_{ll}$$

since B is an orthonormal basis. Therefore we can write

$$\operatorname{Tr}(\varphi) = \sum\nolimits_{j=1}^n \langle\varphi(b_l), b_l\rangle. \tag{3.5.1}$$

We use this simple identity to write down a first trace formula for Hecke operators.

Theorem 3.5.1. *Let $k \geqslant 3$ be an integer and Γ be a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Further, let $T = \Gamma g \Gamma$ be a Hecke operator acting on $S_k(\Gamma)$ where $g \in \mathrm{GL}_2^+(\mathbb{Q})$. Then*

$$\mathrm{Tr}(T \circlearrowleft S_k(\Gamma)) = \frac{\det(g)^{k-1}}{|Z(\Gamma)|} \int_{\Gamma\backslash\mathcal{H}} \sum_{\alpha \in T} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k d\nu(z).$$

Proof. Let $\{f_1, \ldots, f_n\}$ be an orthonormal basis of the finite dimensional Hilbert space $S_k(\Gamma)$. By equation (3.5.1) we have

$$\mathrm{Tr}(T \circlearrowleft S_k(\Gamma)) = \sum_{j=1}^{n} \langle T f_j, f_j \rangle_\Gamma .$$

Let $g_1, \ldots, g_d \in \mathrm{GL}_2^+(\mathbb{Q})$ such that $T = \bigsqcup_{j=1}^d \Gamma g_j$. Then

$$(Tf)(z) = \sum_{i=1}^{d} (f|_k g_i)(z) = \sum_{i=1}^{d} \det(g_i)^{k-1} j(g_i, z)^{-k} f(g_i z)$$

for any $f \in S_k(\Gamma)$. Note that $\det(\alpha) = \det(g)$ for all $\alpha \in T$. Hence we have

$$\begin{aligned}\mathrm{Tr}(T \circlearrowleft S_k(\Gamma)) &= \sum_{j=1}^{n} \int_{\Gamma\backslash\mathcal{H}} \left(\sum_{i=1}^{d} \det(g_i)^{k-1} j(g_i, z)^{-k} f_j(g_i z) \right) \overline{f_j(z)} \operatorname{Im}(z)^k d\nu(z) \\ &= \det(g)^{k-1} \int_{\Gamma\backslash\mathcal{H}} \sum_{i=1}^{d} \left(\sum_{j=1}^{n} f_j(g_i z) \overline{f_j(z)} \right) j(g_i, z)^{-k} \operatorname{Im}(z)^k d\nu(z).\end{aligned}$$

We can replace the sum in the brackets according to Proposition 2.3.6, which says that $K_k^\Gamma(z, w) = \sum_{j=1}^n f_j(z) \overline{f_j(w)}$. Hence

$$\begin{aligned}\mathrm{Tr}(T \circlearrowleft S_k(\Gamma)) &= \det(g)^{k-1} \int_{\Gamma\backslash\mathcal{H}} \sum_{i=1}^{d} K_k^\Gamma(g_i z, z) j(g_i, z)^{-k} \operatorname{Im}(z)^k d\nu(z) \\ &= \frac{\det(g)^{k-1}}{|Z(\Gamma)|} \int_{\Gamma\backslash\mathcal{H}} \sum_{i=1}^{d} \left(\sum_{\gamma \in \Gamma} K_k(\gamma g_i z, z) j(\gamma, g_i z)^{-k} \right) j(g_i, z)^{-k} \operatorname{Im}(z)^k d\nu(z) \\ &= \frac{\det(g)^{k-1}}{|Z(\Gamma)|} \int_{\Gamma\backslash\mathcal{H}} \sum_{i=1}^{d} \sum_{\gamma \in \Gamma} K_k(\gamma g_i z, z) j(\gamma g_i, z)^{-k} \operatorname{Im}(z)^k d\nu(z) \\ &= \frac{\det(g)^{k-1}}{|Z(\Gamma)|} \int_{\Gamma\backslash\mathcal{H}} \sum_{\alpha \in T} K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k d\nu(z).\end{aligned}$$

The last equality follows from $T = \bigsqcup_{j=1}^d \Gamma g_j$. □

4 Simplification of the trace formula

Throughout this chapter we assume $k \geqslant 3$ to be a fixed integer, Γ to be a modular group and $T = \Gamma g \Gamma$ to be a Hecke operator acting on $S_k(\Gamma)$ where g is an element of $\mathrm{GL}_2^+(\mathbb{Q})$. The goal of this chapter is to simplify the trace formula given in Theorem 3.5.1 following Section 6.4 of [Miy06].

In Section 4.1 we will try to interchange summation and integration in the formula, which turns out to be quite involved. Afterwards we will calculate different types of integrals in Section 4.2, before we summarise our results in Section 4.3.

A further simplification can be found in the following chapter, where we focus on T_p-operators acting on $S_k(\Gamma_0(N))$.

We also note that this chapter and the corresponding section in Miyake are based on [Shi63], Section 2 and 3. In particular, the formula presented in Section 4.3 (including the two finishing lemmata) is Theorem 1 in Shimizu's paper for $n = 1$ and trivial character.

4.1 Interchanging summation and integration

We start by recalling the trace formula shown at the end of the previous section. We have

$$\mathrm{Tr}(T \circlearrowleft S_k(\Gamma)) = \frac{\det(g)^{k-1}}{|Z(\Gamma)|} \int_{\Gamma\backslash\mathcal{H}} \sum_{\alpha \in T} \kappa(z,\alpha) d\nu(z) \tag{4.1.1}$$

where

$$\kappa(z,\alpha) = K_k(\alpha z, z) j(\alpha, z)^{-k} \operatorname{Im}(z)^k.$$

We want to simplify the integral in equation (4.1.1) by interchanging summation and integration. Therefore we divide the integral into an integral on a compact set which behaves nicely and an integral on neighbourhoods of cusps where we have to argue more carefully.

First we need to note that $\kappa(z,\alpha)$ will in general not be Γ-invariant, and thus the integral

$$\int_{\Gamma\backslash\mathcal{H}} \kappa(z,\alpha) d\nu(z)$$

might not be well-defined. Therefore we have to fix a fundamental domain F of Γ while we interchange summation and integration. Note that the sum of all integrals will not depend on the choice of F since the trace of T acting on $S_k(\Gamma)$ as in (4.1.1) is unique. Hence we may choose $F = \bigcup_{j=1}^{l} g_j D$ where $g_1, \ldots, g_l$ are fixed coset representatives of $\Gamma\backslash\mathcal{H}$, and

$$D = \{z \in \mathcal{H} \colon |\operatorname{Re}(z)| \leqslant 1/2 \text{ and } |z| \geqslant 1\}$$

is the usual fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$. We will be able to replace F by some appropriate quotient at the end of this section.

Further, we remark that Miyake does not fix such a fundamental domain in the corresponding section in his book which causes some formal problems. For example equation (6.4.7) on page 235 will in general not be well-defined for the mentioned reason. However, this is a purely formal issue, as we will show that all the arguments Miyake is using still work when we use a fixed fundamental domain. Moreover, we will be able to recover Miyake's notation in (our) Theorem 4.1.13 which corresponds to Theorem 6.4.8 in [Miy06].

Notation. Recall that $C(\Gamma)$ denotes the set of Γ-orbits in $\mathbb{Q} \cup \{\infty\}$, which is usually called the set of cusps of Γ. In addition, one may define the **total set of cusps** to be $\mathbb{Q} \cup \{\infty\}$ itself. As we will be mainly working with the latter in this section, we fix the following notation to avoid confusion: A cusp x will denote a single element of $\mathbb{Q} \cup \{\infty\}$ and a cusp $c = [x]$ will denote a Γ-orbit of x in $\mathbb{Q} \cup \{\infty\}$.

Definition 4.1.1. For any cusp x of Γ we define

$$T_x = \{\alpha \in T \colon \alpha x = x\}, \quad U_x = \sigma_x U_\infty \quad \text{and} \quad F_x = F \cap U_x$$

where $\sigma_x \in \mathrm{SL}_2(\mathbb{Z})$ with $\sigma_x \infty = x$ and $U_\infty = \{z \in \mathcal{H} \colon \operatorname{Im}(z) > \delta\}$ for some $\delta > 1$.

Note that U_∞ is a neighbourhood of the cusp ∞ and thus U_x is a neighbourhood of the cusp x. We have $U_{\gamma x} = \gamma\sigma_x U_\infty = \gamma U_x$ for any $\gamma \in \Gamma$. Recall that we denote the stabilizer of a cusp x in Γ by Γ_x and note that $\Gamma_x = \sigma_x \Gamma_\infty \sigma_x^{-1}$. The neighbourhoods U_x are stable under Γ_x since U_∞ is stable under Γ_∞. Furthermore, the following two Lemmata hold:

Lemma 4.1.2. *For any cusps $x \neq y$ of Γ we have $U_x \cap U_y = \varnothing$.*

Proof. Suppose there is $z \in U_x \cap U_y$, then there are $u, v \in U_\infty$ such that $z = \sigma_x u = \sigma_y v$, so $u = \sigma_x^{-1}\sigma_y v$. Let $\tau = \sigma_x^{-1}\sigma_y$, then

$$1 < \operatorname{Im}(u) = \operatorname{Im}(\tau v) = \operatorname{Im}(v)|j(\tau, v)|^{-2}.$$

Further we have $|j(\tau, v)|^2 \geqslant (c_\tau \operatorname{Im}(v))^2 \geqslant c_\tau^2 \operatorname{Im}(v)$ where c_τ denotes the lower left entry of the matrix τ. If $c_\tau \neq 0$ then

$$1 < \operatorname{Im}(v)|j(\tau, v)|^{-2} \leqslant c_\tau^{-2}$$

gives a contradiction since $c_\tau \in \mathbb{Z}$, so $c_\tau = 0$, and thus $\infty = \tau\infty = \sigma_x^{-1}\sigma_y\infty = \sigma_x^{-1}y$. Hence $y = \sigma_x\infty = x$. □

Lemma 4.1.3. *For all but finitely many cusps $x \in \mathbb{Q} \cup \{\infty\}$ of Γ the set F_x is empty.*

Proof. Let D be the usual fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$. By construction $D \cap U_x$ is non-empty for some cusp $x \in \mathbb{Q} \cup \{\infty\}$ if and only if $x = \infty$. We have $F = \bigcup_{j=1}^{l} g_j D$ with $g_1, \ldots, g_l$ being the fixed coset representatives of $\Gamma\backslash\mathcal{H}$. So there is a unique cusp x_j for every j such that $(g_j D) \cap U_{x_j}$ is non-empty, namely $x_j = g_j\infty$. Therefore F_x is non-empty if and only if $x = g_j\infty$ for some $j \in \{1, \ldots, l\}$. □

We will now consider the integral $\int_{F_x} \sum_{\alpha \in T} \kappa(z, \alpha) d\nu(z)$ for some cusp x.

Proposition 4.1.4. *We have for any cusp x of Γ*

$$\int_{F_x} \sum_{\alpha \in T \backslash T_x} \kappa(z, \alpha) d\nu(z) \;=\; \sum_{\alpha \in T \backslash T_x} \int_{F_x} \kappa(z, \alpha) d\nu(z).$$

We need two lemmata to prove this proposition. (These correspond to Lemma 6.4.3 and Lemma 6.4.4 in [Miy06].) Moreover, we quickly refer to Corollary 5.13 on page 143 in [Lan93] at this point, which gives a sufficient condition to interchange summation and integration in a general setting. We will use this corollary several times, but since it is very well-known, we will use it silently without further notice of the statement or the reference.

Notation. For $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ we write c_α for the entry c of α and d_α for the entry d.

Lemma 4.1.5. *The sum*

$$\sum_{\alpha \in \Gamma_\infty \backslash (T \backslash T_\infty)/\Gamma_\infty} |c_\alpha|^{-k}$$

is convergent.

Proof. Note that the sum is well-defined since firstly $c_\alpha = 0$ if and only if $\alpha \in T_\infty$, and secondly $|c_\alpha| = |c_\beta|$ for all $\beta \in \Gamma_\infty \alpha \Gamma_\infty$. Let h be the width of the cusp $[\infty]$ for Γ, and let $\mathcal{A}$ be a set of double coset representatives for $\Gamma_\infty \backslash (T \backslash T_\infty)/\Gamma_\infty$ such that $|d_\alpha| < |hc_\alpha|$ for all $\alpha \in \mathcal{A}$. This is possible since

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \alpha \left[(\pm 1)^m \begin{pmatrix} 1 & h\,m \\ 0 & 1 \end{pmatrix} \right] = (\pm 1)^m \begin{pmatrix} * & * \\ c_\alpha & hc_\alpha\, m + d_\alpha \end{pmatrix}.$$

Then we see for $\alpha \in \mathcal{A}$ that $|j(\alpha, z)| \leqslant |c_\alpha z| + |d_\alpha| < |c_\alpha|(|z| + |h|)$ and hence

$$|c_\alpha|^{-k} < (|h| + |z|)^k |j(\alpha, z)|^{-k}.$$

Therefore it sufficies to show that the sum $\sum_{\alpha \in \mathcal{A}} |j(\alpha, z)|^{-k}$ is convergent for some $z \in \mathcal{H}$. Note that $\alpha \in \mathcal{A}$ representing the double coset $\Gamma_\infty \alpha \Gamma_\infty$ also represents the coset $\Gamma_\infty \alpha$, and that $\alpha, \alpha' \in \mathcal{A}$, $\alpha \neq \alpha'$, represent different cosets in $\Gamma_\infty \backslash T$, since $\Gamma_\infty \alpha = \Gamma_\infty \alpha'$ would imply $\Gamma_\infty \alpha \Gamma_\infty = \Gamma_\infty \alpha' \Gamma_\infty$. Hence

$$\sum_{\alpha \in \mathcal{A}} |j(\alpha, z)|^{-k} \leqslant \sum_{\alpha \in \Gamma_\infty \backslash T} |j(\alpha, z)|^{-k} \leqslant \sum_{\alpha \in \Gamma_\infty^+ \backslash T} |j(\alpha, z)|^{-k}.$$

Let $g_1, \ldots, g_d \in T$ such that $T = \bigsqcup_{j=1}^d \Gamma g_j$, then we can write the above sum as

$$\begin{aligned} \sum_{\alpha \in \Gamma_\infty^+ \backslash (\bigsqcup_{j=1}^d \Gamma g_j)} |j(\alpha, z)|^{-k} &= \sum_{\gamma \in \Gamma_\infty^+ \backslash \Gamma} \sum_{j=1}^d |j(\gamma g_j, z)|^{-k} \\ &= \sum_{j=1}^d |j(g_j, z)|^{-k} \left(\sum_{\gamma \in \Gamma_\infty^+ \backslash \Gamma} |j(\gamma, g_j z)|^{-k} \right). \end{aligned}$$

Finally recall that the sum $G_{k,\Gamma,\infty}(z) = \sum_{\gamma \in \Gamma_\infty^+ \backslash \Gamma} j(\gamma, z)^{-k}$ converges absolutely for any $z \in \mathcal{H}$. Thus we are done since the remaining sum is finite. $\square$

Lemma 4.1.6. *For $h > 0$ and $l > \frac{1}{2}$ there is a constant $C_{h,l} > 0$ such that*

$$\sum_{n \in \mathbb{Z}} \left[(a + nh)^2 + b^2\right]^{-l} < C_{h,l} \left(|b|^{-2l+1} + |b|^{-2l}\right)$$

for all $a, b \in \mathbb{R}$.

Proof. Let $h > 0$, $l > \frac{1}{2}$ and $a, b \in \mathbb{R}$. Define $f(x) = \left[(a + hx)^2 + b^2\right]^{-l}$ for $x \in \mathbb{R}$. Note that $f \geqslant 0$, and that $f'(x) > 0$ for $x < -a/h$, $f'(x) = 0$ for $x = -a/h$ and $f'(x) < 0$ for $x > -a/h$. Thus there is $N \in \mathbb{Z}$ such that

$$\sum_{n \in \mathbb{Z}, n \neq N} f(n) \cdot 1 \leqslant \int_{\mathbb{R}} f(x) dx = \frac{1}{h} \int_{\mathbb{R}} (y^2 + b^2)^{-l} dy.$$

Here we used the substitution $x = (y - a)/h$. To estimate the integral on the right, we divide it into two parts. We see $\int_{|y| \leqslant |b|} (y^2 + b^2)^{-l} dy \leqslant 2|b| \cdot |b|^{-2l}$ and

$$\int_{|y| > |b|} (y^2 + b^2)^{-l} dy \leqslant 2 \int_{|b|}^{\infty} y^{-2l} dy = \frac{2}{2l - 1} |b|^{-2l+1}.$$

Finally we note that $f(N) = \left[(a + hN)^2 + b^2\right]^{-l} \leqslant |b|^{-2l}$. Therefore

$$\begin{aligned} \sum_{n \in \mathbb{Z}} f(n) &\leqslant \frac{1}{h} \int_{\mathbb{R}} (y^2 + b^2)^{-l} dy + f(N) \\ &\leqslant \frac{2}{h} |b|^{-2l+1} + \frac{2}{h(2l - 1)} |b|^{-2l+1} + |b|^{-2l} \end{aligned}$$

which gives the claimed estimate. $\square$

We will now use the previous lemmata to interchange summation and integration in Proposition 4.1.4. Our argumentation follows the proof of Theorem 6.4.5 on page 233/234 in [Miy06], but we include more details for the generalisation to arbitrary cusps.

Proof of Proposition 4.1.4. We will proof the statement first for the cusp $x = \infty$. Put $S(z) = \sum_{\alpha \in T \backslash T_\infty} |\kappa(z, \alpha)|$. We have to show that the sum defining S converges for every $z \in F_\infty$ and that S is integrable on F_∞.

Let $\mathcal{A}$ be a set of coset representatives of $\Gamma_\infty \backslash T$, so $T = \bigsqcup_{\alpha \in \mathcal{A}} \Gamma_\infty \alpha$. One can check that $\gamma_\infty \alpha \in T_\infty$ for $\gamma_\infty \in \Gamma_\infty$, $\alpha \in T$, if and only if $\alpha \in T_\infty$. Thus we have for $\alpha \in \mathcal{A}$ either $\Gamma_\infty \alpha \subseteq T_\infty$ or $\Gamma_\infty \alpha \cap T_\infty = \emptyset$. Put $\mathcal{A}_0 = \mathcal{A} \backslash T_\infty$, then $T \backslash T_\infty = \bigsqcup_{\alpha \in \mathcal{A}_0} \Gamma_\infty \alpha$. Hence

$$\begin{aligned} S(z) &= \sum_{\alpha \in \mathcal{A}_0} \sum_{\gamma_\infty \in \Gamma_\infty} |K_k(\gamma_\infty \alpha z, z)| \cdot |j(\gamma_\infty \alpha, z)|^{-k} \cdot \mathrm{Im}(z)^k \\ &= \mathrm{Im}(z)^k \sum_{\alpha \in \mathcal{A}_0} |j(\alpha, z)|^{-k} |Z(\Gamma)| \sum_{m \in \mathbb{Z}} |K_k(\alpha z + hm, z)|. \end{aligned}$$

where h is the width of the cusp $[\infty]$ for Γ as usual. We use Lemma 4.1.6 to estimate the inner sum:

$$\begin{aligned}\sum_{m\in\mathbb{Z}} |K_k(\alpha z + hm, z)| &= \frac{k-1}{4\pi}|2i|^k \sum_{m\in\mathbb{Z}} |\alpha z + hm - \overline{z}|^{-k} \\ &= \frac{2^k(k-1)}{4\pi} \sum_{m\in\mathbb{Z}} \left[(\mathrm{Re}(\alpha z - \overline{z}) + hm)^2 + \mathrm{Im}(\alpha z - \overline{z})^2\right]^{-k/2} \\ &< \frac{2^k(k-1)}{4\pi} \cdot C_{h,k}\left(|\mathrm{Im}(\alpha z - \overline{z})|^{-k+1} + |\mathrm{Im}(\alpha z - \overline{z})|^{-k}\right).\end{aligned}$$

Moreover, $\mathrm{Im}(\alpha z - \overline{z}) = \mathrm{Im}(\alpha z) + \mathrm{Im}(z) \geqslant \mathrm{Im}(z)$, and thus

$$S(z) < C_{\Gamma,k}(1 + \mathrm{Im}(z)) \sum_{\alpha\in\mathcal{A}_0} |j(\alpha, z)|^{-k}.$$

Now we choose a set $\mathcal{A}'$ of double coset representatives for $\Gamma_\infty\backslash T/\Gamma_\infty$ and adjust the choice of $\mathcal{A}$ such that $\mathcal{A} \subseteq \mathcal{A}'\Gamma_\infty$. To see that this is indeed possible, suppose that there is $\alpha \in \mathcal{A}$ with $\alpha \notin \mathcal{A}'\Gamma_\infty$. As $\alpha \in T$ there is $\alpha' \in \mathcal{A}'$ such that $\alpha \in \Gamma_\infty\alpha'\Gamma_\infty$, so $\alpha = \gamma_\infty\alpha'\gamma'_\infty$ for some $\gamma_\infty, \gamma'_\infty \in \Gamma_\infty$. Thus we can replace the representative α by $\gamma_\infty^{-1}\alpha = \alpha'\gamma'_\infty$ which still represents the coset $\Gamma_\infty\alpha$ but is now also an element of $\mathcal{A}'\Gamma_\infty$. As before we can put $\mathcal{A}'_0 = \mathcal{A}'\backslash T_\infty$, such that $T\backslash T_\infty = \bigsqcup_{\alpha'\in\mathcal{A}'_0} \Gamma_\infty\alpha'\Gamma_\infty$. Hence

$$\begin{aligned}\sum_{\alpha\in\mathcal{A}_0} |j(\alpha, z)|^{-k} &\leqslant \sum_{\alpha'\in\mathcal{A}'_0} \sum_{\gamma_\infty\in\Gamma_\infty} |j(\alpha'\gamma_\infty, z)|^{-k} \\ &= \sum_{\alpha'\in\mathcal{A}'_0} |Z(\Gamma)| \sum_{m\in\mathbb{Z}} |c_{\alpha'}(z + hm) + d_{\alpha'}|^{-k}.\end{aligned}$$

Also as before we use Lemma 4.1.6 to estimate the inner sum:

$$\begin{aligned}\sum_{m\in\mathbb{Z}} |c_{\alpha'}(z + hm) + d_{\alpha'}|^{-k} &= |c_{\alpha'}|^{-k} \sum_{m\in\mathbb{Z}} \left[\left(\mathrm{Re}(z) + hm + \frac{d_{\alpha'}}{c_{\alpha'}}\right)^2 + \mathrm{Im}(z)^2\right]^{-k/2} \\ &< |c_{\alpha'}|^{-k} \cdot C_{h,k}\left(\mathrm{Im}(z)^{-k+1} + \mathrm{Im}(z)^{-k}\right)\end{aligned}$$

Finally we can use Lemma 4.1.5 since $\mathcal{A}'_0$ is a set of double coset representatives for $\Gamma_\infty\backslash(T\backslash T_\infty)/\Gamma_\infty$, so

$$S(z) \;<\; C'_{\Gamma,k}\,\mathrm{Im}(z)^{-k}(1 + \mathrm{Im}(z))^2 \sum_{\alpha'\in\mathcal{A}'_0} |c_{\alpha'}|^{-k} \;\leqslant\; C''_{\Gamma,k}\,\mathrm{Im}(z)^{-k}(1 + \mathrm{Im}(z))^2.$$

The estimate on the right is bounded for $z \in F_\infty$ since we assume $k \geqslant 3$, so S is convergent and bounded on F_∞, and thus also integrable on F_∞ since $\nu(F_\infty) \leqslant \nu(F)$ is finite. Therefore we have shown that

$$\int_{F_\infty} \sum_{\alpha\in T\backslash T_\infty} |\kappa(z, \alpha)| d\nu(z) < \infty,$$

so we can interchange the order of summation and integration as claimed.

It remains to generalise to arbitrary cusps. Let x be a cusp of Γ and $\sigma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\sigma\infty = x$. Put $\Gamma' = \sigma^{-1}\Gamma\sigma$, $g' = \sigma^{-1}g\sigma$ and $T' = \Gamma' g' \Gamma' = \sigma^{-1}T\sigma$. Then $F' := \sigma^{-1}F$ is a fundamental domain for the action of Γ' on $\mathcal{H}$, and thus $\sigma^{-1}F_x = \sigma^{-1}(F \cap \sigma U_\infty) = F'_\infty$. Hence

$$\int_{F_x} \sum_{\alpha \in T \backslash T_x} |\kappa(z,\alpha)| d\nu(z) = \int_{F'_\infty} \sum_{\alpha \in T \backslash T_x} |\kappa(\sigma z, \alpha)| d\nu(z).$$

Note that $T_x = \sigma T'_\infty \sigma^{-1}$ and $T \backslash T_x = \sigma(T' \backslash T'_\infty)\sigma^{-1}$. Therefore we get using Proposition 3.2.1 with $\alpha = \sigma$ and the fact that $j(\sigma\alpha'\sigma^{-1}, \sigma z) = j(\sigma, \alpha' z) j(\alpha', z) j(\sigma, z)^{-1}$:

$$\begin{aligned}
\sum_{\alpha \in T \backslash T_x} |\kappa(\sigma z, \alpha)| &= \sum_{\alpha' \in T' \backslash T'_\infty} |\kappa(\sigma z, \sigma\alpha'\sigma^{-1})| \\
&= \sum_{\alpha' \in T' \backslash T'_\infty} |K_k(\sigma\alpha' z, \sigma z)| \cdot |j(\sigma\alpha'\sigma^{-1}, \sigma z)|^{-k} \operatorname{Im}(\sigma z)^k \\
&= \sum_{\alpha' \in T' \backslash T'_\infty} |K_k(\alpha' z, z)| \cdot |j(\alpha', z)|^{-k} \operatorname{Im}(z)^k.
\end{aligned}$$

Thus we have shown that

$$\int_{F_x} \sum_{\alpha \in T \backslash T_x} |\kappa(z,\alpha)| d\nu(z) = \int_{F'_\infty} \sum_{\alpha' \in T' \backslash T'_\infty} |\kappa(z, \alpha')| d\nu(z).$$

Here the right-hand side is finite, as shown in the first part of the proof, so we can interchange the order of summation and integration also for arbitrary cusps. □

Originally we wanted to study the integral

$$\int_{F_x} \sum_{\alpha \in T} \kappa(z,\alpha) d\nu(z)$$

for some cusp x of Γ, but so far we have only discussed the $(T \backslash T_x)$-part of the sum. The following proposition deals with the remaining part, which needs special treatment:

Proposition 4.1.7. *We have for any cusp x of Γ*

$$\int_{F_x} \sum_{\alpha \in T_x} \kappa(z,\alpha) d\nu(z) \;=\; \lim_{s \searrow 0} \sum_{\alpha \in T_x} \int_{F_x} \kappa(z,\alpha) \operatorname{Im}(z)^{-s} |j(\sigma_x^{-1}, z)|^{2s} d\nu(z).$$

Here $s \searrow 0$ means that $s \to 0$ monotonically and $s > 0$, and σ_x is any element of $\mathrm{SL}_2(\mathbb{Z})$ with $\sigma_x \infty = x$.

We start with a small lemma that will help us to generalise from the cusp ∞ to arbitrary cusps during the proof of this proposition.

Lemma 4.1.8. *For any cusp x of Γ, the subgroup Γ_x is of finite index in T_x.*

Proof. Let $\alpha, \beta \in T_x$ such that $\alpha = \beta$ in $\Gamma \backslash T$, then there is $\gamma \in \Gamma$ such that $\gamma\alpha = \beta$, so $\gamma x = \beta\alpha^{-1}x = x$ since $\alpha, \beta \in T_x$. Thus $\gamma \in \Gamma_x$ and hence $\alpha = \beta$ in $\Gamma_x \backslash T_x$. Therefore we have $|\Gamma_x \backslash T_x| \leqslant |\Gamma \backslash T| < \infty$. □

Proof of Proposition 4.1.7. We will proof the statement first for the cusp $x = \infty$. Fix $s > 0$ and put $S_s(z) = \sum_{\alpha \in T_\infty} |\kappa(z, \alpha)| \operatorname{Im}(z)^{-s}$. We have to show that the sum defining S_s converges for every $z \in F_\infty$ and that S is integrable on F_∞.

Let $\mathcal{A}$ be a set of coset representatives of $\Gamma_\infty \backslash T$, and put $\mathcal{A}_0 = \mathcal{A} \cap T_\infty$ such that $T_\infty = \bigsqcup_{\alpha \in \mathcal{A}_0} \Gamma_\infty \alpha$. Following the proof of Proposition 4.1.4 using this new $\mathcal{A}_0$ we get

$$S_s(z) < C_{\Gamma,k}(1 + \operatorname{Im}(z)) \operatorname{Im}(z)^{-s} \sum_{\alpha \in \mathcal{A}_0} |j(\alpha, z)|^{-k}.$$

This time we do not have to use Lemma 4.1.6 a second time since the sum is already finite by Lemma 4.1.8. Moreover, $c_\alpha = 0$ for every $\alpha \in \mathcal{A}_0 \subseteq T_\infty$, so the sum is independent of z. Hence $S_s(z) < C'_{\Gamma,k}(\operatorname{Im}(z)^{-s} + \operatorname{Im}(z)^{1-s})$. One can easily check that this is integrable on F_∞ with respect to ν for all $s > 0$, and thus

$$\int_{F_\infty} \sum_{\alpha \in T_\infty} \kappa(z, \alpha) \operatorname{Im}(z)^{-s} d\nu(z) = \sum_{\alpha \in T_\infty} \int_{F_\infty} \kappa(z, \alpha) \operatorname{Im}(z)^{-s} d\nu(z). \tag{4.1.2}$$

Now let $(s_n)_n$ be a sequence in $(0, \infty)$ that converges to 0 monotonically from above. Then $\kappa(z, \alpha) \operatorname{Im}(z)^{-s_n} \to \kappa(z, \alpha)$ monotonically as $n \to \infty$ for fixed z, so

$$\int_{F_\infty} \sum_{\alpha \in T_\infty} \kappa(z, \alpha) d\nu(z) = \lim_{n \to \infty} \int_{F_\infty} \sum_{\alpha \in T_\infty} \kappa(z, \alpha) \operatorname{Im}(z)^{-s_n} d\nu(z) \tag{4.1.3}$$

by the Monotone Convergence Theorem (see Theorem 5.5 on page 139 in [Lan93]). Combining equation (4.1.2) and equation (4.1.3) yields the claimed statement for $x = \infty$.

It remains to generalise to arbitrary cusps. Let x be a cusp of Γ and $\sigma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\sigma\infty = x$. As in the proof of 4.1.4 we put $\Gamma' = \sigma^{-1}\Gamma\sigma$, $g' = \sigma^{-1}g\sigma$ and $T' = \Gamma' g' \Gamma' = \sigma^{-1}T\sigma$. Then $T_x = \sigma T'_\infty \sigma^{-1}$, $F' := \sigma^{-1}F$ is a fundamental domain for Γ' and $\sigma^{-1}F_x = F'_\infty$. Using similar arguments as in the proof of Proposition 4.1.4 we get

$$\int_{F_x} \sum_{\alpha \in T_x} \kappa(z, \alpha) d\nu(z) = \int_{F'_\infty} \sum_{\alpha' \in T'_\infty} \kappa(z, \alpha') d\nu(z).$$

Thus we have by equation (4.1.2) and (4.1.3)

$$\int_{F_x} \sum_{\alpha \in T_x} \kappa(z, \alpha) d\nu(z) = \lim_{s \searrow 0} \sum_{\alpha' \in T'_\infty} \int_{F'_\infty} \kappa(z, \alpha') \operatorname{Im}(z)^{-s} d\nu(z).$$

Using once more similar arguments as in the proof of Proposition 4.1.4 one can check that

$$\begin{aligned} \sum_{\alpha' \in T'_\infty} \int_{F'_\infty} \kappa(z, \alpha') \operatorname{Im}(z)^{-s} d\nu(z) &= \sum_{\alpha \in T_x} \int_{F_x} \kappa(\sigma^{-1}z, \sigma^{-1}\alpha\sigma) \operatorname{Im}(\sigma^{-1}z)^{-s} d\nu(z) \\ &= \sum_{\alpha \in T_x} \int_{F_x} \kappa(z, \alpha) \operatorname{Im}(z)^{-s} |j(\sigma^{-1}, z)|^{2s} d\nu(z). \end{aligned}$$

Therefore we are done. □

Combining Proposition 4.1.4 and Proposition 4.1.7 we see

$$\int_{F_x} \sum_{\alpha \in T} \kappa(z,\alpha) d\nu(z) = \sum_{\alpha \in T \backslash T_x} \int_{F_x} \kappa(z,\alpha) d\nu(z) + \lim_{s \searrow 0} \sum_{\alpha \in T_x} \int_{F_x} \kappa(z,\alpha) \operatorname{Im}(z)^{-s} |j(\sigma_x^{-1}, z)|^{2s} d\nu(z). \tag{4.1.4}$$

Define $F^0 := \bigcup_{x \in \mathbb{Q} \cup \{\infty\}} F_x$ and $F^1 := F \backslash F^0$. Then the above equality deals with F^0, and it remains to consider F^1, which is done by the following proposition.

Proposition 4.1.9. *We have*

$$\int_{F^1} \sum_{\alpha \in T} \kappa(z,\alpha) d\nu(z) = \sum_{\alpha \in T} \int_{F^1} \kappa(z,\alpha) d\nu(z).$$

In Miyake's book this statement is remarked in the middle of page 232, but not explicitly stated as a theorem. Hence Miyake does not provide a proper proof, but only sketches the argument in two sentences. We fill in the details at this point, starting with a quick and obvious lemma.

Lemma 4.1.10. *The set F^1 is compact in $\mathcal{H}$.*

Proof. Let D be the usual fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$ and write $F = \bigcup_{j=1}^{l} g_j D$ where $g_1, \ldots, g_l$ are coset representatives of $\Gamma \backslash \mathcal{H}$. As remarked earlier in the proof of Lemma 4.1.3 the set F_x is non-empty if and only if $x = g_j \infty$ for some $j \in \{1, \ldots, l\}$. Further, one can easily check that $(g_j D) \cap F_x$ is non-empty if and only if $x = g_j \infty$. Put $x_j = g_j \infty$. Then

$$F^1 = \left(\bigcup_{j=1}^{l} g_j D \right) \backslash \left(\bigcup_{j=1}^{l} F_{x_j} \right) = \bigcup_{j=1}^{l} g_j \left(D \backslash F_\infty \right).$$

The right-hand side is compact since $D \backslash F_\infty$ is clearly compact by construction. □

Proof of Proposition 4.1.9. Let $g_1, \ldots, g_d \in T$ such that $T = \bigsqcup_{j=1}^{d} g_j \Gamma$. Then

$$\begin{aligned} \int_{F^1} \sum_{\alpha \in T} |\kappa(z,\alpha)| d\nu(z) &= \int_{F^1} \sum_{j=1}^{d} \sum_{\gamma \in \Gamma} |K_k(g_j \gamma z, z)| |j(g_j \gamma, z)|^{-k} \operatorname{Im}(z)^k d\nu(z) \\ &= \sum_{j=1}^{d} \int_{F^1} |j(g_j^{-1}, z)|^{-k} \operatorname{Im}(z)^k \sum_{\gamma \in \Gamma} |K_k(\gamma z, g_j^{-1} z)| |j(\gamma, z)|^{-k} d\nu(z). \end{aligned}$$

Here we used Proposition 3.2.1 again. Now recall that the sum defining $K_k^\Gamma(z,w)$ converges uniformly on any compact subset of $\mathcal{H} \times \mathcal{H}$ as shown in Lemma 3.4.4, and note that $K := F^1 \times g_j^{-1}(F^1)$ is compact in $\mathcal{H} \times \mathcal{H}$ for every j since F^1 is compact by Lemma 4.1.10. Hence we find a constant $C_j > 0$ for every j such that

$$\sup_{z \in F^1} \sum_{\gamma \in \Gamma} |K_k(\gamma z, g_j^{-1} z)| |j(\gamma, z)|^{-k} \leqslant \sup_{(z,w) \in K} |Z(\Gamma)| \left| K_k^\Gamma(z,w) \right| \leqslant C_j.$$

Moreover, the continuous function $|j(g_j^{-1}, z)|^{-k} \operatorname{Im}(z)^k$ is clearly bounded on the compact set F^1. Finally we note that $\nu(F^1)$ is finite as $\nu(F)$ is. Thus

$$\int_{F^1} \sum_{\alpha \in T} |\kappa(z, \alpha)| d\nu(z) < \infty$$

and hence we can interchange summation and integration. □

Next we deduce a new trace-formula which combines all the previous results. To state it we first need to define $Z(T) = \left\{\alpha \in T \colon \alpha = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \text{ for some } a \in \mathbb{R}^\times\right\}$,

$$T^2 = \bigcup_{x \in \mathbb{Q} \cup \{\infty\}} T_x \backslash Z(T), \quad T^1 = T \backslash T^2.$$

Recall that $\det(\alpha) = \det(g)$ for all $\alpha \in T$. Hence we either have $Z(T) = \{\pm\sqrt{\det(g)} \cdot \mathrm{id}\}$ or $Z(T) = \varnothing$. In particular, $Z(T)$ is finite.

The following proposition corresponds to equation (6.4.7) on page 235 in [Miy06], which is stated as a direct corollary of the previous results, without proof. We add a formal proof here.

Proposition 4.1.11. *We have*

$$\operatorname{Tr}(T \circlearrowleft S_k(\Gamma)) = \frac{\det(g)^{k-1}}{|Z(\Gamma)|} \left[\sum_{\alpha \in T^1} \int_F \kappa(z, \alpha) d\nu(z) + \lim_{s \searrow 0} \sum_{\alpha \in T^2} \int_F \kappa(z, \alpha, s) d\nu(z) \right]$$

where

$$\kappa(z, \alpha, s) = \begin{cases} \kappa(z, \alpha) \operatorname{Im}(z)^{-s} |j(\sigma_x^{-1}, z)|^{2s}, & z \in U_x \textit{ and } \alpha x = x \textit{ for some cusp } x, \\ \kappa(z, \alpha), & \textit{otherwise.} \end{cases}$$

Before we start with the proof, we want to remark that the definition of $\kappa(z, \alpha, s)$ is independent of the choice σ_x for cusps x of Γ. To see this let $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that $\sigma\infty = x$ for some cusp x of Γ. If $x = \infty$, then $c = 0$ and $a = \pm 1$, so $|j(\sigma^{-1}, z)| = 1$ is independent of σ. If $x = p/q \in \mathbb{Q}$ with p, q coprime, then $a = \pm p$ and $c = \pm q$, so $|j(\sigma^{-1}, z)| = |-qz + p|$, which is again independent of σ.

Proof of Proposition 4.1.11. We start with equation (4.1.1) and split the integral in an integral over F^0 and an integral over F^1:

$$\operatorname{Tr}(T \circlearrowleft S_k(\Gamma)) = \frac{\det(g)^{k-1}}{|Z(\Gamma)|} \left[\int_{F^0} \sum_{\alpha \in T} \kappa(z, \alpha) d\nu(z) + \int_{F^1} \sum_{\alpha \in T} \kappa(z, \alpha) d\nu(z) \right].$$

Note that the union defining F^0 is actually a finite union since all but finitely many neighbourhoods F_x are empty, as shown in Lemma 4.1.3. Hence we get using the equality in (4.1.4)

$$\int_{F^0} \sum_{\alpha\in T} \kappa(z,\alpha)d\nu(z) = \sum_{j=1}^{l} \int_{F_{x_j}} \sum_{\alpha\in T} \kappa(z,\alpha)d\nu(z)$$
$$= \sum_{j=1}^{l} \left[\sum_{\alpha\in T\backslash T_{x_j}} \int_{F_{x_j}} \kappa(z,\alpha)d\nu(z) + \lim_{s\searrow 0} \sum_{\alpha\in T_{x_j}} \int_{F_{x_j}} \kappa(z,\alpha)\operatorname{Im}(z)^{-s}|j(\sigma_{x_j}^{-1},z)|^{2s}d\nu(z) \right]. \tag{4.1.5}$$

Note that for $\alpha \in T\backslash T_x$, $z \in F_x$ we have $\alpha x \neq x$ and cannot have $z \in F_y$ for any other cusp y by Lemma 4.1.2, so for any s

$$\sum_{\alpha\in T\backslash T_{x_j}} \int_{F_{x_j}} \kappa(z,\alpha)d\nu(z) = \sum_{\alpha\in T\backslash T_{x_j}} \int_{F_{x_j}} \kappa(z,\alpha,s)d\nu(z).$$

For the second sum in (4.1.5) we have by definition

$$\sum_{\alpha\in T_{x_j}} \int_{F_{x_j}} \kappa(z,\alpha)\operatorname{Im}(z)^{-s}|j(\sigma_{x_j}^{-1},z)|^{2s}d\nu(z) = \sum_{\alpha\in T_{x_j}} \int_{F_{x_j}} \kappa(z,\alpha,s)d\nu(z).$$

Therefore

$$\begin{aligned}\int_{F^0} \sum_{\alpha\in T} \kappa(z,\alpha)d\nu(z) &= \sum_{j=1}^{l} \lim_{s\searrow 0} \sum_{\alpha\in T} \int_{F_{x_j}} \kappa(z,\alpha,s)d\nu(z) \\ &= \lim_{s\searrow 0} \sum_{\alpha\in T} \int_{F^0} \kappa(z,\alpha,s)d\nu(z). \end{aligned} \tag{4.1.6}$$

We will now consider $\int_{F^1} \sum_{\alpha\in T} \kappa(z,\alpha)d\nu(z)$. By Proposition 4.1.4 we can interchange summation and integration, and as $z \in F^1$ implies $z \notin U_x$ for any cusp x, we can write

$$\int_{F^1} \sum_{\alpha\in T} \kappa(z,\alpha)d\nu(z) = \sum_{\alpha\in T} \int_{F^1} \kappa(z,\alpha,s)d\nu(z)$$

for any s. Combining this with (4.1.6) yields

$$\int_{F^0} \sum_{\alpha\in T} \kappa(z,\alpha)d\nu(z) + \int_{F^1} \sum_{\alpha\in T} \kappa(z,\alpha)d\nu(z) = \lim_{s\searrow 0} \sum_{\alpha\in T} \int_{F} \kappa(z,\alpha,s)d\nu(z).$$

To conclude the claimed formula it remains to separate some safe terms. Let $\alpha \in T^1$. We have to distinguish between two cases: Either $\alpha \in Z(T)$, or $\alpha \notin T_x$ for any cusp x. In the second case we have $\kappa(z,\alpha,s) = \kappa(z,\alpha)$ for any $z \in \mathcal{H}$, so we can easily separate these terms. Suppose that $\alpha \in Z(T)$, then $\alpha = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ for some $a \in \mathbb{R}^\times$. Thus

$$\int_F |\kappa(z,\alpha)|d\nu(z) = \int_F |K_k(z,z)||a|^{-k}\operatorname{Im}(z)^k d\nu(z) = \frac{k-1}{4\pi}|a|^{-k}\nu(F) < \infty$$

since $K_k(z,z) = (k-1)/(4\pi)\,\mathrm{Im}(z)^{-k}$. So we can interchange integral and limit by the Dominated Convergence Theorem giving us

$$\lim_{s\searrow 0}\int_F \kappa(z,\alpha,s)d\nu(z) = \int_F \lim_{s\searrow 0}\kappa(z,\alpha,s)d\nu(z) = \int_F \kappa(z,\alpha)d\nu(z).$$

It remains to recall that $Z(T)$ is finite as remarked earlier. Hence we have

$$\lim_{s\searrow 0}\sum_{\alpha\in T}\int_F \kappa(z,\alpha,s)d\nu(z) \;=\; \sum_{\alpha\in T^1}\int_F \kappa(z,\alpha)d\nu(z) \;+\; \lim_{s\searrow 0}\sum_{\alpha\in T^2}\int_F \kappa(z,\alpha,s)d\nu(z)$$

as claimed. □

We will finish this section with some group theoretic considerations, which allow us to rearrange our trace formula in such a way that we are finally able to replace the fundamental domain F by some appropriate quotient.

Let G be a group and H a subgroup of G. Elements $g_1, g_2 \in G$ are called H**-conjugate**, denoted by $g_1 \sim_H g_2$, if there exists $h \in H$ such that $g_2 = h^{-1}g_1h$. This gives an equivalence relation on G. We call the corresponding equivalence class of $g \in G$, H-conjugacy class and denote it by $[g]_H$.

For a subset M of G which is stable under conjugation by elements in H, which means $h^{-1}Mh = M$ for all $h \in H$, we define $M//H$ as the set of all H-conjugacy classes in M, so

$$M//H = \{[g]_H : g \in M\}.$$

Note that $M//H$ gives a partition of M, since $[g]_H \subseteq M$ for all $g \in M$.

Lemma 4.1.12. *The subsets T^1 and T^2 of T are stable under conjugation by elements in Γ.*

Proof. First note that $[\alpha]_\Gamma = \{\alpha\}$ for all $\alpha \in Z(T)$, as they commute with any other element. Secondly, we see for any $\gamma \in \Gamma$

$$T_{\lambda^{-1}x} = \{\alpha \in T : \gamma\alpha\gamma^{-1}x = x\} = \gamma^{-1}\{\alpha' \in \gamma T\gamma^{-1} : \alpha' x = x\}\gamma = \gamma^{-1}T_x\gamma.$$

Here we used that $\gamma^{-1}T\gamma = T$ which is obvious. Therefore we have $\gamma^{-1}T^2\gamma = T^2$ and thus also $\gamma^{-1}T^1\gamma = T^1$. □

Recall that we defined $Z(\alpha) = \{\beta \in \mathrm{GL}_2^+(\mathbb{Q}) : \alpha\beta = \beta\alpha\}$ in Section 2.2, and define

$$\Gamma(\alpha) = \{\gamma \in \Gamma : \gamma\alpha = \alpha\gamma\}.$$

Then $\Gamma(\alpha) = Z(\alpha) \cap \Gamma$. We are now able to state Theorem 6.4.8 on page 235 in [Miy06], which will be the starting point for further considerations in the next section. For the sake of convenience we recall the complete notation used in the theorem.

Theorem 4.1.13. *Let $k \geqslant 3$ be an integer, and let $T = \Gamma g \Gamma$ with Γ being a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and g being an element of $\mathrm{GL}_2^+(\mathbb{Q})$. Then*

$$\mathrm{Tr}(T \circlearrowleft S_k(\Gamma)) = \frac{\det(g)^{k-1}}{|Z(\Gamma)|} \cdot \left[\sum_{\alpha \in T^1//\Gamma} \int_{\Gamma(\alpha)\backslash\mathcal{H}} \kappa(z,\alpha) d\nu(z) + \lim_{s \searrow 0} \sum_{\alpha \in T^2//\Gamma} \int_{\Gamma(\alpha)\backslash\mathcal{H}} \kappa(z,\alpha,s) d\nu(z) \right]$$

where we use the following notation:

- *Put $Z(\Gamma) = \Gamma \cap \{\pm 1\}$, $T^2 = \bigcup_{x \in \mathbb{Q}\cup\{\infty\}} T_x \backslash Z(T)$ where $T_x = \{\alpha \in T \colon \alpha x = x\}$ and $Z(T) = \{\alpha \in T \colon \alpha$ is scalar$\}$, $T^1 = T\backslash T^2$ and $\Gamma(\alpha) = \{\gamma \in \Gamma \colon \gamma\alpha = \alpha\gamma\}$.*
- *For $j \in \{1,2\}$, $T^j//\Gamma$ denotes the set of Γ-conjugacy classes in T^j.*
- *The limit $s \searrow 0$ means $s \to 0$ monotonically from above.*
- *We have*

$$\kappa(z,\alpha,s) = \begin{cases} \kappa(z,\alpha)\,\mathrm{Im}(z)^{-s} |j(\sigma_x^{-1}, z)|^{2s}, & z \in U_x \text{ and } \alpha x = x \text{ for some cusp } x, \\ \kappa(z,\alpha), & \text{otherwise} \end{cases}$$

where $\kappa(z,\alpha) = K_k(\alpha z, z) j(\alpha,z)^{-k}\,\mathrm{Im}(z)^k$, σ_x is any element of $\mathrm{SL}_2(\mathbb{Z})$ with $\sigma_x\infty = x$ and $U_x = \sigma_x\{z \in \mathcal{H} \colon \mathrm{Im}(z) > \delta\}$ for some $\delta > 1$.

Proof. We want to modify the formula given by Proposition 4.1.11. Note that

$$T^1 = \bigsqcup_{\alpha \in T^1//\Gamma} [\alpha]_\Gamma = \bigsqcup_{\alpha \in T^1//\Gamma} \bigcup_{\gamma \in \Gamma} \{\gamma^{-1}\alpha\gamma\} = \bigsqcup_{\alpha \in T^1//\Gamma} \bigsqcup_{\gamma \in \Gamma(\alpha)\backslash\Gamma} \{\gamma^{-1}\alpha\gamma\}.$$

The last inner union is indeed disjoint since $\gamma_1^{-1}\alpha\gamma_1 = \gamma_2^{-1}\alpha\gamma_2$ if and only if $\gamma_1\gamma_2^{-1} \in \Gamma(\alpha)$. Therefore we have

$$\sum_{\alpha \in T^1} \int_F \kappa(z,\alpha) d\nu(z) = \sum_{\alpha \in T^1//\Gamma} \sum_{\gamma \in \Gamma(\alpha)\backslash\Gamma} \int_F \kappa(z, \gamma^{-1}\alpha\gamma) d\nu(z).$$

Let $\alpha \in T^1$ and $\gamma \in \Gamma$. We have $\int_F \kappa(z,\gamma^{-1}\alpha\gamma) d\nu(z) = \int_{\gamma F} \kappa(\gamma^{-1}z, \gamma^{-1}\alpha\gamma) d\nu(z)$. Using Proposition 3.2.1 one can check that $\kappa(\gamma^{-1}z, \gamma^{-1}\alpha\gamma) = \kappa(z,\alpha)$. Thus we see

$$\int_F \kappa(z,\gamma^{-1}\alpha\gamma) d\nu(z) = \int_{\gamma F} \kappa(\gamma^{-1}z, \gamma^{1}\alpha\gamma) d\nu(z) = \int_{\gamma F} \kappa(z,\alpha) d\nu(z).$$

Let $\gamma \in \Gamma(\alpha)$. Then $\kappa(\gamma z, \alpha) = \kappa(z, \gamma^{-1}\alpha\gamma) = \kappa(z,\alpha)$ as γ and α commute. So the integrand $\kappa(z,\alpha)$ is $\Gamma(\alpha)$-invariant. Therefore we have

$$\sum_{\gamma \in \Gamma(\alpha)\backslash\Gamma} \int_{\gamma F} \kappa(z,\alpha) d\nu(z) = \int_{\Gamma(\alpha)\backslash\mathcal{H}} \kappa(z,\alpha) d\nu(z),$$

and hence

$$\sum_{\alpha\in T^1}\int_F \kappa(z,\alpha)d\nu(z) = \sum_{\alpha\in T^1//\Gamma}\int_{\Gamma(\alpha)\backslash\mathcal{H}} \kappa(z,\alpha)d\nu(z). \tag{4.1.7}$$

It remains to consider the second term of the claimed formula. As before we see

$$\sum_{\alpha\in T^2}\int_F \kappa(z,\alpha,s)d\nu(z) = \sum_{\alpha\in T^2//\Gamma}\sum_{\gamma\in\Gamma(\alpha)\backslash\Gamma}\int_{\gamma F}\kappa(\gamma^{-1}z,\gamma^{-1}\alpha\gamma,s)d\nu(z)$$

for any $s>0$. Let $\alpha\in T^2$, $\gamma\in\Gamma$ and $z\in\gamma F$. If $(\gamma^{-1}z)\in U_x$ and $(\gamma^{-1}\alpha\gamma)x = x$ for some cusp x, one can check that

$$\kappa(\gamma^{-1}z,\gamma^{-1}\alpha\gamma,s) = \kappa(z,\alpha)\,\mathrm{Im}(z)^{-s}|j((\gamma\sigma_x)^{-1},z)|^{2s}.$$

On the other hand these conditions imply $z\in\gamma U_x = U_{\gamma x}$ and $\alpha(\gamma x) = \gamma x$, so by definition of $\kappa(z,\alpha,s)$ we also have

$$\kappa(z,\alpha,s) = \kappa(z,\alpha)\,\mathrm{Im}(z)^{-s}|j(\sigma_{\gamma x}^{-1},z)|^{2s}.$$

Since $(\gamma\sigma_x)\infty = \gamma x$ we can choose $\sigma_{\gamma x} = \gamma\sigma_x$ and hence $\kappa(\gamma^{-1}z,\gamma^{-1}\alpha\gamma,s) = \kappa(z,\alpha,s)$. Therefore

$$\begin{aligned}\sum_{\alpha\in T^2}\int_F \kappa(z,\alpha,s)d\nu(z) &= \sum_{\alpha\in T^2//\Gamma}\sum_{\gamma\in\Gamma(\alpha)\backslash\Gamma}\int_{\gamma F}\kappa(z,\alpha,s)d\nu(z)\\ &= \sum_{\alpha\in T^2//\Gamma}\int_{\Gamma(\alpha)\backslash\mathcal{H}}\kappa(z,\alpha,s)d\nu(z). \qquad (4.1.8)\end{aligned}$$

Using (4.1.7) and (4.1.8) with Proposition 4.1.11 gives the claimed expression. □

4.2 Calculation of integrals

We aim to further simplify the trace formula obtained by Theorem 4.1.13. More precisely, we want to compute integrals of the form

$$\int_{\Gamma(\alpha)\backslash\mathcal{H}}\kappa(z,\alpha)d\nu(z) \quad\text{and}\quad \int_{\Gamma(\alpha)\backslash\mathcal{H}}\kappa(z,\alpha,s)d\nu(z).$$

It turns out that it is convenient to use the classification of elements in $\mathrm{GL}_2^+(\mathbb{R})$ introduced in Section 2.2 for this purpose. Recall that $Z(T)$ is the set of scalar elements in T, and that the (total) set of cusps of Γ is given by $\mathbb{Q}\cup\{\infty\}$. We define

$$\begin{aligned}T^e &= \{\alpha\in T\colon \alpha \text{ elliptic}\}, \quad T^p = \{\alpha\in T\colon \alpha\text{ parabolic}\},\\ T^{h_1} &= \{\alpha\in T\colon \alpha\text{ hyperbolic with fixed points in } \mathbb{R}\backslash\mathbb{Q}\},\\ T^{h_2} &= \{\alpha\in T\colon \alpha\text{ hyperbolic with fixed points in } \mathbb{Q}\cup\{\infty\}\}.\end{aligned}$$

By Corollary 2.2.1 we know that $T = Z(T) \cup T^e \cup T^p \cup T^{h_1} \cup T^{h_2}$, and by definition of T^1 and T^2 we have $T^1 = Z(T) \cup T^e \cup T^{h_1}$ and $T^2 = T^p \cup T^{h_2}$. Clearly all these unions are disjoint. Moreover, we note that all these sets are stable under conjugation by elements in Γ. This is obvious for $Z(T)$, and also clear for T^e, T^p and $T^{h_1} \cup T^{h_2}$ since trace and determinant are stable under conjugation. Further, it can be checked for T^{h_1} and T^{h_2}. Therefore we may split the two sums given in the trace formula of Theorem 4.1.13 as follows:

$$\sum_{\alpha \in T^1//\Gamma} \int_{\Gamma(\alpha)\backslash\mathcal{H}} \kappa(z,\alpha) d\nu(z) = \sum_{\alpha \in Z(\Gamma)//\Gamma} \dots + \sum_{\alpha \in T^e//\Gamma} \dots + \sum_{\alpha \in T^{h_1}//\Gamma} \dots$$

$$\sum_{\alpha \in T^2//\Gamma} \int_{\Gamma(\alpha)\backslash\mathcal{H}} \kappa(z,\alpha,s) d\nu(z) = \sum_{\alpha \in T^p//\Gamma} \dots + \sum_{\alpha \in T^{h_2}//\Gamma} \dots$$

In the following subsections we will study all of these five terms separately, closely following the argumentation in [Miy06], pages 236 to 240. In the process we will fill in many (often technical) details omitted in Miyake's book. At then of each section we summarise our results, combining them in the end in Section 4.3 within one big trace formula.

4.2.1 The scalar terms

Let $\alpha = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right) \in Z(T)$. Then $\Gamma(\alpha) = \Gamma$ and one can check that $\kappa(z,\alpha) = \frac{k-1}{4\pi}\lambda^{-k}$. Hence

$$\int_{\Gamma(\alpha)\backslash\mathcal{H}} \kappa(z,\alpha) d\nu(z) = \frac{k-1}{4\pi}\lambda^{-k}\nu(\Gamma\backslash\mathcal{H}).$$

By Lemma 2.1.2 we have that $\nu(\Gamma\backslash\mathcal{H}) = d_\Gamma \pi/3$ where $d_\Gamma = [\mathrm{SL}_2(\mathbb{Z})/\{\pm 1\} : \Gamma/Z(\Gamma)]$, and since $\alpha \in T$ we see $\det(\alpha) = \det(g)$, so $\lambda = \mathrm{sign}(\lambda)\sqrt{\det(g)}$. Therefore

$$\int_{\Gamma(\alpha)\backslash\mathcal{H}} \kappa(z,\alpha) d\nu(z) = \frac{k-1}{12}\,\mathrm{sign}(\lambda)^k \det(g)^{-k/2} d_\Gamma.$$

Note that $Z(T) = T \cap \{\pm\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right)\}$ with $\lambda = \sqrt{\det(g)}$, so if $\det(g)$ does not have a rational square root, then $Z(T)$ is empty. In addition, if both $\pm\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right) \in T$, then these elements are clearly not Γ-conjugates of each other. Finally, one can check that the quotient $d_\Gamma/|Z(\Gamma)|$ equals $1/2 \cdot [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$. We summarise our results for the scalar terms as:

Lemma 4.2.1. *We have*

$$\frac{\det(g)^{k-1}}{|Z(\Gamma)|} \sum_{\alpha \in Z(T)//\Gamma} \int_{\Gamma(\alpha)\backslash\mathcal{H}} \kappa(z,\alpha) d\nu(z)$$
$$= \frac{k-1}{24} \det(g)^{k/2-1} [\mathrm{SL}_2(\mathbb{Z}) : \Gamma] \sum_{\alpha \in Z(T)} \mathrm{sign}(\lambda_\alpha)^k.$$

where $Z(T) = T \cap \{\pm\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right)\}$, $\lambda = \sqrt{\det(g)}$ and λ_α denotes the eigenvalue of α. In particular, the sum is empty if $\det(g)$ does not have a rational square root.

4.2.2 The elliptic terms

Let $\alpha \in T^e$. Since α is elliptic there is $z_0 \in \mathcal{H}$ such that z_0 and $\overline{z_0}$ are the unique fixed points of α. Furthermore, there is $\lambda \in \mathbb{C}\backslash\mathbb{R}$ such that λ and $\overline{\lambda}$ are the eigenvalues of α. Put $\sigma = \left(\begin{smallmatrix} 1 & -z_0 \\ 1 & -\overline{z_0} \end{smallmatrix}\right)$, then $\sigma z_0 = 0$ and thus $\sigma\alpha\sigma^{-1}0 = 0$. Hence $\sigma\alpha\sigma^{-1}$ is of the form $\left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right)$. Moreover, $\sigma\overline{z_0} = \infty$, so $\sigma\alpha\sigma^{-1}\infty = \infty$, and therefore $\sigma\alpha\sigma^{-1}$ is of the form $\left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)$. As α and $\sigma\alpha\sigma^{-1}$ have the same eigenvalues we get $\sigma\alpha\sigma^{-1} = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{smallmatrix}\right)$. Note that we might have to replace λ by $\overline{\lambda}$ at this point. Moreover, this fixes λ since σ is unique.

Using the equality we can express α in terms of its fixed points and its eigenvalues:

$$\alpha = \sigma^{-1}\begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix}\sigma = \frac{1}{z_0 - \overline{z_0}}\begin{pmatrix} \overline{\lambda}z_0 - \lambda\overline{z_0} & |z_0|^2(\lambda - \overline{\lambda}) \\ \overline{\lambda} - \lambda & \lambda z_0 - \overline{\lambda}\overline{z_0} \end{pmatrix}.$$

Now we fix $z \in \mathcal{H}$ and put $w = \sigma z$, $w' = \sigma\overline{z}$. One can check that $\overline{w}w' = 1$, so $w' = \frac{w}{|w|^2}$. Moreover, we see

$$\frac{w - w'}{\lambda/\overline{\lambda} \cdot w - w'} = \frac{\sigma z - \sigma\overline{z}}{(\sigma\alpha\sigma^{-1})(\sigma z) - \sigma\overline{z}} = \frac{z - \overline{z}}{\alpha z - \overline{z}}\frac{\alpha z - \overline{z_0}}{z - \overline{z_0}}.$$

Here we used that $\beta a - \beta b = \det(\beta)(a - b)j(\beta, a)^{-1}j(\beta, b)^{-1}$ for any $\beta \in \mathrm{GL}_2(\mathbb{C})$ and for any $a, b \in \mathbb{C}$ such that $j(\beta, a) \neq 0$ and $j(\beta, b) \neq 0$.Further, one can check that $(\alpha z - \overline{z_0})j(\alpha, z) = \overline{\lambda}(z - \overline{z_0})$. Therefore

$$\kappa(z, \alpha) = \frac{k-1}{4\pi}(2i\,\mathrm{Im}(z))^k\left[(\alpha z - \overline{z})j(\alpha, z)\right]^{-k} = \frac{k-1}{4\pi}\,\overline{\lambda}^{-k}\left(\frac{w - w'}{\lambda/\overline{\lambda} \cdot w - w'}\right)^k.$$

Since $w' = \frac{w}{|w|^2}$ as noted earlier, and since $\det(g) = \det(\alpha) = \lambda\overline{\lambda}$, we can write

$$\kappa(z, \alpha) = \frac{k-1}{4\pi}\lambda^k\det(g)^{-k}\left(\frac{1 - |w|^2}{1 - \lambda/\overline{\lambda} \cdot |w|^2}\right)^k.$$

Next we note that $\Gamma(\alpha) = \Gamma_{z_0}$ by Corollary 2.2.4, and that Γ_{z_0} is a finite group by Lemma 2.2.6. Since any non-scalar element in Γ_{z_0} has the unique fixed point z_0 in $\mathcal{H}$, all but one Γ_{z_0}-orbit in $\mathcal{H}$ consists of exactly $|\Gamma_{z_0}/Z(\Gamma)|$ elements. Therefore we get

$$\int_{\mathcal{H}}\kappa(z, \alpha)d\nu(z) = |\Gamma_{z_0}/Z(\Gamma)| \cdot \int_{\Gamma(\alpha)\backslash\mathcal{H}}\kappa(z, \alpha)d\nu(z).$$

Combining these results we see

$$\int_{\Gamma(\alpha)\backslash\mathcal{H}}\kappa(z, \alpha)d\nu(z) = \frac{k-1}{4\pi}\lambda^k\det(g)^{-k}\frac{1}{|\Gamma_{z_0}/Z(\Gamma)|}\int_{\mathcal{H}}\left(\frac{1 - |\sigma z|^2}{1 - \lambda/\overline{\lambda} \cdot |\sigma z|^2}\right)^k d\nu(z).$$

Obviously we want to substitute $w = \sigma z$. One can check that $\sigma\mathcal{H} = \mathbb{D}$ where $\mathbb{D}$ denotes the open unit disk in the complex plane. Moreover, one can check that the substitution transforms $d\nu(z) = \mathrm{Im}(z)^{-2}dz$ into $d\nu_{\mathbb{D}}(w) = 4(1-|w|^2)^{-2}dw$. (See Section 1.4 in [Miy06] for some details on this matter.) Hence

$$\begin{aligned}\int_{\mathcal{H}}\left(\frac{1-|\sigma z|^2}{1-\lambda/\overline{\lambda}\cdot|\sigma z|^2}\right)^k d\nu(z) &= 4\int_{\mathbb{D}}\frac{(1-|w|^2)^{k-2}}{\left(1-\lambda/\overline{\lambda}\cdot|w|^2\right)^k}dw \\ &= 4\int_0^1\frac{(1-r^2)^{k-2}}{\left(1-\lambda/\overline{\lambda}\cdot r^2\right)^k}rdr\int_0^{2\pi}d\varphi \\ &= 8\pi\int_0^1\frac{(1-s)^{k-2}}{\left(1-\lambda/\overline{\lambda}\cdot s\right)^k}\frac{ds}{2} \\ &= 4\pi\left[\frac{1}{k-1}\frac{1}{\lambda/\overline{\lambda}-1}\left(\frac{1-s}{1-\lambda/\overline{\lambda}\cdot s}\right)^{k-1}\right]_0^1 \\ &= -\frac{4\pi}{k-1}\frac{\overline{\lambda}}{\lambda-\overline{\lambda}}.\end{aligned}$$

Therefore we have using again that $\det(g) = \lambda\overline{\lambda}$

$$\int_{\Gamma(\alpha)\backslash\mathcal{H}}\kappa(z,\alpha)d\nu(z) = \frac{\lambda^k\,\overline{\lambda}}{\overline{\lambda}-\lambda}\frac{\det(g)^{-k}}{|\Gamma_{z_0}/Z(\Gamma)|} = \frac{\lambda^{k-1}}{\overline{\lambda}-\lambda}\frac{\det(g)^{1-k}}{|\Gamma_{z_0}/Z(\Gamma)|}.$$

Note that $|\Gamma_{z_0}/Z(\Gamma)|\cdot|Z(\Gamma)| = |\Gamma_{z_0}| = |\Gamma(\alpha)|$. We summarise our observations:

Lemma 4.2.2. *For $\alpha\in T^e$ with unique fixed point $z\in\mathcal{H}$ we can write*

$$\sigma_\alpha\alpha\sigma_\alpha^{-1} = \begin{pmatrix}\lambda_\alpha & 0\\ 0 & \overline{\lambda_\alpha}\end{pmatrix}$$

where $\sigma_\alpha = \left(\begin{smallmatrix}1 & -z\\ 1 & -\overline{z}\end{smallmatrix}\right)$ and $\lambda_\alpha,\overline{\lambda_\alpha}$ are the eigenvalues of α. Using this notation we have

$$\frac{\det(g)^{k-1}}{|Z(\Gamma)|}\sum_{\alpha\in T^e//\Gamma}\int_{\Gamma(\alpha)\backslash\mathcal{H}}\kappa(z,\alpha)d\nu(z) \;=\; \sum_{\alpha\in T^e//\Gamma}\frac{1}{|\Gamma(\alpha)|}\frac{\lambda_\alpha^{k-1}}{\overline{\lambda_\alpha}-\lambda_\alpha}.$$

4.2.3 The hyperbolic terms of type one

Let $\alpha\in T^{h_1}$. Then there are distinct $x_1,x_2\in\mathbb{R}\backslash\mathbb{Q}$ such that x_1 and x_2 are the unique fixed points of α. Furthermore, α has two distinct real eigenvalues, say λ_1 and λ_2. We can assume that $x_2 > x_1$ without loss of generality. Put $\sigma = (x_2-x_1)^{-1/2}\left(\begin{smallmatrix}1 & -x_2\\ 1 & -x_1\end{smallmatrix}\right)$, then $\sigma x_1 = \infty$, $\sigma x_2 = 0$ and $\sigma\in\mathrm{SL}_2(\mathbb{R})$. Since $\sigma\alpha\sigma^{-1}0 = 0$ and $\sigma\alpha\sigma^{-1}\infty = \infty$ we can argue as for elliptic α that $\sigma\alpha\sigma^{-1} = \left(\begin{smallmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{smallmatrix}\right)$. (Again, we might have to swap λ_1 and λ_2 at this point.) As before we use this equation to express α by

$$\alpha = \sigma^{-1}\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\sigma = \frac{1}{x_2-x_1}\begin{pmatrix}\lambda_2x_2-\lambda_1x_1 & x_1x_2(\lambda_1-\lambda_2)\\ \lambda_2-\lambda_1 & \lambda_1x_2-\lambda_2x_1\end{pmatrix}.$$

Now we fix $z \in \mathcal{H}$ and put $w = \sigma z$, $w' = \sigma\overline{z}$. Since σ has entries in $\mathbb{R}$ we have $w' = \overline{w}$. Exactly as in the previous subsection we find

$$\frac{w-\overline{w}}{\lambda_1/\lambda_2 \cdot w - \overline{w}} = \frac{z-\overline{z}}{\alpha z - \overline{z}}\frac{\alpha z - x_1}{z - x_1}, \quad (\alpha z - x_1) j(\alpha, z) = \lambda_2(z - x_1),$$

and thus

$$\kappa(z,\alpha) = \frac{k-1}{4\pi}\ \lambda_2^{-k}\left(\frac{w-\overline{w}}{\lambda_1/\lambda_2 \cdot w - \overline{w}}\right)^k.$$

Note that $\lambda_2 \neq 0$ since $\lambda_1\lambda_2 = \det(\alpha) > 0$. We put $\lambda = \lambda_1/\lambda_2$.

Next we consider $\Gamma(\alpha)$. By Corollary 2.2.4 we have $\Gamma(\alpha) = \Gamma_{x_1} \cap \Gamma_{x_1}$. First suppose that this intersection is trivial, so $\Gamma(\alpha) = Z(\Gamma)$. Then $\Gamma(\alpha)\backslash\mathcal{H} = \mathcal{H}$. Recall that elements in $\mathrm{SL}_2(\mathbb{R})$ act as automorphisms on the upper half-plane, and that $d\nu(z)$ is $\mathrm{SL}_2(\mathbb{R})$-invariant. Thus we get using the substitution $w = \sigma z$ where $\sigma \in \mathrm{SL}_2(\mathbb{R})$ by construction

$$\begin{aligned}\int_{\Gamma(\alpha)\backslash\mathcal{H}} \kappa(z,\alpha) d\nu(z) &= \frac{k-1}{4\pi}\lambda_2^{-k}\int_{\mathcal{H}}\left(\frac{w-\overline{w}}{\lambda w - \overline{w}}\right)^k d\nu(w) \\ &= \frac{k-1}{4\pi}\lambda_2^{-k}\int_0^\infty\int_0^\pi\left(\frac{re^{i\varphi}-re^{-i\varphi}}{\lambda re^{i\varphi}-re^{-i\varphi}}\right)^k\frac{r\cdot drd\varphi}{(r\sin(\varphi))^2}.\end{aligned}$$

Taking absolut values yields

$$\int_{\Gamma(\alpha)\backslash\mathcal{H}} |\kappa(z,\alpha)| d\nu(z) = \frac{k-1}{\pi}|\lambda_2|^{-k}\int_0^\infty \frac{1}{r}dr\int_0^\pi\frac{1}{(\sin(\varphi))^2}\left|\frac{e^{i\varphi}-e^{-i\varphi}}{\lambda e^{i\varphi}-e^{-i\varphi}}\right|^k d\varphi,$$

which is a contradiction since we know that the left-hand side is convergent, but the integral $\int_0^\infty r^{-1}dr$ does not converge, and the last integral is convergent and non-zero. Thus we cannot have $\Gamma(\alpha) = Z(\Gamma)$, and may therefore use Lemma 2.2.6: There is $u > 0$ such that

$$\left(\sigma(\Gamma_{x_1}\cap\Gamma_{x_2})\sigma^{-1}\right)\cdot\{\pm 1\} = \left\{\pm\begin{pmatrix}u^m & 0\\ 0 & u^{-m}\end{pmatrix} : m \in \mathbb{Z}\right\}.$$

Note that we may replace u by u^{-1} without changing the set, and that $u \neq 1$ since $\Gamma_{x_1} \cap \Gamma_{x_1} \neq Z(\Gamma)$ by assumption. Thus we can assume $u > 1$. Furthermore, we note that $\pm\left(\begin{smallmatrix}u^m & 0\\ 0 & u^{-m}\end{smallmatrix}\right)z = u^{2m}z$ for any $z \in \mathbb{C}$ and any $m \in \mathbb{Z}$. Hence a fundamental domain of the quotient $(\sigma(\Gamma_{x_1}\cap\Gamma_{x_2})\sigma^{-1})\backslash\mathcal{H}$ is given by $\{w \in \mathcal{H} \colon 1 \leqslant |w| < u^2\}$. (The sign does not matter since ± 1 acts trivially.) So we get using the substitution $w = \sigma z$

$$\begin{aligned}\int_{\Gamma(\alpha)\backslash\mathcal{H}} \kappa(z,\alpha) d\nu(z) &= \frac{k-1}{4\pi}\lambda_2^{-k}\int_{(\sigma\Gamma(\alpha)\sigma^{-1})\backslash\mathcal{H}}\left(\frac{w-\overline{w}}{\lambda w - \overline{w}}\right)^k d\nu(w) \\ &= \frac{k-1}{4\pi}\lambda_2^{-k}\int_1^{u^2}\int_0^\pi\left(\frac{re^{i\varphi}-re^{-i\varphi}}{\lambda re^{i\varphi}-re^{-i\varphi}}\right)^k\frac{r\cdot drd\varphi}{(r\sin(\varphi))^2} \\ &= \frac{k-1}{4\pi}\lambda_2^{-k}\int_1^{u^2}\frac{1}{r}dr\ \int_0^\pi\left(\frac{e^{i\varphi}-e^{-i\varphi}}{\lambda e^{i\varphi}-e^{-i\varphi}}\right)^k\frac{d\varphi}{(\sin(\varphi))^2}.\end{aligned} \tag{4.2.1}$$

For the first integral we compute $\int_1^{u^2} r^{-1}dr = 2\ln(u)$, and for the second one we note that $(e^{i\varphi} - e^{-i\varphi})^2 = -4(\sin(\varphi))^2$. Thus we have to consider

$$\int_0^\pi \frac{(e^{i\varphi} - e^{-i\varphi})^{k-2}}{(\lambda e^{i\varphi} - e^{-i\varphi})^k} d\varphi.$$

Let $f(\varphi)$ be the integrand, then f is π-periodic, so $f(\varphi + \pi) = f(\varphi)$ for all $\varphi \in \mathbb{C}$, and f is a meromorphic function on $\mathbb{C}$ with singularities at $\pi n + i\ln(\lambda)/2$ for $n \in \mathbb{Z}$. (Note that $\lambda = \det(\alpha)/\lambda_2^2 > 0$ since $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$.) Suppose that $\lambda \in (0,1)$, then f is holomorphic on the extended upper halp-plane $\{\varphi \in \mathbb{C} \colon \mathrm{Im}(\varphi) > -R\}$ for sufficiently small $R > 0$. Put

$$A_R := \{\varphi \in \mathbb{C} \colon 0 \leqslant \mathrm{Re}(\varphi) < \pi,\ \mathrm{Im}(\varphi) > -R\},$$

and let Φ denote the map $\varphi \mapsto 2i\varphi$, then

$$\Phi(A_R) = \{z \in \mathbb{C} \colon \mathrm{Re}(z) < 2R,\ 0 \leqslant \mathrm{Im}(\varphi) < 2\pi i\}.$$

Recall that the complex exponential function is bijective on $\Phi(A_R)$. We denote its inverse defined on the punctured disc $B_R := \exp(\Phi(A_R)) = \{q \in \mathbb{C}^\times \colon |q| < e^{2R}\}$ by log, and recall that this inverse is holomorphic except for a $2\pi i$-skip while crossing the positive real axis. We define

$$\tilde{f} \colon B_R \to \mathbb{C},\ q \mapsto f\left(\frac{\log(q)}{2i}\right).$$

Then $\tilde{f}$ is by construction well-defined. Moreover, $\tilde{f}$ is holomorphic on B_R since f being π-periodic compensates for the $2\pi i$-skip. We claim that $\tilde{f}$ has a removable singularity at 0. To see this note that

$$f(\varphi) = e^{2i\varphi}\ \frac{(1 - e^{2i\varphi})^{k-2}}{(1 - \lambda e^{2i\varphi})^k}.$$

Hence $\tilde{f}(q) = q(1-q)^{k-2}(1-\lambda q)^{-k}$ and thus $\lim_{q\to 0} \tilde{f}(q) = 0$. Therefore we can write

$$f\left(\frac{\log(q)}{2i}\right) = \tilde{f}(q) = \sum_{n=1}^{\infty} a_n q^n, \quad q \in B_R$$

for some $a_n \in \mathbb{C}$, $n \in \mathbb{N}$. (The constant term vanishes since $\lim_{q\to 0} \tilde{f}(q) = 0$.) The series converges absolutely and locally uniformly. In particular, it converges absolutely uniformly on the smaller annulus $B_{R/2}$. Substituting $q = e^{2i\varphi}$ gives $f(\varphi) = \sum_{n=1}^{\infty} a_n e^{2in\varphi}$ for $\varphi \in A_R$, which correspondingly converges absolutely uniformly on $A_{R/2}$. Finally we can compute using uniform convergence of the sum

$$\int_0^\pi f(\varphi)d\varphi = \sum_{n=1}^{\infty} a_n \int_0^\pi e^{2in\varphi} d\varphi = 0.$$

It remains to consider the case $\lambda > 1$. (Clearly $\lambda \neq 1$ since $\lambda_1 \neq \lambda_2$.) We will use a similar argument. Put

$$A_R := \{\varphi \in \mathbb{C} \colon 0 \leqslant \mathrm{Re}(\varphi) < \pi,\ \mathrm{Im}(\varphi) < R\},$$

then f is holomorphic on A_R for sufficiently small $R > 0$, and $\tilde{f}$ defined as before is holomorphic on $B_R := \exp(\Phi(A_R)) = \{q \in \mathbb{C}^\times : |q| > e^{-2R}\}$. Further, we define $\hat{f}(q) = \tilde{f}(1/q)$ for $q \in B'_R := \{q \in \mathbb{C}^\times : |q| < e^{2R}\}$. Then $\hat{f}$ is holomorphic on B'_R, and we claim that $\hat{f}$ has a removable singularity at 0. To see this note that

$$\hat{f}(q) = \frac{1/q \cdot (1 - 1/q)^{k-2}}{(1 - \lambda/q)^k} = q\,\frac{(1-q)^{k-2}}{(\lambda - q)^k}.$$

Hence $\lim_{q\to 0} \hat{f}(q) = 0$, and we can write

$$f\left(\frac{\log(1/q)}{2i}\right) = \tilde{f}(1/q) = \hat{f}(q) = \sum_{n=1}^{\infty} a_n q^n, \quad q \in B'_R,$$

for some $a_n \in \mathbb{C}$, $n \in \mathbb{N}$. Substituting $q = e^{-2i\varphi}$ yields $f(\varphi) = \sum_{n=1}^{\infty} a_n e^{-2in\varphi}$ for $\varphi \in A_R$, converging absolutely uniformly on $A_{R/2}$, so

$$\int_0^\pi f(\varphi)d\varphi = \sum_{n=1}^{\infty} a_n \int_0^\pi e^{-2in\varphi} d\varphi = 0.$$

Therefore we have shown in general that $\int_0^\pi f(\varphi)d\varphi = 0$, and thus by equation (4.2.1)

$$\int_{\Gamma(\alpha)\backslash\mathcal{H}} \kappa(z,\alpha)d\nu(z) = \frac{-2(k-1)\ln(u)}{\pi} \lambda_2^{-k} \int_0^\pi f(\varphi)d\varphi = 0.$$

Hence the terms in the trace formula of Theorem 4.1.13 corresponding to hyperbolic α with fixed points in $\mathbb{R}\backslash\mathbb{Q}$ do not contribute anything:

Lemma 4.2.3. *We have*

$$\frac{\det(g)^{k-1}}{|Z(\Gamma)|} \sum_{\alpha \in T^{h_1}//\Gamma} \int_{\Gamma(\alpha)\backslash\mathcal{H}} \kappa(z,\alpha)d\nu(z) \;=\; 0.$$

4.2.4 The hyperbolic terms of type two

Let $\alpha \in T^{h_2}$. Then there are distinct $x_1, x_2 \in \mathbb{Q} \cup \{\infty\}$ such that x_1 and x_2 are the unique fixed points of α. Furthermore, α has two distinct real eigenvalues, say λ_1 and λ_2. If $x_1, x_2 \neq \infty$ we may choose choose $\sigma \in \mathrm{SL}_2(\mathbb{R})$ as in Subsection 4.2.3 (assuming that $x_2 > x_1$). If $x_1 = \infty$ we choose $\sigma = \left(\begin{smallmatrix} 1 & -x_2 \\ 0 & 1 \end{smallmatrix}\right)$, and if $x_2 = \infty$ we choose $\sigma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -x_1 \end{smallmatrix}\right)$. Using exactly the same arguments as in the previous subsection we get

$$\kappa(z,\alpha) = \frac{k-1}{4\pi}\,\lambda_2^{-k} \left(\frac{\sigma z - \overline{\sigma z}}{\lambda\sigma z - \overline{\sigma z}}\right)^k$$

where $\lambda = \lambda_1/\lambda_2 > 0$. By Corollary 2.2.4 we have $\Gamma(\alpha) = Z(\Gamma)$, so

$$\int_{\Gamma(\alpha)\backslash\mathcal{H}} \kappa(z,\alpha,s)d\nu(z) = \int_{\mathcal{H}} \kappa(z,\alpha,s)d\nu(z).$$

Let $\sigma_1, \sigma_2 \in \mathrm{SL}_2(\mathbb{Z})$ such that $\sigma_1\infty = x_1$, $\sigma_2\infty = x_2$, and let $U_1 := \sigma_1 U_\infty$, $U_2 := \sigma_2 U_\infty$ where $U_\infty = \{z \in \mathcal{H}\colon \mathrm{Im}(z) > \delta\}$ for some $\delta > 1$ as in Section 4.1. Since the only fixed points of α are x_1 and x_2 we have by definition

$$\kappa(z,\alpha,s) = \begin{cases} \kappa(z,\alpha)\,\mathrm{Im}(z)^{-s}|j(\sigma_j^{-1},z)|^{2s} & , z \in U_j,\ j \in \{1,2\}, \\ \kappa(z,\alpha) & , \text{otherwise.} \end{cases}$$

We split the integral $\int_{\mathcal{H}} \kappa(z,\alpha,s)d\nu(z)$ into three parts, an integral over U_1, an integral over U_2 and an integral over $H' := \mathcal{H}\backslash(U_1 \cup U_2)$. For the first two integrals we get substituting $w = \sigma z$ as before

$$\begin{aligned}\int_{U_j} \kappa(z,\alpha,s)d\nu(z) &= \frac{k-1}{4\pi}\ \lambda_2^{-k} \int_{\sigma U_j} \left(\frac{w-\overline{w}}{\lambda w - \overline{w}}\right)^k \mathrm{Im}(\sigma^{-1}w)^{-s}|j(\sigma_j^{-1},\sigma^{-1}w)|^{2s}d\nu(w) \\ &= \frac{k-1}{4\pi}\ \lambda_2^{-k} \int_{\sigma U_j} \left(\frac{w-\overline{w}}{\lambda w - \overline{w}}\right)^k \mathrm{Im}(w)^{-s}|j((\sigma\sigma_j)^{-1},w)|^{2s}d\nu(w).\end{aligned}$$

Since $\sigma\sigma_1\infty = \infty$ and $\sigma\sigma_1 \in \mathrm{SL}_2(\mathbb{R})$ we have that $\sigma\sigma_1$ is of the form $\left(\begin{smallmatrix} a & * \\ 0 & 1/a\end{smallmatrix}\right)$, $a \in R^\times$. Hence $|j((\sigma\sigma_1)^{-1},w)| = |a|$ is constant, and

$$\sigma U_1 = (\sigma\sigma_1)U_\infty = \left\{z \in \mathcal{H}\colon \mathrm{Im}(z) > a^2\delta\right\}.$$

Therefore we get

$$\begin{aligned}\int_{U_1} \kappa(z,\alpha,s)d\nu(z) &= \frac{k-1}{4\pi}\ \lambda_2^{-k}\ a^{2s} \int_{\sigma U_1} \left(\frac{w-\overline{w}}{\lambda w - \overline{w}}\right)^k \mathrm{Im}(w)^{-s}d\nu(w) \\ &= \frac{k-1}{4\pi}\ \lambda_2^{-k}\ a^{2s} \int_0^\pi \int_{a^2\delta/\sin(\varphi)}^\infty \left(\frac{re^{i\varphi}-re^{-i\varphi}}{\lambda re^{i\varphi} - re^{-i\varphi}}\right)^k (r\sin(\varphi))^{-s} \frac{r\cdot drd\varphi}{(r\sin(\varphi))^2} \\ &= \frac{k-1}{4\pi}\ \lambda_2^{-k}\ a^{2s} \int_0^\pi \left(\frac{e^{i\varphi}-e^{-i\varphi}}{\lambda e^{i\varphi} - e^{-i\varphi}}\right)^k (\sin(\varphi))^{-s-2} \int_{a^2\delta/\sin(\varphi)}^\infty \frac{1}{r^{s+1}}dr\ d\varphi.\end{aligned}$$

Similarly to the previous subsection we define

$$f(\varphi) = \left(\frac{e^{i\varphi}-e^{-i\varphi}}{\lambda e^{i\varphi} - e^{-i\varphi}}\right)^k \frac{1}{(\sin(\varphi))^2} = (-4)\frac{(e^{i\varphi}-e^{-i\varphi})^{k-2}}{(\lambda e^{i\varphi} - e^{-i\varphi})^k},$$

and recall that we have shown $\int_0^\pi f(\varphi)d\varphi = 0$. Further, we note that

$$\int_{a^2\delta/\sin(\varphi)}^\infty \frac{1}{r^{s+1}}dr = \frac{1}{s}\left(\frac{\sin(\varphi)}{a^2\delta}\right)^s.$$

Hence we get

$$\int_{U_1} \kappa(z,\alpha,s)d\nu(z) = \frac{k-1}{4\pi}\ \lambda_2^{-k}\ \frac{1}{s\delta^s} \int_0^\pi f(\varphi)d\varphi = 0.$$

Consider now the integral over U_2. Since $\sigma\sigma_2\infty = 0$ and $\sigma\sigma_2 \in \mathrm{SL}_2(\mathbb{R})$ we have that $\sigma\sigma_2$ is of the form $\left(\begin{smallmatrix} 0 & b \\ -1/b & * \end{smallmatrix}\right)$, $b \in R^\times$. Thus $|j((\sigma\sigma_2)^{-1}, w)| = |w/b|$, and one can check that

$$\sigma U_2 = \sigma\sigma_2 U_\infty = \left\{ re^{i\varphi} \colon r \in (0,\infty), \varphi \in (0,\pi), \frac{\sin(\varphi)}{r} > \frac{\delta}{b^2} \right\}.$$

Therefore we get

$$\begin{aligned}
\int_{U_2} \kappa(z,\alpha,s) d\nu(z) &= \frac{k-1}{4\pi}\ \lambda_2^{-k} \int_{\sigma U_2} \left(\frac{w-\overline{w}}{\lambda w - \overline{w}}\right)^k \mathrm{Im}(w)^{-s} |j((\sigma\sigma_2)^{-1}, w)|^{2s} d\nu(w) \\
&= \frac{k-1}{4\pi}\ \lambda_2^{-k}\ b^{-2s} \int_0^\pi \int_0^{b^2\sin(\varphi)/\delta} \left(\frac{re^{i\varphi} - re^{-i\varphi}}{\lambda re^{i\varphi} - re^{-i\varphi}}\right)^k (r\sin(\varphi))^{-s} r^{2s} \frac{r \cdot dr d\varphi}{(r\sin(\varphi))^2} \\
&= \frac{k-1}{4\pi}\ \lambda_2^{-k}\ b^{-2s} \int_0^\pi f(\varphi)(\sin(\varphi))^{-s} \int_0^{b^2\sin(\varphi)/\delta} r^{s-1} dr\ d\varphi \\
&= \frac{k-1}{4\pi}\ \lambda_2^{-k}\ b^{-2s} \int_0^\pi f(\varphi)(\sin(\varphi))^{-s} \left[\frac{1}{s}\left(\frac{b^2\sin(\varphi)}{\delta}\right)^s\right] d\varphi \\
&= \frac{k-1}{4\pi}\ \lambda_2^{-k}\ \frac{1}{s\delta^s} \int_0^\pi f(\varphi) d\varphi \\
&= 0.
\end{aligned}$$

It remains to compute the integral over H'. Note that

$$\sigma H' = \mathcal{H} \backslash (\sigma U_1 \cup \sigma U_2) = \left\{ re^{i\varphi} \colon r \in (0,\infty), \varphi \in (0,\pi), \frac{b^2\sin(\varphi)}{\delta} \leqslant r \leqslant \frac{a^2\delta}{\sin(\varphi)} \right\}.$$

Therefore we see

$$\begin{aligned}
\int_{H'} \kappa(z,\alpha) d\nu(z) &= \frac{k-1}{4\pi}\ \lambda_2^{-k} \int_{\sigma H'} \left(\frac{w-\overline{w}}{\lambda w - \overline{w}}\right)^k d\nu(w) \\
&= \frac{k-1}{4\pi}\ \lambda_2^{-k} \int_0^\pi \int_{b^2\sin(\varphi)/\delta}^{a^2\delta/\sin(\varphi)} \left(\frac{re^{i\varphi} - re^{-i\varphi}}{\lambda re^{i\varphi} - re^{-i\varphi}}\right)^k \frac{r \cdot dr d\varphi}{(r\sin(\varphi))^2} \\
&= \frac{k-1}{4\pi}\ \lambda_2^{-k} \int_0^\pi f(\varphi) \int_{b^2\sin(\varphi)/\delta}^{a^2\delta/\sin(\varphi)} \frac{1}{r} dr\ d\varphi.
\end{aligned}$$

Note that

$$\int_{b^2\sin(\varphi)/\delta}^{a^2\delta/\sin(\varphi)} \frac{1}{r} dr = \ln\left(\frac{a^2\delta^2}{b^2(\sin(\varphi))^2}\right) = 2\left[\ln\left(\frac{a\delta}{b}\right) - \ln(\sin(\varphi))\right].$$

Hence we have that

$$\begin{aligned}
\int_{H'} \kappa(z,\alpha) d\nu(z) &= \frac{k-1}{2\pi}\ \lambda_2^{-k} \left[\ln\left(\frac{a\delta}{b}\right) \int_0^\pi f(\varphi) d\varphi - \int_0^\pi f(\varphi)\ln(\sin(\varphi)) d\varphi\right] \\
&= \frac{k-1}{-2\pi}\ \lambda_2^{-k} \int_0^\pi f(\varphi)\ln(\sin(\varphi)) d\varphi
\end{aligned}$$

since the first term vanishes as before. One can check that

$$\frac{d}{d\varphi}\left[\left(\frac{e^{i\varphi}-e^{-i\varphi}}{\lambda e^{i\varphi}-e^{-i\varphi}}\right)^{k-1}\right] = 2i(k-1)(\lambda-1)\frac{\left(e^{i\varphi}-e^{-i\varphi}\right)^{k-2}}{\left(\lambda e^{i\varphi}-e^{-i\varphi}\right)^{k}} = \frac{i(k-1)(\lambda-1)f(\varphi)}{-2}.$$

Hence we can use integration by parts to see

$$\int_{H'} \kappa(z,\alpha)d\nu(z) = \frac{-i}{\pi}\frac{\lambda_2^{-k}}{\lambda-1}\int_0^\pi \frac{i(k-1)(\lambda-1)f(\varphi)}{-2}\ln(\sin(\varphi))d\varphi$$
$$= \frac{-i}{\pi}\frac{\lambda_2^{-k}}{\lambda-1}\left(\left[\left(\frac{e^{i\varphi}-e^{-i\varphi}}{\lambda e^{i\varphi}-e^{-i\varphi}}\right)^{k-1}\ln(\sin(\varphi))\right]_0^\pi - \int_0^\pi \left(\frac{e^{i\varphi}-e^{-i\varphi}}{\lambda e^{i\varphi}-e^{-i\varphi}}\right)^{k-1}\frac{\cos(\varphi)}{\sin(\varphi)}d\varphi\right).$$

The first term vanishes, roughly since $\lim_{x\searrow 0} x^{k-1}\ln(x) = 0$. So we are left with

$$\int_{H'} \kappa(z,\alpha)d\nu(z) = \frac{-1}{\pi}\frac{\lambda_2^{-k}}{\lambda-1}\int_0^\pi \left(\frac{e^{i\varphi}-e^{-i\varphi}}{\lambda e^{i\varphi}-e^{-i\varphi}}\right)^{k-1}\frac{e^{i\varphi}+e^{-i\varphi}}{e^{i\varphi}-e^{-i\varphi}}d\varphi.$$

Let $g(\varphi)$ be the integrand, then

$$g(\varphi) = -\frac{\left(1-e^{2i\varphi}\right)^{k-2}}{\left(1-\lambda e^{2i\varphi}\right)^{k-1}}\left(1+e^{2i\varphi}\right),$$

and g is π-periodic, and meromorphic on $\mathbb{C}$ with singularities at $\pi n + i\ln(\lambda)/2$ for $n \in \mathbb{Z}$. Hence we can argue for g as we did for the function f in the previous subsection. First we suppose that $\lambda \in (0,1)$, then we can define a function $\tilde{g}$ by $\tilde{g}(q) = g(\log(q)/(2i))$ for $q \in B_R := \{q \in \mathbb{C}^\times : |q| < R\}$ which will be well-defined and holomorphic on B_R for sufficiently small $R > 0$. Since $\tilde{g}(q) = -(1-q)^{k-2}(1-\lambda q)^{-k+1}(1+q)$ we see that $\tilde{g}$ has a removable singularity at 0, so we can write $\tilde{g}(q) = \sum_{n=0}^\infty a_n q^n$ for some $a_n \in \mathbb{C}$, $n \in \mathbb{N}_0$. Substituting $q = e^{2i\varphi}$ gives $g(\varphi) = \sum_{n=0}^\infty a_n e^{2in\varphi}$, and since the series converges absolutely and locally uniformly we have

$$\int_0^\pi g(\varphi)d\varphi = \sum_{n=0}^\infty a_n \int_0^\pi e^{2in\varphi}d\varphi = \pi a_0.$$

The constant term a_0 is given by $\lim_{q\to 0}\tilde{g}(q) = -1$, so $\int_0^\pi g(\varphi)d\varphi = -\pi$, and thus we have for $\lambda \in (0,1)$ as $\det(g) = \det(\alpha) = \lambda_1\lambda_2$ that

$$\int_{\Gamma(\alpha)\backslash\mathcal{H}} \kappa(z,\alpha,s)d\nu(z) = \int_{H'} \kappa(z,\alpha)d\nu(z) = \frac{\lambda_2^{-k}}{\lambda-1} = \frac{\lambda_1^{k-1}}{\lambda_1-\lambda_2}\det(g)^{1-k}.$$

It remains to consider the case $\lambda > 1$. (Again we have $\lambda \neq 1$ since the given eigenvalues are distinct.) As in the previous subsection we define $\hat{g} = \tilde{g}(1/q)$ for $q \in B_R$ and sufficiently small $R > 0$. Since

$$\hat{g}(q) = -\frac{(1-1/q)^{k-2}(1+1/q)}{(1-\lambda/q)^{k-1}} = -\frac{(q-1)^{k-2}(q+1)}{(q-\lambda)^{k-1}}$$

we see that $\hat{g}$ has a removable singularity at 0, and thus we can write $\hat{g}(q) = \sum_{n=0}^{\infty} a_n q^n$ for some $a_n \in \mathbb{C}$, $n \in \mathbb{N}_0$. Substituting $q = e^{-2i\varphi}$ gives $g(\varphi) = \sum_{n=0}^{\infty} a_n e^{-2in\varphi}$, and since the series converges absolutely and locally uniformly we have

$$\int_0^\pi g(\varphi) d\varphi = \sum_{n=0}^{\infty} a_n \int_0^\pi e^{-2in\varphi} d\varphi = \pi a_0.$$

The constant term a_0 is given by $\lim_{q \to 0} \hat{g}(q) = \lambda^{-k+1}$, so $\int_0^\pi g(\varphi) d\varphi = \lambda^{-k+1}\pi$, and thus we have for $\lambda > 1$

$$\int_{\Gamma(\alpha)\backslash\mathcal{H}} \kappa(z, \alpha, s) d\nu(z) = -\frac{\lambda_2^{-k}\lambda^{-k+1}}{\lambda - 1} = \frac{\lambda_1^{-k}}{1/\lambda - 1} = \frac{\lambda_2^{k-1}}{\lambda_2 - \lambda_1} \det(g)^{1-k}.$$

To combine these results in a single formula we note that $\lambda \in (0, 1)$ if and only if $|\lambda_1| < |\lambda_2|$, and correspondingly $\lambda > 1$ if and only if $|\lambda_1| > |\lambda_2|$. Further, we have $\lambda_1 > 0$ if and only if $\lambda_2 > 0$ since $\lambda_1\lambda_2 = \det(\alpha) > 0$. One can check that this yields

$$\int_{\Gamma(\alpha)\backslash\mathcal{H}} \kappa(z, \alpha, s) d\nu(z) = -\frac{\operatorname{sign}(\lambda_1)^k \cdot \min\{|\lambda_1|, |\lambda_2|\}^{k-1}}{|\lambda_2 - \lambda_1|} \det(g)^{1-k}.$$

Note that this formula is indeed independent of the ordering of the eigenvalues of α, so it gives the same result if we replace λ_1 by λ_2 and vice versa.

Lemma 4.2.4. *We have*

$$\begin{aligned} &\frac{\det(g)^{k-1}}{|Z(\Gamma)|} \lim_{s \searrow 0} \sum_{\alpha \in T^{h_2}//\Gamma} \int_{\Gamma(\alpha)\backslash\mathcal{H}} \kappa(z, \alpha, s) d\nu(z) \\ &\qquad = \frac{-1}{|Z(\Gamma)|} \sum_{\alpha \in T^{h_2}//\Gamma} \frac{\operatorname{sign}(\lambda_{\alpha,1})^k \cdot \min\{|\lambda_{\alpha,1}|, |\lambda_{\alpha,2}|\}^{k-1}}{|\lambda_{\alpha,2} - \lambda_{\alpha,1}|} \end{aligned}$$

where $\lambda_{\alpha,1}$ and $\lambda_{\alpha,2}$ are the distinct eigenvalues of $\alpha \in T^{h_2}$.

4.2.5 The parabolic terms

Let $\alpha \in T^p$. Then there is $x \in \mathbb{Q} \cup \{\infty\}$ such that x is the unique fixed points of α. Furthermore, α has exactly one eigenvalue which is rational, say $\lambda \in \mathbb{Q}^\times$. Let $\sigma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\sigma\infty = x$, then $\sigma^{-1}\alpha\sigma = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$, and since there is only one eigenvalue we have $\sigma^{-1}\alpha\sigma = \left(\begin{smallmatrix} \lambda & B \\ 0 & \lambda \end{smallmatrix}\right)$ for some $B \in \mathbb{Q}^\times$. (Note that $B \neq 0$ since α cannot be scalar.) Further we observe that $\lambda^2 = \det(\alpha) = \det(g)$, so $\lambda = \pm\sqrt{\det(g)}$ and T does not contain any parabolic elements if $\det(g)$ does not have a rational square root.

Put $\mu = B/(2\lambda)$. Using that $\beta a - \beta b = (a - b)j(\beta, a)^{-1}j(\beta, b)^{-1}$ for any $\beta \in \mathrm{SL}_2(\mathbb{R})$ and for any $a, b \in \mathbb{C}\backslash\mathbb{R}$, one can check for $z \in \mathcal{H}$ that

$$\kappa(\sigma z, \alpha) = \frac{k-1}{4\pi}\lambda^{-k} \left(\frac{z - \overline{z}}{z - \overline{z} + B/\lambda}\right)^k = \frac{k-1}{4\pi}\lambda^{-k} \left(\frac{\mathrm{Im}(z)}{\mathrm{Im}(z) - i\mu}\right)^k.$$

By Corollary 2.2.4 we have $\Gamma(\alpha) = \Gamma_x$, and by Lemma 2.2.6

$$\left(\sigma^{-1}\Gamma_x\sigma\right) \cdot \{\pm 1\} = \left\{\pm \begin{pmatrix} 1 & hm \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z}\right\}$$

where h is the width of the cusp $[x]$ for Γ. Therefore a fundamental domain of the quotient $\sigma^{-1}(\Gamma(\alpha)\backslash\mathcal{H}) = (\sigma^{-1}\Gamma_x\sigma)\backslash\mathcal{H}$ is given by $\{w \in \mathcal{H} \colon |\operatorname{Re}(w)| \leqslant h/2\}$. Further, we have by definition

$$\kappa(z,\alpha,s) = \begin{cases} \tilde{\kappa}(z,\alpha,s), & z \in U_x, \\ \kappa(z,\alpha), & \text{otherwise}, \end{cases}$$

where

$$\tilde{\kappa}(z,\alpha,s) = \kappa(z,\alpha)\operatorname{Im}(z)^{-s}|j(\sigma^{-1},z)|^{2s}$$

and $U_x = \sigma U_\infty$, $U_\infty = \{z \in \mathcal{H} \colon \operatorname{Im}(z) > \delta\}$ for some $\delta > 1$ as before. So

$$\int_{\Gamma(\alpha)\backslash\mathcal{H}} \kappa(z,\alpha,s)d\nu(z) = \int_{\Gamma(\alpha)\backslash U_x} \tilde{\kappa}(z,\alpha,s)d\nu(z) + \int_{\Gamma(\alpha)\backslash(\mathcal{H}\backslash U_x)} \kappa(z,\alpha)d\nu(z).$$

We want to reunite the two integrals on the right by introducing a limit for the second term. One can check that

$$\int_{\Gamma(\alpha)\backslash(\mathcal{H}\backslash U_x)} \kappa(z,\alpha)d\nu(z) \;=\; \lim_{s\searrow 0}\int_{\Gamma(\alpha)\backslash(\mathcal{H}\backslash U_x)} \tilde{\kappa}(z,\alpha,s)d\nu(z),$$

but this does not help since we are missing a limit for the first term. Therefore we have to consider the sum of all integrals in the trace formula of Theorem 4.1.13 coming from parabolic elements and the corresponding limit, so

$$\lim_{s\searrow 0}\sum_{\alpha\in T^p//\Gamma}\int_{\Gamma(\alpha)\backslash\mathcal{H}} \kappa(z,\alpha,s)d\nu(z).$$

Put $T^p_x = T^p \cap T_x$ for all $x \in \mathbb{Q}\cup\{\infty\}$, then we can write T^p as the disjoint union of all T^p_x since every $\alpha \in T^p$ fixes a unique $x \in \mathbb{Q}\cup\{\infty\}$. We claim that no two elements in T^p_x are Γ-conjugate. To see this suppose that $\alpha,\beta \in T^p_x$ such that $\alpha = \gamma^{-1}\beta\gamma$ for some $\gamma \in \Gamma$. Then $\beta(\gamma x) = \gamma(\alpha x) = \gamma x$, so $\gamma x = x$ since x is the unique fixed point of β. Hence $\gamma \in \Gamma_x = \Gamma(\alpha)$, and thus $\beta = \gamma\alpha\gamma^{-1} = \alpha$. This proves the claim. Next let $\alpha \in T^p_x$ and $\beta \in T^p_y$. If $\alpha = \gamma^{-1}\beta\gamma$ for some $\gamma \in \Gamma$, then $\beta(\gamma x) = \gamma x$ as before, and thus $\gamma x = y$ since y is the only fixed point of β. So distinct parabolic elements in T can only be Γ-conjugate if their fixed points do not agree, but lie in the same Γ-orbit. Moreover, we see for $\gamma \in \Gamma$ that

$$\gamma\, T^p_x\, \gamma^{-1} = \left(\gamma\, T^p\, \gamma^{-1}\right) \cap \left(\gamma\, T_x\, \gamma^{-1}\right) = T^p \cap T_{\gamma x} = T^p_{\gamma x}$$

since α is parabolic if and only if $\tau^{-1}\alpha\tau$ is parabolic for any invertible τ. Recall that $C(\Gamma)$ denotes the set of Γ-orbits in $\mathbb{Q}\cup\{\infty\}$. By the above observations a complete set of representatives for the set of Γ-conjugacy classes $T^p//\Gamma$ is given by $\bigcup_{x\in C(\Gamma)} T^p_x$. Further we recall that $C(\Gamma)$ is a finite set. Hence we can write

$$\begin{aligned}\lim_{s\searrow 0}\sum_{\alpha\in T^p//\Gamma}\int_{\Gamma(\alpha)\backslash\mathcal{H}}\kappa(z,\alpha,s)d\nu(z) &= \sum_{x\in C(\Gamma)}\lim_{s\searrow 0}\sum_{\alpha\in T^p_x}\int_{\Gamma(\alpha)\backslash\mathcal{H}}\kappa(z,\alpha,s)d\nu(z)\\ &= \sum_{x\in C(\Gamma)}\left[\lim_{s\searrow 0}\sum_{\alpha\in T^p_x}\int_{\Gamma(\alpha)\backslash U_x}\tilde{\kappa}(z,\alpha,s)d\nu(z)+\sum_{\alpha\in T^p_x}\int_{\Gamma(\alpha)\backslash(\mathcal{H}\backslash U_x)}\kappa(z,\alpha)d\nu(z)\right].\end{aligned}$$

We still want to introduce the limit $s\searrow 0$ for the second term to reunite the two sums. Fix some $x\in C(\Gamma)$ and let $\sigma_x\in \mathrm{SL}_2(\mathbb{Z})$ such that $\sigma_x\infty = x$. For $\alpha\in T^p_x$ we have $\sigma_x^{-1}\alpha\sigma_x = \left(\begin{smallmatrix}\lambda_\alpha & B_\alpha\\ 0 & \lambda_\alpha\end{smallmatrix}\right)$ with $\lambda_\alpha = \pm\sqrt{\det(g)}$ and $B_\alpha\in\mathbb{Q}^\times$ as before. Put $\mu_\alpha = B_\alpha/(2\lambda_\alpha)$ and let h_x be the width of the cusp $[x]$ for Γ. We note that a fundamental domain of the quotient $\sigma_x^{-1}(\Gamma(\alpha)\backslash(\mathcal{H}\backslash U_x)) = (\sigma_x^{-1}\Gamma_x\sigma_x)\backslash(\mathcal{H}\backslash U_\infty)$ is given by

$$F_1 := \{w\in\mathcal{H}\colon |\operatorname{Re}(w)|\leqslant h_x/2, \operatorname{Im}(w)\leqslant\delta\}.$$

Hence

$$\begin{aligned}&\sum_{\alpha\in T^p_x}\int_{\Gamma(\alpha)\backslash(\mathcal{H}\backslash U_x)}|\tilde{\kappa}(z,\alpha,s)|d\nu(z)\\ &= \sum_{\alpha\in T^p_x}\int_{\sigma^{-1}(\Gamma(\alpha)\backslash(\mathcal{H}\backslash U_x))}|\kappa(\sigma w,\alpha)|\operatorname{Im}(w)^{-s}d\nu(w)\\ &= \frac{k-1}{4\pi}\sum_{\alpha\in T^p_x}|\lambda_\alpha|^{-k}\int_{F_1}\left|\frac{\operatorname{Im}(w)}{\operatorname{Im}(w)-i\mu_\alpha}\right|^k\operatorname{Im}(w)^{-s}d\nu(w)\\ &= \frac{k-1}{4\pi}\det(g)^{-k/2}\sum_{\alpha\in T^p_x}\int_{-h_x/2}^{h_x/2}dx\int_0^\delta\frac{y^{k-s-2}}{|y-i\mu_\alpha|^k}dy\\ &\leqslant \frac{k-1}{4\pi}\det(g)^{-k/2}h_x\sum_{\alpha\in T^p_x}\delta\cdot\sup_{y\in[0,\delta]}\frac{y^{k-s-2}}{|y-i\mu_\alpha|^k}\\ &= \frac{k-1}{4\pi}\det(g)^{-k/2}h_x\delta^{k-s-1}\sum_{\alpha\in T^p_x}|\mu_\alpha|^{-k}.\end{aligned}$$

Since $g\in \mathrm{GL}_2^+(\mathbb{Q})$ we may choose $N\in\mathbb{N}$ such that Ng has integer entries. Then NB_α is an integer, so $B_\alpha\in 1/N\cdot(\mathbb{Z}\backslash\{0\})$. Clearly $|\mu_\alpha|^{-k} = 2^k\det(g)^{k/2}|B_\alpha|^{-k}$, and $B_\alpha = B_\beta$ for distinct $\alpha,\beta\in T^p_x$ if and only if $\lambda_\alpha = -\lambda_\beta$. Hence

$$\sum_{\alpha\in T^p_x}|\mu_\alpha|^{-k}\leqslant 2^{k+1}\det(g)^{k/2}\sum_{m\in\mathbb{Z}\backslash\{0\}}\left|\frac{m}{N}\right|^{-k}<\infty.$$

Therefore the whole sum we started with is finite, and we get using Dominated Convergence Theorem

$$\sum_{\alpha\in T^p_x}\int_{\Gamma(\alpha)\backslash(\mathcal{H}\backslash U_x)}\kappa(z,\alpha)d\nu(z) = \lim_{s\searrow 0}\sum_{\alpha\in T^p_x}\int_{\Gamma(\alpha)\backslash(\mathcal{H}\backslash U_x)}\tilde{\kappa}(z,\alpha,s)d\nu(z).$$

Hence

$$\lim_{s\searrow 0}\sum_{\alpha\in T^p//\Gamma}\int_{\Gamma(\alpha)\backslash\mathcal{H}}\kappa(z,\alpha,s)d\nu(z) \;=\; \sum_{x\in C(\Gamma)}\lim_{s\searrow 0}\sum_{\alpha\in T^p_x}\int_{\Gamma(\alpha)\backslash\mathcal{H}}\tilde{\kappa}(z,\alpha,s)d\nu(z).$$

Now we can compute the integral. As above we get for fixed $x \in C(\Gamma)$ and $\alpha \in T^p_x$

$$\begin{aligned}\int_{\Gamma(\alpha)\backslash\mathcal{H}}\tilde{\kappa}(z,\alpha,s)d\nu(z) &= \int_{(\sigma_x^{-1}\Gamma_x\sigma_x)\backslash\mathcal{H}}\kappa(\sigma_x w,\alpha)\operatorname{Im}(w)^{-s}d\nu(w)\\ &= \frac{k-1}{4\pi}\lambda_\alpha^{-k}h_x\int_0^\infty\frac{y^{k-s-2}}{(y-i\mu_\alpha)^k}dy.\end{aligned}$$

Substituting $y = \mu_\alpha t$ yields

$$\int_0^\infty\frac{y^{k-s-2}}{(y-i\mu_\alpha)^k}dy = \int_0^{\pm\infty}\frac{(\mu_\alpha t)^{k-s-2}}{(\mu_\alpha t-i\mu_\alpha)^k}(\mu_\alpha dt) = \frac{1}{\mu_\alpha^{1+s}}\int_0^{\pm\infty}\frac{(it)^{k-s-2}i^{2+s}}{(it+1)^k}dt.$$

Note that the sign of ∞ is determined by the sign of μ_α. Next we will use the substitution $it = (1-u)/u$ which gives

$$\frac{i^{2+s}}{\mu_\alpha^{1+s}}\int_0^{\pm\infty}\frac{(it)^{k-s-2}}{(it+1)^k}dt = \frac{-i^s}{\mu_\alpha^{1+s}}\int_\gamma\frac{(1-u)^{k-s-2}u^k}{u^{k-s-2}}\frac{idu}{u^2} = \frac{-i^{1+s}}{\mu_\alpha^{1+s}}\int_\gamma u^s(1-u)^{k-s-2}du$$

where γ denotes the transformation of the straight line from 0 to $\pm\infty$ by $u = (1+it)^{-1}$. Thus $\gamma(0) = 1$ and $\gamma(1) = 0$. Since the integrand $u^s(1-u)^{k-2-s}$ is holomorphic for small $s > 0$, we can replace γ by any curve γ' which has the same endpoints as γ. Let $\gamma'\colon r \mapsto 1-r$, $r \in [0,1]$, then

$$\int_\gamma u^s(1-u)^{k-s-2}du = \int_{\gamma'}u^s(1-u)^{k-s-2}du = -\int_{(\gamma')^{-1}}u^s(1-u)^{k-s-2}du$$

where $(\gamma')^{-1}$ denotes the inverse path of γ', so $(\gamma')^{-1}\colon r \mapsto r$, $r \in [0,1]$. Hence

$$-\int_\gamma u^s(1-u)^{k-s-2}du = \int_0^1 u^s(1-u)^{k-s-2}du = B(s+1,k-s-1)$$

where $B(a,b)$ is the beta function as introduced in Section 3.2. (Compare Section 2.1 in [BW10].) As stated before it satisfies the identity $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ where $\Gamma(z)$ denotes the gamma function. Therefore we finally have for some fixed $x \in C(\Gamma)$

$$\begin{aligned}&\lim_{s\searrow 0}\sum_{\alpha\in T^p_x}\int_{\Gamma(\alpha)\backslash\mathcal{H}}\tilde{\kappa}(z,\alpha,s)d\nu(z)\\ &= \frac{k-1}{4\pi}h_x\lim_{s\searrow 0}\sum_{\alpha\in T^p_x}\lambda_\alpha^{-k}\,\frac{i^{1+s}}{\mu_\alpha^{1+s}}\,\frac{\Gamma(s+1)\Gamma(k-s-1)}{\Gamma(k)}\\ &= \frac{k-1}{4\pi}h_x\lim_{s\searrow 0}\left[\frac{\Gamma(s+1)\Gamma(k-s-1)}{\Gamma(k)}\sum_{\alpha\in T^p_x}\operatorname{sign}(\lambda_\alpha)^{-k}\det(g)^{-k/2}\left(\frac{2i\lambda_\alpha}{B_\alpha}\right)^{1+s}\right]\\ &= \frac{h_x\det(g)^{-k/2}}{2\pi}\lim_{s\searrow 0}\sum_{\alpha\in T^p_x}\operatorname{sign}(\lambda_\alpha)^k\left(\frac{i\lambda_\alpha}{B_\alpha}\right)^{1+s}.\end{aligned}$$

Here we used that $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$ where $0! = 1$, and that Γ is continuous (even holomorphic) on $\{z \in \mathbb{C} \colon \operatorname{Re}(z) > 0\}$. (This is shown in Theorem 2.1.1 on page 19 in [BW10].) Hence we have for the sum over all parabolic elements

$$\begin{aligned}
&\lim_{s \searrow 0} \sum_{\alpha \in T^p//\Gamma} \int_{\Gamma(\alpha)\backslash\mathcal{H}} \kappa(z,\alpha,s)d\nu(z) \\
&= \sum_{x \in C(\Gamma)} \frac{h_x \det(g)^{-k/2}}{2\pi} \lim_{s \searrow 0} \sum_{\alpha \in T^p_x} \operatorname{sign}(\lambda_\alpha)^k \left(\frac{i\lambda_\alpha}{B_\alpha}\right)^{1+s} \\
&= \frac{\det(g)^{-k/2}}{2\pi} \lim_{s \searrow 0} \sum_{\alpha \in T^p//\Gamma} \operatorname{sign}(\lambda_\alpha)^k \left(\frac{ih_\alpha\lambda_\alpha}{B_\alpha}\right)^{1+s}
\end{aligned}$$

where h_α denotes the width of the cusp $[x_\alpha]$ for Γ with x_α being the unique fixed point of $\alpha \in T^p$.

As in the previous subsections we finish with a summarising lemma.

Lemma 4.2.5. *For $\alpha \in T^p$ with unique fixed point $x \in \mathbb{Q} \cup \{\infty\}$ we have*

$$\sigma^{-1}\alpha\sigma = \begin{pmatrix} \lambda_\alpha & B_\alpha \\ 0 & \lambda_\alpha \end{pmatrix}$$

where $\sigma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\sigma\infty = x$ and λ_α is the only eigenvalue of α. Further, we write h_α for the width of the cusp corresponding to the fixed point x of α. We then have

$$\begin{aligned}
&\frac{\det(g)^{k-1}}{|Z(\Gamma)|} \lim_{s \searrow 0} \sum_{\alpha \in T^p//\Gamma} \int_{\Gamma(\alpha)\backslash\mathcal{H}} \kappa(z,\alpha,s)d\nu(z) \\
&\qquad = \frac{\det(g)^{k/2-1}}{2\pi|Z(\Gamma)|} \lim_{s \searrow 0} \sum_{\alpha \in T^p//\Gamma} \operatorname{sign}(\lambda_\alpha)^k \left(\frac{ih_\alpha\lambda_\alpha}{B_\alpha}\right)^{1+s}.
\end{aligned}$$

In particular, T^p is empty if $\det(g)$ *does not have a rational square root.*

4.3 The final trace formula

Combining the formulae from Lemma 4.2.1 to Lemma 4.2.5 with Theorem 4.1.13 yields:

Theorem 4.3.1. *Let $T = \Gamma g \Gamma$ with Γ being a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and g being an element of $\mathrm{GL}_2^+(\mathbb{Q})$. Then*

$$\operatorname{Tr}(T \circlearrowleft S_k(\Gamma)) = t_s + t_e + t_h + t_p$$

with

$$
\begin{aligned}
t_s &= \frac{k-1}{24}\det(g)^{k/2-1}\,[\mathrm{SL}_2(\mathbb{Z}):\Gamma]\sum_{\alpha\in Z(T)}\mathrm{sign}(\lambda_\alpha)^k,\\
t_e &= \sum_{\alpha\in T^e//\Gamma}\frac{1}{|\Gamma(\alpha)|}\,\frac{\lambda_\alpha^{k-1}}{\overline{\lambda_\alpha}-\lambda_\alpha},\\
t_h &= \frac{-1}{|Z(\Gamma)|}\sum_{\alpha\in T^{h_2}//\Gamma}\frac{\mathrm{sign}(\lambda_{\alpha,1})^k\cdot\min\{|\lambda_{\alpha,1}|,|\lambda_{\alpha,2}|\}^{k-1}}{|\lambda_{\alpha,2}-\lambda_{\alpha,1}|},\\
t_p &= \frac{\det(g)^{k/2-1}}{2\pi|Z(\Gamma)|}\lim_{s\searrow 0}\sum_{\alpha\in T^p//\Gamma}\mathrm{sign}(\lambda_\alpha)^k\left(\frac{ih_\alpha\lambda_\alpha}{B_\alpha}\right)^{1+s}
\end{aligned}
$$

where we use the following notation:

- *For $\alpha\in Z(T)$, λ_α denotes the eigenvalue of α.*
- *For $\alpha\in T^e$ we choose λ_α such that $\sigma\alpha\sigma^{-1}=\left(\begin{smallmatrix}\lambda_\alpha & 0\\ 0 & \overline{\lambda_\alpha}\end{smallmatrix}\right)$ where $\sigma=\left(\begin{smallmatrix}1 & -z\\ 1 & -\overline{z}\end{smallmatrix}\right)$ and z is the unique fixed point of α in $\mathcal{H}$.*
- *For $\alpha\in T^{h_2}$, $\lambda_{\alpha,1}$ and $\lambda_{\alpha,2}$ denote the distinct eigenvalues of α.*
- *For $\alpha\in T^p$ we choose λ_α and B_α such that $\sigma^{-1}\alpha\sigma=\left(\begin{smallmatrix}\lambda_\alpha & B_\alpha\\ 0 & \lambda_\alpha\end{smallmatrix}\right)$ where $\sigma\in\mathrm{SL}_2(\mathbb{Z})$ such that $\sigma\infty=x$ and x is the unique fixed point of α. Further, we write h_α for the width of the cusp corresponding to the fixed point x of α.*

The following lemmata simplify the elliptic and parabolic terms in many situations. They are both combined in Theorem 6.4.10 in [Miy06]. We only give a proof for the first lemma since this is the one we will need in the next chapter.

Lemma 4.3.2. *If there is $\omega\in\mathrm{GL}_2(\mathbb{R})$ with $\det(\omega)=-1$ and such that $\omega\alpha\omega^{-1}\in T$ for all $\alpha\in T$, then we have*

$$
t_e = -\frac{1}{2}\sum_{\alpha\in T^e//\Gamma}\frac{1}{|\Gamma(\alpha)|}\frac{\lambda_\alpha^{k-1}-\overline{\lambda_\alpha}^{k-1}}{\lambda_\alpha-\overline{\lambda_\alpha}}
$$

in the situation of Theorem 4.3.1.

This is an improvement since the expression $(\lambda_\alpha^{k-1}-\overline{\lambda_\alpha}^{k-1})/(\lambda_\alpha-\overline{\lambda_\alpha})$ is now independent of the choice of eigenvalue of α. Moreover, writing $\lambda_\alpha=re^{i\varphi}$ one can easily check that

$$
\frac{\lambda_\alpha^{k-1}-\overline{\lambda_\alpha}^{k-1}}{\lambda_\alpha-\overline{\lambda_\alpha}} = r^{k-2}\,\frac{\sin((k-1)\varphi)}{\sin(\varphi)}. \tag{4.3.1}
$$

Note that the right-hand side is real.

Proof. Let $\alpha \in T^e$ with fixed point $z \in \mathcal{H}$, and put $\sigma = \left(\begin{smallmatrix} 1 & -z \\ 1 & -\overline{z} \end{smallmatrix}\right)$. Then $\sigma\alpha\sigma^{-1} = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{smallmatrix}\right)$ for some $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Put $\beta = \omega\alpha\omega^{-1}$, then β is elliptic since α is, and $\beta \in T$ by assumption, so $\beta \in T^e$. The unique fixed points of β are $w = \omega\overline{z}$ and $w' = \omega z$. Clearly $w' = \overline{w}$ and $w \in \mathcal{H}$ since $\det(\omega)$ is negative. Put $\tau = \left(\begin{smallmatrix} 1 & -w \\ 1 & -\overline{w} \end{smallmatrix}\right)$, then $\tau\beta\tau^{-1} = \left(\begin{smallmatrix} \mu & 0 \\ 0 & \overline{\mu} \end{smallmatrix}\right)$. Since β is a conjugate of α their eigenvalues agree, so either $\mu = \lambda$ or $\mu = \overline{\lambda}$.

Suppose that $\mu = \lambda$. Then $\sigma\alpha\sigma^{-1} = \tau\beta\tau^{-1}$, so $\beta = (\tau^{-1}\sigma)\alpha(\tau^{-1}\sigma)^{-1}$. Consider

$$\tau^{-1}\sigma = \frac{1}{w - \overline{w}} \begin{pmatrix} w - \overline{w} & z\overline{w} - \overline{z}w \\ 0 & z - \overline{z} \end{pmatrix}.$$

Let $\omega = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. One can check that

$$z\overline{w} - \overline{z}w = \frac{ac|z|^2 + ad(z + \overline{z}) + bd}{|cz + d|^2} \, (z - \overline{z}).$$

Hence $\tau^{-1}\sigma$ has entries in $\mathbb{R}$ since $(z - \overline{z})/(w - \overline{w}) = \operatorname{Im}(z)/\operatorname{Im}(w)$ is real. Moreover, we have $\det(\tau^{-1}\sigma) = \operatorname{Im}(z)/\operatorname{Im}(w) > 0$ as $z, w \in \mathcal{H}$. Therefore α and β are conjugate by an element of $\mathrm{GL}_2^+(\mathbb{R})$ which contradicts Lemma 2.2.5 as $\beta = \omega\alpha\omega^{-1}$. Therefore we must have $\mu = \overline{\lambda}$, so $\tau\beta\tau^{-1} = \left(\begin{smallmatrix} \overline{\lambda} & 0 \\ 0 & \lambda \end{smallmatrix}\right)$.

Again by Lemma 2.2.5, α and β represent different elements in $T^e//\Gamma$. Further we have $|\Gamma(\alpha)| = |\Gamma(\beta)|$ since the map $\gamma \mapsto \omega\gamma\omega^{-1}$ gives a bijection $\Gamma(\alpha) \to \Gamma(\beta)$. Adding the terms in the sum over $T^e//\Gamma$ corresponding to α and β yields

$$\frac{1}{|\Gamma(\alpha)|} \frac{\lambda^{k-1}}{\overline{\lambda} - \lambda} + \frac{1}{|\Gamma(\beta)|} \frac{\overline{\lambda}^{k-1}}{\lambda - \overline{\lambda}} = \frac{-1}{|\Gamma(\alpha)|} \frac{\lambda^{k-1} - \overline{\lambda}^{k-1}}{\lambda - \overline{\lambda}}.$$

This proves the claimed formula for t_e. □

Lemma 4.3.3. *If there is $\omega \in \mathrm{GL}_2(\mathbb{R})$ with $\det(\omega) = -1$ and such that $\omega\alpha\omega^{-1} \in T$ for all $\alpha \in T$, then we have*

$$t_p = -\frac{\det(g)^{k/2-1}}{4|Z(\Gamma)|} \lim_{s \searrow 0} s \sum_{\alpha \in T^p//\Gamma} \operatorname{sign}(\lambda_\alpha)^k \left| \frac{h_\alpha \lambda_\alpha}{B_\alpha} \right|^{1+s}$$

in the situation of Theorem 4.3.1.

For a proof of this we refer to Theorem 6.4.10 on page 241 in [Miy06].

5 A trace formula for the Hecke operators T_p acting on $S_k(\Gamma_0(N))$

In this final chapter we present an explicit formula for the trace of the Hecke operator T_p acting on $S_K(\Gamma_0(N))$ as developed by H. Hijikata in [Hij74]. Though we will not prove this formula, we will discuss it and calculate two examples. We also mention that S. L. Ross II slightly simplified Hijikata's formula in [RI92] replacing some terms by tables such that the computation of the trace of a Hecke operator "essentially reduces [...] to looking up values in a table" as he writes in the abstract of the corresponding paper.

Miyake presents two trace formulae in Section 6.8 of [Miy06]. The first one is Theorem 6.8.4 on pages 262 - 264 which is defined for rather general groups Γ and looks still quite similar to Hijikata's formula. In the second formula on page 265 Miyake specialises to the group $\Gamma_0(N)$ with $N = pq^\nu$ for some odd primes p, q and some $\nu \in \mathbb{N}_0$, which results in another "ready to compute" formula.

We concentrate on Hijikata's formula since out of the four mentioned formulae it is probably the one closest to our formula as stated at the end of the previous chapter. Moreover, it is the most original formula, too, out of the mentioned four.

5.1 Motivating observations

We start with a simple application of Theorem 4.3.1:

Corollary 5.1.1. *Let $k \geqslant 4$ be even, $\Gamma = \Gamma_0(N)$ for some $N \in \mathbb{N}$, and $T = \Gamma\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)\Gamma$ for some prime p. Then*

$$\mathrm{Tr}(T \circlearrowleft S_k(\Gamma)) = t_e + t_h$$

with

$$\begin{aligned} t_e &= -\frac{1}{2} \sum_{\alpha \in T^e//\Gamma} \frac{1}{|\Gamma(\alpha)|} \frac{\lambda_\alpha^{k-1} - \overline{\lambda_\alpha}^{k-1}}{\lambda_\alpha - \overline{\lambda_\alpha}}, \\ t_h &= -\frac{1}{2} \sum_{\alpha \in T^{h_2}//\Gamma} \frac{\min\{|\lambda_{\alpha,1}|, |\lambda_{\alpha,2}|\}^{k-1}}{|\lambda_{\alpha,2} - \lambda_{\alpha,1}|} \end{aligned}$$

where λ_α denotes an eigenvalue of $\alpha \in T^e$, and $\lambda_{\alpha,1}$, $\lambda_{\alpha,2}$ denote the two distinct eigenvalues of $\alpha \in T^{h_2}$.

Proof. By Theorem 4.3.1 we have $\mathrm{Tr}(T \circlearrowleft S_k(\Gamma)) = t_s + t_e + t_h + t_p$. Since $\det(\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)) = p$ and p is a prime, which does not have a rational square root, we have $t_s = t_p = 0$ as remarked in Lemma 4.2.1 and Lemma 4.2.5. Further, we have $Z(\Gamma) = 2$ and k even, which gives the term t_h. Put $\omega = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and let $\alpha \in T$, so $\alpha = \gamma_1 \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \gamma_2$ for some $\gamma_1, \gamma_2 \in \Gamma$. Then

$$\omega\alpha\omega^{-1} = (\omega\gamma_1\omega)(\omega\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)\omega)(\omega\gamma_2\omega) = \gamma_1'\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)\gamma_2' \in T$$

since one can easily check that $\gamma_j' := \omega\gamma_j\omega \in \Gamma$, $j = 1, 2$. Therefore we can use Lemma 4.3.2 to get t_e, which proves the claimed formula. □

In the following we will rearrange the terms of the trace formula given by the previous corollary. This will on the one hand lead to a nicer statement, and on the other hand motivate the trace formula of H. Hijikata, which will be presented afterwards.

Let $k \geqslant 4$ be even, $\Gamma = \Gamma_0(N)$ for some $N \in \mathbb{N}$, and $T = \Gamma\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)\Gamma$ for some prime p as in the previous theorem. Note that the case k odd is trivial, since the only modular form of odd weight and level $\Gamma_0(N)$ is the zero-function.

Let $\alpha \in T$, then $\det(\alpha) = p$. Further let t be the trace of α. By definition we know that α is elliptic if and only if $t^2 < 4p$, and that α is hyperbolic if and only if $t^2 > 4p$. Recall that $\alpha \in T^{h_2}$ if and only if α has two distinct fixed points in $\mathbb{Q} \cup \{\infty\}$. This is the case if either α fixes ∞, so if α is of the form $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$, or if $\sqrt{t^2 - 4p}$ is rational. In the first case we get that $ad = p$, and thus either $a = 1$ and $d = p$, or the other way round. Hence we have $t^2 - 4p = (p+1)^2 - 4p = (p-1)^2$. In the second case $t^2 - 4p$ needs to be a square. Therefore we have shown that for $\alpha \in T$

$$\begin{aligned} \alpha \in T^e \quad &\iff \quad |\operatorname{Tr}(\alpha)| < 2\sqrt{p}, \\ \alpha \in T^{h_2} \quad &\iff \quad \operatorname{Tr}(\alpha)^2 - 4p = U^2 \text{ for some } U \in \mathbb{N}. \end{aligned}$$

Clearly there are only finitely many possible values for the trace of an element in T^e. Moreover, let t_0 be the largest positive integer such that $t_0^2 - 4p \leqslant (t_0 - 1)^2$. Then one can easily see that $2\sqrt{p} < |\operatorname{Tr}(\alpha)| \leqslant t_0$ for any $\alpha \in T^{h_2}$. Hence the trace of an element in T^{h_2} is bounded as well.

Next we consider eigenvalues. Let $\alpha \in T$ with trace $t \in \mathbb{Z}$. The eigenvalues of α are given by the zeros of the polynomial $\Phi_t(x) = x^2 - tx + p$. Hence they are uniquely determined by the trace of α. We denote the zeros of $\Phi_t(x)$ by $\lambda_1(t)$ and $\lambda_2(t)$.

We may now rewrite the formula given by Corollary 5.1.1. Keeping notation as before we get

$$\operatorname{Tr}(T \circlearrowleft S_k(\Gamma)) = -\frac{1}{2}\left[\sum_{\substack{t \in \mathbb{Z}, \\ |t| < 2\sqrt{p}}} \frac{(\lambda_1(t))^{k-1} - (\lambda_2(t))^{k-1}}{\lambda_1(t) - \lambda_2(t)} \sum_{\substack{\alpha \in T^e//\Gamma \\ \operatorname{Tr}(\alpha) = t}} \frac{1}{|\Gamma(\alpha)|} + \sum_{\substack{t \in \mathbb{Z}, \\ t^2 - 4p = U^2 \\ \text{for some } U \in \mathbb{N}}} \frac{\min\{|\lambda_1(t)|, |\lambda_2(t)|\}^{k-1}}{|\lambda_1(t) - \lambda_2(t)|} \sum_{\substack{\alpha \in T^{h_2}//\Gamma \\ \operatorname{Tr}(\alpha) = t}} 1\right]. \qquad (5.1.1)$$

Define

$$a(t) = \begin{cases} \left((\lambda_1(t))^{k-1} - (\lambda_2(t))^{k-1}\right)(\lambda_1(t) - \lambda_2(t))^{-1} & , |t| < 2\sqrt{p}, \\ \min\{|\lambda_1(t)|, |\lambda_2(t)|\}^{k-1} |\lambda_1(t) - \lambda_2(t)|^{-1} & , |t| > 2\sqrt{p}, \end{cases}$$

and

$$B(t) = \begin{cases} B_1(t) & , |t| < 2\sqrt{p}, \\ B_2(t) & , |t| > 2\sqrt{p}, \end{cases}$$

where

$$B_1(t) = \sum_{\substack{\alpha \in T^e//\Gamma \\ \mathrm{Tr}(\alpha)=t}} \frac{1}{|\Gamma(\alpha)|}, \quad B_2(t) = \sum_{\substack{\alpha \in T^{h_2}//\Gamma \\ \mathrm{Tr}(\alpha)=t}} 1.$$

Then we may write (5.1.1) as

$$\mathrm{Tr}(T \circlearrowleft S_k(\Gamma)) = -\frac{1}{2}\sum_t a(t)B(t) \tag{5.1.2}$$

where the sum runs over all integers $t \in \mathbb{Z}$ such that either $|t| < 2\sqrt{p}$, or $t^2 - 4p$ is a positive square. In particular, the sum is finite as mentioned earlier.

The difficult part is now the evaluation of B_1 and B_2 in terms of the given operator T_p, the given level $\Gamma_0(N)$ and the current trace t. As the corresponding studies go beyond the scope of this thesis, we only quote and explain the trace formula given in the paper by H. Hijikata. Since it is stated in a more general context as we are working in, we will adjust it to our situation.

5.2 Hijikata's trace formula for $\Gamma_0(N)$

Consider Theorem 0.1 on page 57 of [Hij74]. We may use the following simplifications:

- Since we only consider operators T_n for n being prime, N and n are coprime if and only if n does not divide N.
- The second term of the formula vanishes as we assume $k \geqslant 4$. Moreover, n is never a square since n is prime. Hence the third term of the formula vanishes as well, and we are left with the first one.
- We only consider $\Gamma = \Gamma_0(N)$ itself. Therefore we may take $M = 1$, let $\mathfrak{h}$ be the trivial group and assume χ to be the trivial character that maps everything to 1.
- Again since n is prime, $s^2 - 4n$ will never be 0, so the parabolic case (p) does not happen and might therefore be removed.
- We do not have the factors $n^{1-k/2}$ in the definition of $a(s)$ as we defined the general action of Hecke operators slightly differently. Further, $\mathrm{sign}(x)^k = 1$ in the definition of $a(s)$ since we assume k to be even.

- Finally, $\mathfrak{h}$ is given by a trivial direct product, and hence we can use the slightly simplified definition of $c(s,f)$. (Note that $\chi(x) = \chi(y) = 1$ for any x, y as χ is trivial.)

Theorem (Hijikata's trace formula for $\Gamma_0(N)$)**.** *Let $k \geqslant 4$ be even, $\Gamma = \Gamma_0(N)$ for some $N \in \mathbb{N}$, and $T = \Gamma\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)\Gamma$ for some prime p not dividing N. Then*

$$\operatorname{Tr}(T \circlearrowleft S_k(\Gamma)) = -\frac{1}{2}\sum_t a(t) \sum_{f\in\mathbb{N},\, f|U_t} b(t,f) \prod_{\substack{q \text{ prime} \\ q|N}} c(t,f,q)$$

where the following notation is used:

- *Put $D(t) := t^2 - 4p$. The sum $\sum_t$ runs over all integers $t \in \mathbb{Z}$ such that either $D(t)$ is a positive square, or $D(t)$ is negative. For every such t we choose $U_t \in \mathbb{N}$ and if $D(t) < 0$ some negative squarefree integer m_t such that $D(t)$ is of one of the following forms:*
  (1) $D(t) = U_t^2$
  (2) $D(t) = U_t^2 m_t$ *with* $m_t = 1 \textit{ mod } 4$
  (3) $D(t) = U_t^2 4 m_t$ *with* $m_t = 2, 3 \textit{ mod } 4$

  In the following we say that t is of type (h) if $D(t)$ is of the form in (1), and t is of type (e) if $D(t)$ is of the form in (2) or (3).

- *Put $\Phi_t(x) = x^2 - tx + p$, and let λ_1, λ_2 be the solutions of $\Phi_t(x) = 0$. We define*

$$a(t) = \begin{cases} \min\{|\lambda_1|,|\lambda_2|\}^{k-1} |\lambda_1 - \lambda_2|^{-1} & , t \text{ of type } (h), \\ \left(\lambda_1^{k-1} - \lambda_2^{k-1}\right)(\lambda_1 - \lambda_2)^{-1} & , t \text{ of type } (e). \end{cases}$$

- *We define*

$$b(t,f) = \begin{cases} \varphi(U_t/f) & , t \text{ of type } (h), \\ h\left(D(t)/f^2\right)/w\left(D(t)/f^2\right) & , t \text{ of type } (e), \end{cases}$$

  where $\varphi(n)$ denotes Euler's totient function, so $\varphi(n)$ is the order of the unit group of the ring $\mathbb{Z}/n\mathbb{Z}$. Further, $h(d)$ denotes the class number of the order of the number field $\mathbb{Q}(\sqrt{d}\,)$ with discriminant d, and $w(d)$ denotes $1/2$ of the order of the unit group of this order.

- *Fix $t \in \mathbb{Z}$ and $f \in \mathbb{N}$ dividing U_t, and let q be a prime diving N. Further, let ν be the order of q dividing N, so $\nu \in \mathbb{N}_0$ such that q^ν divides N but $q^{\nu+1}$ does not, and let μ be the order of q dividing f. Put*

$$\tilde{A} = \left\{n \in \mathbb{Z}\colon \Phi_t(n) = 0 \textit{ mod } q^{\nu+2\mu},\ 2n = t \textit{ mod } q^\mu\right\},$$

  and if $D(t)/f^2 = 0$ mod q put

$$\tilde{B} = \left\{n \in \tilde{A}\colon \Phi_t(n) = 0 \textit{ mod } q^{\nu+2\mu+1}\right\}.$$

Then

$$c(t,f,q) = \begin{cases} |A| & , D(t)/f^2 \neq 0 \ mod\ q, \\ |A| + |B| & , D(t)/f^2 = 0 \ mod\ q, \end{cases}$$

where A and B are complete sets of representatives for $\tilde{A}$ and $\tilde{B}$ mod $q^{\nu+\mu}$, respectively.

In the following we will use Hijikata's formula to compute some traces of Hecke operators for specific N and p. These examples give detailed explanations for all the terms appearing in the presented formula. We start by making some general observations on Hijikata's trace formula which will save us some work while computing examples.

(1) Put for $t \in \mathbb{Z}$ such that either $D(t)$ is negative or $D(t)$ is a positive square

$$\mathcal{A}(t) := a(t) \sum_{f \in \mathbb{N},\, f | U_t} b(t,f) \prod_{\substack{q \text{ prime} \\ q|N}} c(t,f,q).$$

We claim $\mathcal{A}(t) = \mathcal{A}(-t)$. First note that $D(t) = D(-t)$, so if $t \in \mathbb{Z}$ is a valid value for the first sum in the trace formula then $-t$ is as well. Further, t and $-t$ are obviously of the same type, and $U_t = U_{-t}$. Thus f takes the same values for t and $-t$, and $b(t,f) = b(-t,f)$. We claim $a(t) = a(-t)$. To see this let λ_1, λ_2 be the solutions of $\Phi_t(x) = 0$, then $x^2 - tx + p = (x - \lambda_1)(x - \lambda_2)$ and thus

$$\Phi_{-t}(x) = x^2 - (-t)x + p = (x + \lambda_1)(x + \lambda_2).$$

So $-\lambda_1$, $-\lambda_2$ are the solutions of $\Phi_{-t}(x) = 0$. Hence we clearly have $a(t) = a(-t)$ if t is of type (h). If t is of type (e) we see

$$a(-t) = \frac{(-\lambda_1)^{k-1} - (-\lambda_2)^{k-1}}{(-\lambda_1) - (-\lambda_2)} = \frac{(-1)^{k-1}}{-1} \cdot \frac{\lambda_1^{k-1} - \lambda_2^{k-1}}{\lambda_1 - \lambda_2} = a(t)$$

since k is even. It remains to show that $c(t,f,q) = c(-t,f,q)$. Fix $f \in \mathbb{N}$ dividing U_t and a prime q dividing N. Then $\Phi_t(n) = \Phi_{-t}(-n)$, so for $m \in \mathbb{N}_0$

$$\Phi_t(n) = 0 \text{ mod } q^m \quad \Leftrightarrow \quad \Phi_{-t}(-n) = 0 \text{ mod } q^m.$$

Further we clearly have $2n = t$ mod q^m if and only if $2(-n) = -t$ mod q^m for $m \in \mathbb{N}_0$. This proves the claim, so $\mathcal{A}(t) = \mathcal{A}(-t)$ for all valid $t \in \mathbb{Z}$.

(2) Let λ_1, λ_2 be the solutions of $\Phi_t(x) = x^2 - tx + p = 0$. Then $\lambda_1 + \lambda_2 = t$ and $\lambda_1\lambda_2 = p$. Put $a_k(t) = (\lambda_1^{k-1} - \lambda_2^{k-1})(\lambda_1 - \lambda_2)^{-1}$. Then

$$\begin{aligned} a_k(t) &= \frac{(\lambda_1 + \lambda_2) \cdot \left(\lambda_1^{k-2} - \lambda_2^{k-2}\right) - \lambda_1\lambda_2 \cdot \left(\lambda_1^{k-3} - \lambda_2^{k-3}\right)}{\lambda_1 - \lambda_2} \\ &= t \cdot a_{k-1}(t) - p \cdot a_{k-2}(t). \end{aligned}$$

Note that $a_2(t) = 1$ and $a_3(t) = \lambda_1 + \lambda_2 = t$. Hence we may use the above recurrence formula to write down expressions for $a_k(t)$ for fixed integers k, namely

$$a_4(t) = t^2 - p, \quad a_5(t) = t^3 - 2pt, \quad a_6(t) = t^4 - 3pt^2 + p^2, \quad \ldots$$

Continuing we get for example

$$\begin{aligned} a_{24}(t) &= t^{22} - 21pt^{20} + 190p^2t^{18} - 969p^3t^{16} + 3060p^4t^{14} - 6188p^5t^{12} \\ &\quad + 8008p^6t^{10} - 6435p^7t^8 + 3003p^8t^6 - 715p^9t^4 + 66p^{10}t^2 - p^{11}. \end{aligned} \tag{5.2.1}$$

Though this expression looks fairly messy it well be useful in the first example. More important, the presented concept of expressing $a_k(t)$ in terms of $a_{k-1}(t)$ and $a_{k-2}(t)$ yields that $a_k(t)$ is an integer for every $t \in \mathbb{Z}$ since p is.

Example 5.2.1. We start with a very basic example whose result we can check afterwards by a direct computation. Let Γ be the full modular group $\mathrm{SL}_2(\mathbb{Z})$ and $k = 24$. We want to compute the trace of the Hecke operator $T_2 = \Gamma\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)\Gamma$ acting on $S_{24}(\Gamma)$. In terms of Hijikata's formula we therefore have $N = 1$ and $p = 2$. So $D(t) = t^2 - 8$ and the sum $\sum_t$ runs over all integers $|t| \leqslant 3$, since $D(\pm 4) = 8$ is not a square and $D(\pm 5) = 17$ is already greater than $(5-1)^2$. Further, we note that the product $\prod_q$ is empty since $N = 1$ does not have any prime divisors, so $\prod_q c(t, f, q) = 1$.

Next we consider the different possible values for t case by case. By the above observations it sufficies to consider non-negative values of t.

- Let $t = 0$. Then $D(0) = 1^2 \cdot 4 \cdot (-2)$, so $U_0 = 1$ and t is of type (e). Further, we have $\Phi_0(x) = x^2 + 2$, so $\lambda_{1,2} = \pm i\sqrt{2}$, and thus

$$a(0) = \frac{\lambda_1^{23} - \lambda_2^{23}}{\lambda_1 - \lambda_2} = -2^{11}.$$

  The sum $\sum_f$ only takes the value $f = 1$ since $U_0 = 1$. To determine $b(0, 1)$ let K be the number field $\mathbb{Q}(\sqrt{-8}\,) = \mathbb{Q}(\sqrt{-2}\,)$. We are looking for an order in K with discriminant -8. By Theorem 2.4.3 this order is the ring of integers $\mathcal{O}_K$ itself. Hence we have

$$b(0, 1) = \frac{h(-8)}{w(-8)} = \frac{h(\mathcal{O}_K)}{|U(\mathcal{O}_K)|/2} = \frac{1}{1} = 1$$

  by Proposition 2.4.4 and Table 2.4.6.

- Let $t = 1$. Then $D(1) = 1^2 \cdot (-7)$, so $U_1 = 1$ and t is of type (e). We have $\Phi_1(x) = x^2 - x + 2$, so $\lambda_{1,2} = (1 \pm i\sqrt{7}\,)/2$, and

$$a(1) = \frac{\lambda_1^{23} - \lambda_2^{23}}{\lambda_1 - \lambda_2}.$$

  This is rather hard to simplify. Recall that we expressed $a(t) = a_{24}(t)$ as a polynomial in t in equation (5.2.1). Using this with $p = 2$ we may compute $a(1) = 967$.

For the second sum we only have to consider $f = 1$ as before. To determine $b(1,1)$ let K be the number field $\mathbb{Q}(\sqrt{-7})$, then $\mathcal{O}_K$ is an order in K with discriminant -7 and thus

$$b(1,1) = \frac{h(-7)}{w(-7)} = \frac{h(\mathcal{O}_K)}{|U(\mathcal{O}_K)|/2} = \frac{1}{1} = 1$$

by Proposition 2.4.4 and Table 2.4.6.

- Let $t = 2$. Then $D(2) = 1^2 \cdot 4 \cdot (-1)$, so $U_2 = 1$ and t is still of type (e). We have $\Phi_2(x) = x^2 - 2x + 2$, so $\lambda_{1,2} = 1 \pm i = re^{\pm i\varphi}$ with $r = \sqrt{2}$ and $\varphi = \pi/4$. Hence we get using equation (4.3.1) that

$$a(2) = r^{22} \cdot \frac{\sin(23\varphi)}{\sin(\varphi)} = 2^{11} \cdot \frac{\sin(-\pi/4)}{\sin(\pi/4)} = -2^{11}.$$

  Clearly equation (5.2.1) would have given the same result.

  As before we only have $f = 1$ for the second sum. To determine $b(2,1)$ let K be $\mathbb{Q}(\sqrt{-4}) = \mathbb{Q}(\sqrt{-1})$, then $\mathcal{O}_K$ is an order in K with discriminant -4 and thus

$$b(2,1) = \frac{h(-4)}{w(-4)} = \frac{h(\mathcal{O}_K)}{|U(\mathcal{O}_K)|/2} = \frac{1}{2}$$

  by Proposition 2.4.4 and Table 2.4.6.

- Let $t = 3$. Then $D(3) = 1^2$, so $U_3 = 1$ and t is of type (h). Further we have $\Phi_3(x) = x^2 - 3x + 2$, so $\lambda_1 = 1$, $\lambda_2 = 2$ and thus

$$a(3) = \frac{\min\{|\lambda_1|, |\lambda_2|\}^{k-1}}{|\lambda_1 - \lambda_2|} = 1.$$

  As before $f = 1$, and hence $b(3,1) = \varphi(1) = 1$.

Combining these results we get

$$\begin{aligned}\mathrm{Tr}(T_2 \circlearrowleft S_{24}(\mathrm{SL}_2(\mathbb{Z}))) &= -\frac{1}{2}\Big[a(0)b(0,1) + 2a(1)b(1,1) + 2a(2)b(2,1) + 2a(3)b(3,1)\Big] \\ &= 2^{10} - 967 + 2^{10} - 1 \\ &= 1080.\end{aligned}$$

We quickly check this result: Let E_k denote the normalised Eisenstein series of weight k, so $E_k = 1/2 \cdot G_{k,\mathrm{SL}_2(\mathbb{Z}),\infty}$, where $G_{k,\mathrm{SL}_2(\mathbb{Z}),\infty}$ is defined as in Subsection 2.1.3. One can easily check that a basis of $S_{24}(\mathrm{SL}_2(\mathbb{Z}))$ is given by $f_1 = E_4^3\Delta$ and $f_2 = \Delta^2$ where $\Delta = (E_4^3 - E_6^2)/1782$. (Note that E_k agrees with the function G_k defined in (4.1.4) and (4.1.5) on the bottom of page 99 in [Miy06], and thus Δ is the function defined in (4.1.14) two pages afterwards.) One can compute

$$T_2(f_1) = 696f_1 + 20736000f_2 \quad \text{and} \quad T_2(f_1) = f_1 + 384f_2.$$

Hence the linear operator T_2 is given by the matrix

$$A = \begin{pmatrix} 696 & 1 \\ 20736000 & 384 \end{pmatrix}$$

with respect to the basis $\{f_1, f_2\}$, and thus we have

$$\mathrm{Tr}\left(T_2 \circlearrowleft S_{24}(\mathrm{SL}_2(\mathbb{Z}))\right) = \mathrm{Tr}(A) = 1080$$

as expected.

Finally, we also give the trace of T_2 acting on $S_k(\mathrm{SL}_2(\mathbb{Z}))$ for general k. Since $b(t, f)$ does not depend on k we can use the corresponding values computed above, so

$$\mathrm{Tr}(T_2 \circlearrowleft S_k(\mathrm{SL}_2(\mathbb{Z}))) = -\frac{1}{2}\Big[a(0) + 2a(1) + a(2) + 2a(3)\Big].$$

Also the eigenvalues of $\Phi_t(x)$ do not depend on k, so we directly see

$$\begin{aligned}&\mathrm{Tr}(T_2 \circlearrowleft S_k(\mathrm{SL}_2(\mathbb{Z}))) \\ &\quad = (-2)^{k/2-2} - \frac{\left(1+i\sqrt{7}\right)^{k-1} - \left(1-i\sqrt{7}\right)^{k-1}}{2^{k-1}\cdot i\sqrt{7}} - \frac{(1+i)^{k-1} - (1-i)^{k-1}}{4i} - 1.\end{aligned}$$

The previous example was simple in two ways: First we did not have to consider terms of the form $c(t, f, q)$ since $N = 1$, and secondly we did not have to work with orders of number fields other than the ring of integers itself. In the following example we will have to deal with these cases. However, we will not argue as detailed as before, since the basic considerations will still be the same.

Example 5.2.2. Let $\Gamma = \Gamma_0(4)$. We want to compute the trace of the Hecke operator $T_3 = \Gamma\left(\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right)\Gamma$ acting on $S_k(\Gamma)$ for some even integer $k \geqslant 4$. In terms of Hijikata's formula we have $N = 4$ and $p = 3$, so $D(t) = t^2 - 12$ and one can easily check that the sum $\sum_t$ runs over all integers $|t| \leqslant 4$. As in the first example we consider all valid and non-negative values of t case by case:

- Let $t = 0$. Then $D(0) = 2^2 \cdot (-3)$, so $U_0 = 2$ and t is of type (e). Further, we have $\Phi_0(x) = x^2 + 3$, so $\lambda_{1,2} = \pm i\sqrt{3}$ and thus

$$a(0) = \frac{\lambda_1^{k-1} - \lambda_2^{k-1}}{\lambda_1 - \lambda_2} = (-3)^{(k-2)/2}.$$

Since $U_0 = 2$ we have to consider $f = 1$ and $f = 2$ for the second sum. Let $f = 1$, then $D(0)/f^2 = -12$. Let $K = \mathbb{Q}(\sqrt{-12}\,) = \mathbb{Q}(\sqrt{-3}\,)$. We are looking for an order in K with discriminant -12. Recall that $\Delta(\mathcal{O}) = n^2 \cdot (-3)$ for an arbitrary order $\mathcal{O} = \mathbb{Z} + n(1 + \sqrt{-3}\,)/2 \cdot \mathbb{Z}$ in K as remarked in Subsection 2.4.5. Hence we want $n = 2$, so $\mathcal{O} = \mathbb{Z} + \sqrt{-3}\,\mathbb{Z}$. Using Theorem 2.4.7 we get

$$h(-12) = h(\mathcal{O}) = \frac{2h(\mathcal{O}_K)}{[U(\mathcal{O}_K) : U(\mathcal{O})]}\left(1 - \frac{L(-3, 2)}{2}\right).$$

By the observations following the mentioned theorem we have $L(-3,2) = -1$, $|U(\mathcal{O})| = 2$ and $[U(\mathcal{O}_K) : U(\mathcal{O})] = 3$, so $h(-12) = 1$ and $w(-12) = 1$, and thus $b(0,1) = 1$. Next we consider the product $\prod_q c(0,1,q)$. The only prime dividing 4 is $q = 2$. Thus the order of q dividing N is $\nu = 2$, and the order of q dividing f is $\mu = 0$. Hence

$$\tilde{A} = \left\{n \in \mathbb{Z} \colon n^2 + 3 = 0 \bmod 4,\ 2n = 0 \bmod 1\right\}.$$

The term mod 1 is redundant, and we easily see $\tilde{A} = \{n \in \mathbb{Z} \colon n \text{ odd}\}$. A set of representatives of $\tilde{A}$ mod 4 is given by $A = \{\pm 1\}$. Since $D(0)/f^2 = -12$ is even we also have to consider

$$\tilde{B} = \left\{n \in \tilde{A} \colon n^2 + 3 = 0 \bmod 8\right\}.$$

One can check that $\tilde{B} = \varnothing$, so $B = \varnothing$ and thus $c(0,1,2) = |A| + |B| = 2$.

It remains to consider the case $f = 2$. We have $D(0)/f^2 = -3$, so $K = \mathbb{Q}(\sqrt{-3}\,)$ as before, but this time we directly see

$$b(0,2) = \frac{h(-3)}{w(-3)} = \frac{h(\mathcal{O}_K)}{|U(\mathcal{O}_K)|/2} = \frac{1}{3}$$

since $\mathcal{O}_K$ has discriminant -3. Consider $\prod_q c(0,2,q)$. As before $q = 2$ and $\nu = 2$, but the order of q dividing f is now $\mu = 1$. Hence

$$\tilde{A} = \left\{n \in \mathbb{Z} \colon n^2 + 3 = 0 \bmod 16,\ 2n = 0 \bmod 2\right\}.$$

Again the term $2n = 0 \bmod 2$ is redundant, and one can check $\tilde{A} = \varnothing$. The set $\tilde{B}$ is not relevant as $D(0)/f^2 = -3$ is not even. So $c(0,2,2) = 0$.

Therefore the 0-term in the trace formula is given by

$$\mathcal{A}(0) := a(0) \cdot [b(0,1)c(0,1,2) + b(0,2)c(0,2,2)] = 2 \cdot (-3)^{(k-2)/2}.$$

- Let $t = 1$. Then $D(1) = 1^2 \cdot (-11)$, so $U_1 = 1$ and t is of type (e). This time we start backwards, so by computing all relevant values $c(1,f,q)$. Clearly $f = 1$ since $U_1 = 1$ and $q = 2$. The order of q dividing N is still $\nu = 2$, and the order of q dividing f is $\mu = 0$. Hence

$$\tilde{A} = \left\{n \in \mathbb{Z} \colon n^2 - n + 3 = 0 \bmod 4,\ 2n = 1 \bmod 1\right\}.$$

  One can check that $\tilde{A} = \varnothing$, so $c(1,1,2) = 0$. Therefore the whole ± 1-term in the trace formula vanishes.

- Let $t = 2$. Then $D(2) = 1^2 \cdot 4 \cdot (-2)$, so $U_2 = 1$ and t is of type (e). Again we start by computing relevant values $c(1,f,q)$. We have $f = 1$ since $U_2 = 1$ and $q = 2$, so as before $\nu = 2$ and $\mu = 0$ and thus

$$\tilde{A} = \left\{n \in \mathbb{Z} \colon n^2 - 2n + 3 = 0 \bmod 4,\ 2n = 2 \bmod 1\right\}.$$

  Again one can check $\tilde{A} = \varnothing$, so $c(2,1,2) = 0$. Therefore the ± 2-term in the trace formula vanishes as well.

- Let $t = 3$. Then $D(3) = 1^2 \cdot (-3)$, so $U_3 = 1$ and t is of type (e). As in the previous cases we only have $f = 1$, $q = 2$, and one can easily check that $c(3, 1, 2) = 0$. Hence the ± 3-term in the trace formula vanishes, too.

- Let $t = 4$. Then $D(4) = 2^2$, so $U_4 = 2$ and t is of type (h). Further, we have $\Phi_4(x) = x^2 - 4x + 3$, so $\lambda_1 = 1$, $\lambda_2 = 3$ and thus

$$a(4) = \frac{\min\{|\lambda_1|, |\lambda_2|\}^{k-1}}{|\lambda_1 - \lambda_2|} = \frac{1}{2}.$$

Since $U_4 = 2$ the second sum runs over $f = 1, 2$. Let $f = 1$, then $b(4, 1) = \varphi(2) = 1$. Let $q = 2$. The order of q dividing N is $\nu = 2$ as before, and the order of q dividing f is $\mu = 0$. Hence

$$\tilde{A} = \left\{n \in \mathbb{Z} \colon n^2 - 4n + 3 = 0 \bmod 4,\ 2n = 4 \bmod 1\right\}.$$

One can check $\tilde{A} = \{n \in \mathbb{Z} \colon n \text{ odd}\}$. A set of representatives of $\tilde{A}$ mod 4 is given by $\{\pm 1\}$. Since $D(4)/f^2$ is even we have to consider

$$\tilde{B} = \left\{n \in \tilde{A} \colon n^2 - 4n + 3 = 0 \bmod 8\right\},$$

and one can check $\tilde{B} = \tilde{A}$. Hence we have $c(4, 1, 2) = |A| + |A| = 4$. Let now $f = 2$, then $b(4, 2) = \varphi(1) = 1$. Let $q = 2$, $\nu = 2$ as before. The order of q dividing f is $\mu = 1$. Hence

$$\tilde{A} = \left\{n \in \mathbb{Z} \colon n^2 - 4n + 3 = 0 \bmod 16,\ 2n = 4 \bmod 2\right\}.$$

One can check that $\tilde{A} = \{n \in \mathbb{Z} \colon n = 1, 3 \bmod 8\}$, and a set of representatives for $\tilde{A}$ mod 8 is given by $A = \{1, 3\}$. Since $D(4)/f^2$ is odd, we do not consider $\tilde{B}$, and thus we have $c(4, 2, 2) = |A| = 2$.

Therefore the ± 4-term in the trace formula is given by

$$\mathcal{A}(\pm 4) := a(4) \cdot \left[b(4, 1)c(4, 1, 2) + b(4, 2)c(4, 2, 2)\right] = 3.$$

Combining all of these results we finally see

$$\operatorname{Tr}(T_3 \circlearrowleft S_k(\Gamma_0(4))) = -\frac{1}{2}\Big[\mathcal{A}(0) + 2\mathcal{A}(4)\Big] = -(-3)^{k/2-1} - 3.$$

As one can check in the table given on page 296 in [Miy06] the space $S_6(\Gamma_0(4))$ is 1-dimensional. By the above formula the trace of T_3 acting on $S_6(\Gamma_0(4))$ is -12. Hence we have $T_3(f) = -12f$ for any $f \in S_6(\Gamma_0(4))$ since the matrix representation of T_3 with respect to any basis in $S_6(\Gamma_0(4))$ is simply (-12). (This can also be verified using Sage.)

As a final example we compute the eigenvectors of the Hecke operator T_2 acting on $S_{24}(\mathrm{SL}_2(\mathbb{Z}))$, which we already considered in the first example. The presented method can be generalised following the argumentation on page 266, 267 in [Miy06] to arbitrary T_p operators acting on any space $S_k(\Gamma_0(N))$ such that p does not divide N. (For such cases we would need a more general trace formula.)

Example 5.2.3. Recall that we have shown in Example 5.2.1 that

$$\mathrm{Tr}\left(T_2 \circlearrowleft S_{24}(\mathrm{SL}_2(\mathbb{Z}))\right) = 1080.$$

Further, we now that the space $S_{24}(\mathrm{SL}_2(\mathbb{Z}))$ is 2-dimensional. Let μ_1, μ_2 be the two eigenvalues of T_2, then $\mu_1 + \mu_2 = 1080$. So we need a second equation to determine the eigenvalues. For this purpose we use part (2) of Lemma 4.5.7 on page 140 in [Miy06]. We have $p = 2$ and $N = 1$. Note that $T(2,2) = \mathrm{SL}_2(\mathbb{Z})\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$, so for any cusp form f in $S_{24}(\mathrm{SL}_2(\mathbb{Z}))$ we have

$$T(2,2)(f) = f|_k \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2^{k-2} f.$$

Therefore the mentioned lemma gives

$$(T_2)^2 = T_4 + 2^{k-1} T_1$$

where T_1 denotes the identity operator. Clearly the trace of T_1 acting on $S_{24}(\mathrm{SL}_2(\mathbb{Z}))$ is given by 2 since the corresponding space is 2-dimensional. Further, one can check that $\mathrm{Tr}(T^m) = \sum_{j=1}^n \nu^m$ for a general linear operator T with eigenvalues $\nu_1, \ldots, \nu_n$. Hence

$$\mathrm{Tr}\left((T_2)^2 \circlearrowleft S_{24}(\mathrm{SL}_2(\mathbb{Z}))\right) = \mu_1^2 + \mu_2^2.$$

Therefore it remains to compute the trace of T_4 acting on $\mathrm{SL}_2(\mathbb{Z})$. Unfortunately, the trace formula presented at the beginning of this section is only valid for T_p operators with p being a prime. Using the more general formula given in Hijikata's paper [Hij74] one can compute

$$\mathrm{Tr}\left(T_4 \circlearrowleft S_{24}(\mathrm{SL}_2(\mathbb{Z}))\right) = 25326656.$$

We are not giving any details here as the purpose of this example is the computation of the eigenvalues of T_2. Using that $\mu_2 = 1080 - \mu_1$ we see

$$\begin{aligned} 25326656 + 2^{23} \cdot 2 &= \mathrm{Tr}\left(\left(T_4 + 2^{23} T_1\right) \circlearrowleft S_{24}(\mathrm{SL}_2(\mathbb{Z}))\right) \\ &= \mathrm{Tr}\left((T_2)^2 \circlearrowleft S_{24}(\mathrm{SL}_2(\mathbb{Z}))\right) \\ &= \mu_1^2 + (1080 - \mu_1)^2, \end{aligned}$$

so $0 = \mu_1^2 - 1080\mu_1 - 20468736$. We note that the same equation holds if we replace μ_1 by μ_2. Therefore the eigenvalues μ_1, μ_2 are the roots of $\Psi(x) = x^2 - 1080x - 20468736$. These are given by

$$\mu_{1,2} = 540 \pm \sqrt{540^2 + 20468736} = 540 \pm 12\sqrt{144169}.$$

We may check this result using the matrix representation of T_2 developed at the end of Example 5.2.1.

6 Summary and outlook

In this thesis we developed a trace formula for Hecke operators for modular groups following Section 6.1 to 6.4 of [Miy06]. We began by showing that the space of holomorphic functions on $\mathcal{H}$ being integrable in the sense that $\int_{\mathcal{H}} |f(z)|^2 \operatorname{Im}(z)^k d\nu(z) < \infty$ is a reproducing kernel Hilbert space with kernel

$$K_k = \frac{k-1}{4\pi}\left(\frac{z-\overline{w}}{2i}\right)^{-k}.$$

Afterwards we introduced a similar space for Γ-invariant functions with Γ being a modular group. We denoted this space by $H_k^2(\Gamma)$ and proved that it, too, is a reproducing kernel Hilbert space with kernel

$$K_k^\Gamma(z,w) = |Z(\Gamma)|^{-1}\textstyle\sum_{\gamma\in\Gamma}(K_k(\cdot,w)|_k\gamma)(z).$$

This is Theorem 3.4.5. The reason to consider these spaces is given by Theorem 3.3.3, which states that $H_k^2(\Gamma)$ and $S_k(\Gamma)$ agree as Hilbert spaces. Hence the space of cusp forms of weight k and level Γ is a reproducing kernel Hilbert space with kernel K_k^Γ. We used this fact in Section 3.5 to write down a first trace formula: For $k \geqslant 3$ the trace of an Hecke operator $T = \Gamma g \Gamma$ acting on $S_k(\Gamma)$ is given by

$$\operatorname{Tr}(T \circlearrowleft S_k(\Gamma)) = \frac{\det(g)^{k-1}}{|Z(\Gamma)|}\int_{\Gamma\backslash\mathcal{H}}\sum_{\alpha\in T} K_k(\alpha z, z) j(\alpha,z)^{-k}\operatorname{Im}(z)^k d\nu(z).$$

Chapter 4 is concerned with the simplification of this formula closely following Section 6.4 of [Miy06]. We started by interchanging summation and integration. This turned out to be quite challenging near cusps where we had to introduce an extra term to guarantee convergence. Moreover, we had to work with a fixed fundamental domain for Γ which was replaced by some appropriate quotient at the end of the section using the notation of conjugacy classes. Finally, we got

$$\begin{aligned}\operatorname{Tr}(T \circlearrowleft S_k(\Gamma)) \;=\; \frac{\det(g)^{k-1}}{|Z(\Gamma)|}\cdot\Bigg[&\sum_{\alpha\in T^1//\Gamma}\int_{\Gamma(\alpha)\backslash\mathcal{H}}\kappa(z,\alpha)d\nu(z)\\ &+\lim_{s\searrow 0}\sum_{\alpha\in T^2//\Gamma}\int_{\Gamma(\alpha)\backslash\mathcal{H}}\kappa(z,\alpha,s)d\nu(z)\Bigg].\end{aligned}$$

The notation used in this statement is explained in Theorem 4.1.13.

In the following section we calculated the integrals appearing in the above formula depending on the type of $\alpha \in T$, so whether α is scalar, elliptic, parabolic or hyperbolic. The corresponding calculations were very technical, though most of them yielded fairly nice results. In particular, it turned out that a large subset of the α's in T, namely the hyperbolic elements with fixed points in $\mathbb{R}\backslash\mathbb{Q}$, do not contribute anything to the trace. Finally, we summarised our results in a simplified trace formula (Theorem 4.3.1) at the end of the chapter.

Eventually we presented Hijikata's trace formula in Chapter 5 motivating it with the formula presented at the end of the previous chapter and explaining which terms would have to be studied further to get Hijikata's formula. In the end we computed some explicit traces of Hecke operators using the trace formula of Hijikata.

It remains to comment on possible further studies. The most intuitive extension would be to close the gap between the trace formula presented at the end of Chapter 4 and Hijikata's formula. Therefore one might want to follow Section 6.5 to 6.8 in [Miy06]. Further, one could generalise the concepts introduced in this work to more general groups. Miyake shows in his book that everything works exactly the same if we use Fuchsian groups of the first kind (see Section 1.5 in [Miy06]) possibly in combination with characters of these groups of finite order instead of modular groups. Moreover, one could consider trace formulae for generalised spaces of modular forms. An example of such are spaces of *Siegel modular forms* which are functions holomorphic on the space of symmetric $n \times n$ matrices with positive definite imaginary part that are invariant under the action of some symplectic group. These symplectic groups generalise modular groups.

It would also be interesting to study applications of trace formulae. For example one could use the formula to compute eigenvalues of T_p operators as presented on pages 266, 267 in Miyake's book, or to compute dimensions of spaces of cusp forms. The latter can be done by computing the trace of the trivial operator acting on the corresponding space of cusp forms. However, in both cases it would be useful to either have a look at implementations of trace formulae for computer algebra systems (see for example [Ste12]), or to consider implementing a trace formula on ones own. In this context one might also compare different approaches to compute traces (or eigenvectors) of Hecke operators with respect to their runtimes.

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