Geometric criteria for $C^{1,\alpha}$-rectifiability

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1. Introduction

Aim: Obtain criteria for rectifiability of a set with $C^{1,\alpha}$ maps: (i) in terms of approximate tangent paraboloids (over a fixed plane for every point), (ii) in terms of approximate tangent cylinders (allowed to rotate at different scales for the same point) and (iii) in terms of beta numbers.

Rectifiable sets are useful in many geometric variational problems, where a minimizing sequence of smooth surfaces can converge to a singular limit. Given $k \in \{1, \dots, n\}$, a set $E \subset \mathbb{R}^n$ is **$\mathcal{H}^k$-rectifiable** if there are countably many Lipschitz maps $f_i \colon \mathbb{R}^k \to \mathbb{R}^n$ such that

$$E \subset E_0 \cup \bigcup_i f_i(\mathbb{R}^k), \qquad \mathcal{H}^k(E_0) = 0.$$

Given $\alpha \in [0,1]$, E is $C^{1,\alpha}$ $\mathcal{H}^k$-rectifiable if the maps f_i are of class $C^{1,\alpha}$. Here $\mathcal{H}^k$ is the k-dimensional Hausdorff measure. We will use the **Hausdorff upper and lower k-densities**

$$\Theta^{*k}(E,x) = \limsup_{r\to 0}\,(2r)^{-k}\mathcal{H}^k(E \cap B(x,r)), \quad \Theta^k_*(E,x) = \liminf_{r\to 0}\,(2r)^{-k}\mathcal{H}^k(E \cap B(x,r)).$$

$C^{1,\alpha}$-rectifiability arises naturally in many different contexts: singular sets of convex functions [1], of sets with positive reach [7] and of the distance function [5] are $C^{1,1}$-rectifiable.

2. The first criterion

Throughout we fix $\alpha \in (0,1]$ and $k \in \{1, \dots, n\}$. Given a plane V in $\mathbb{R}^n$ denote by P_V the orthogonal projection on V. Given a point $x \in \mathbb{R}^n$, a k-plane V and $\lambda \geq 0$ we define the **α-paraboloid**

$$Q_\alpha(x,V,\lambda) = \left\{ y \in \mathbb{R}^n \colon |P_{V^\perp}(y-x)| \leq \lambda |P_V(y-x)|^{1+\alpha} \right\}.$$

Theorem 1 ($C^{1,\alpha}$-rectifiability and tangent paraboloids [3, 8])

A subset $E \subset \mathbb{R}^n$ with $\mathcal{H}^k(E) < \infty$ is $C^{1,\alpha}$ $\mathcal{H}^k$-rectifiable if and only if for $\mathcal{H}^k$-a.e. $x \in E$ there exist a k-plane V and $\lambda > 0$ such that

$$\lim_{r\to 0} \frac{1}{r^k}\mathcal{H}^k\big(E \cap B(x,r) \setminus Q_\alpha(x,V,\lambda)\big) = 0. \tag{1}$$

We call any paraboloid satisfying (1) an *approximate tangent paraboloid* for E at x. The case $\alpha = 0$ (i.e. when there is an approximate tangent *cone*) is a classical criterion of rectifiability [6, Chapter 15]. Note that $C^{1,0}$-rectifiable $\iff$ rectifiable.

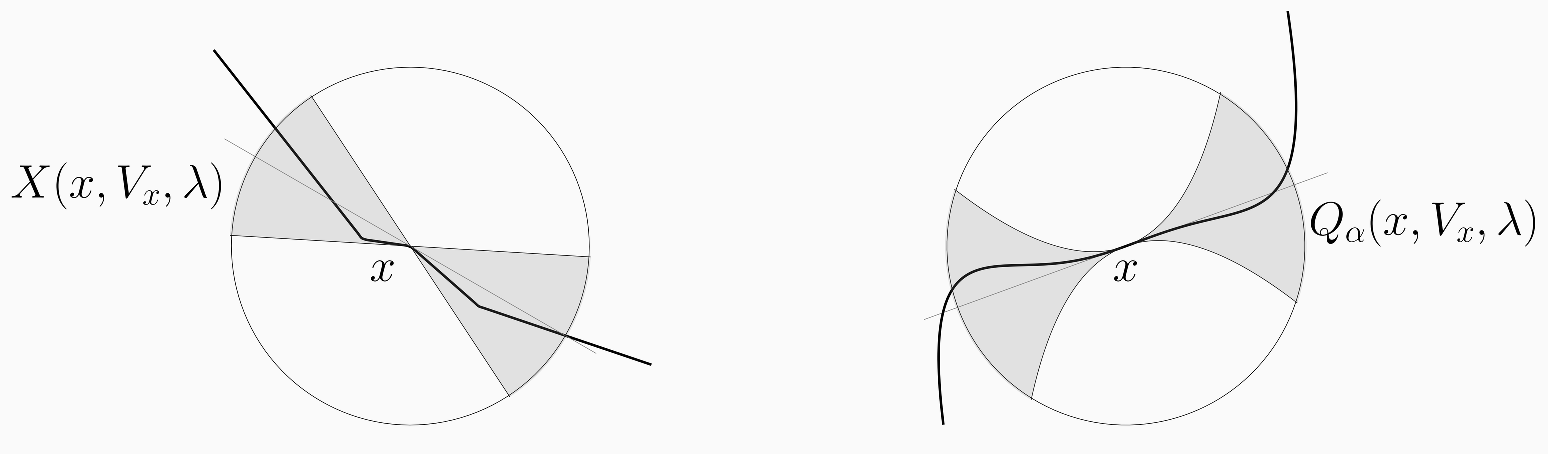

Figure 1. A set is $\mathcal{H}^k$-rectifiable if and only if at almost every point there is an approximate tangent cone (left). It is $C^{1,\alpha}$ $\mathcal{H}^k$-rectifiable if and only if at almost every point there is an approximate tangent α-paraboloid (right).

3. The second criterion

Given a point x, a k-plane V, a number $\lambda > 0$ and a radius $r > 0$ we define the cylinder

$$C^r_\alpha(x,V,\lambda) = \{ y \in \mathbb{R}^n \colon |P_{V^\perp}(y-x)| \leq \lambda r^{1+\alpha} \}.$$

Theorem 2 ($C^{1,\alpha}$-rectifiability from tangent cylinders [3])

Take a subset $E \subset \mathbb{R}^n$ with $\mathcal{H}^k(E) < \infty$ such that for $\mathcal{H}^k$-a.e. $x \in E$ the following hold:

$$\Theta^k_*(E,x) > 0 \tag{2}$$

and there exists $\lambda > 0$ and for every radius $r > 0$ there exists a k-plane $V_{x,r}$ such that

$$\limsup_{r\to 0} \frac{1}{r^k}\mathcal{H}^k\big(E \cap B(x,r) \setminus C^r_\alpha(x,V_{x,r},\lambda)\big) = 0. \tag{3}$$

Then E is $C^{1,\alpha}$ $\mathcal{H}^k$-rectifiable.

Remark The planes are allowed to depend not only on the point x but also on the scale r. However we have to require assumption (2) (which we would like to remove).

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[3] G. Del Nin, K. O. Idu: Geometric criteria for $C^{1,\alpha}$-rectifiability, arXiv:1909.10625 (2019)

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[8] M. Santilli: Rectifiability and approximate differentiability of higher order for sets, *Indiana Univ. Math. J.* 68(3), 1013–1046 (2019)

4. Sketch of proof of Theorem 1 for $k=1$, $n=2$

We prove just one implication, that if E satisfies (1) for $k=1$ then it is $C^{1,\alpha}$ $\mathcal{H}^1$-rectifiable.

1. Up to a standard countable decomposition argument we reduce to the case where E is contained in the graph of a Lipschitz function $f \colon E' \to \mathbb{R}$.
2. In terms of f assumption (1) becomes: for every $x' \in E'$ there exists a linear map $L_{x'} \colon \mathbb{R} \to \mathbb{R}$ such that for every point $y' \in E'$ we have $|f(y') - f(x') - L_{x'}(y'-x')| \leq \lambda |x'-y'|^{1+\alpha}$ $(*)$.
3. Interchanging the role of x' and y' and applying the triangle inequality, for every x', y' in E' we obtain that $\|L_{x'} - L_{y'}\| \leq 2\lambda |y'-x'|^\alpha$ $(**)$.
4. Since $(*)$ and $(**)$ hold we can apply Whitney's extension theorem to obtain that $f \colon E' \to \mathbb{R}$ extends to a $C^{1,\alpha}$ function $\tilde{f} \colon \mathbb{R} \to \mathbb{R}$. □

5. Sketch of proof of Theorem 2 for $k=1$, $n=2$

We want to reduce to the setting of Theorem 1. Fix a point x (w.l.o.g. $x=0$) where (2) and (3) hold. We look at dyadic scales $r_j = 2^{-j}$. Call ℓ_j the lines that satisfy (3) at scale r_j, θ_j the angle between ℓ_j and ℓ_{j+1}, and C_j the cylinder $C^r_\alpha(0,\ell_j,\lambda)$ (see Figure 2). We claim that $|\theta_j| \lesssim r_j^\alpha$. Indeed:

1. From (2) and (3) applied to scale r_j and r_{j+1} we have that
$$\mathcal{H}^1(E \cap B(x,r) \cap C_j \cap C_{j+1}) \geq \delta r_j \quad \text{for some } \delta > 0.$$
2. The diameter of the intersection $C_j \cap C_{j+1}$ is of order $d_j \approx r_j^{1+\alpha}/|\theta_j|$ (see Figure 2).
3. Every set with $\mathcal{H}^1(E) < \infty$ satisfies $\Theta^{*,1}(E,x) \leq 1$ for $\mathcal{H}^1$-a.e. $x \in E$, and thus for small enough radii $\mathcal{H}^1(E \cap B(x,r)) \leq 2r$. Applying this to $E \cap C_j \cap C_{j+1}$ we obtain
$$\delta r \leq \mathcal{H}^1(E \cap B(x,r_j) \cap C_j \cap C_{j+1}) \leq 2d_j \approx r_j^{1+\alpha}/|\theta_j| \quad \Longrightarrow \quad |\theta_j| \lesssim r_j^\alpha.$$

Therefore the sequence $(\ell_j)_{j\in\mathbb{N}}$ converges with rate r_j^α to some ℓ_∞ and as a consequence $C_j \subset C^r_\alpha(0,\ell_\infty,M'')$. We conclude applying Theorem 1. □

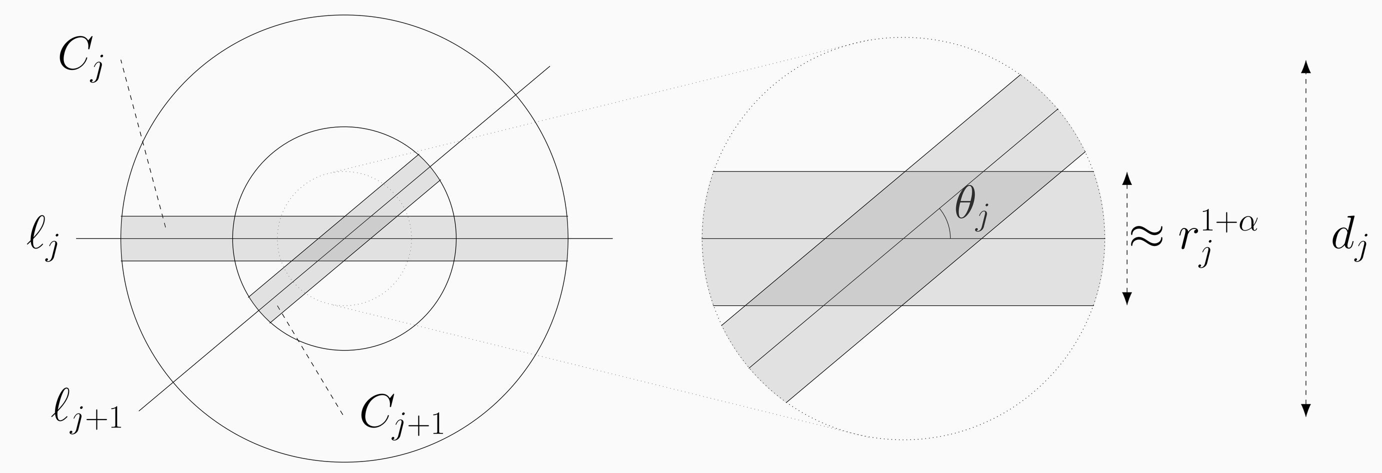

Figure 2. The diameter d_j of $C_j \cap C_{j+1}$ is of order $d_j \lesssim \frac{r_j^{1+\alpha}}{|\sin\theta_j|} \approx \frac{r_j^{1+\alpha}}{|\theta_j|}$. The lines ℓ_j can not tilt much from one scale to the smaller one, otherwise the measure concentrates too much on the intersection (the parallelogram), which is prevented by the upper density bound.

6. Beta numbers

Fix $1 \leq p < \infty$. Given $E \subset \mathbb{R}^n$ with $\mathcal{H}^k(E) < \infty$ define the **beta numbers**

$$\beta_p(x,r) = \inf_{V\ k\text{-plane}} \left(\frac{1}{r^k} \int_{E\cap B(x,r)} \left(\frac{\mathrm{dist}(y,V)}{r} \right)^p d\mathcal{H}^k(y) \right)^{1/p}. \tag{4}$$

The following is an important result due to Azzam and Tolsa:

Theorem (Rectifiability $\iff$ integrability of β numbers [2])

If $\mathcal{H}^k(E) < \infty$ then E is $\mathcal{H}^k$-rectifiable if and only if

$$\int_0^1 \beta_2(x,r)^2 \frac{dr}{r} < \infty \quad \text{for } \mathcal{H}^k\text{-a.e. } x \in E.$$

The following version for $C^{1,\alpha}$-rectifiability is due to Ghinassi:

Theorem ($C^{1,\alpha}$-rectifiability $\Longleftarrow$ higher integrability of β numbers [4])

If $\mathcal{H}^k(E) < \infty$, then E is $C^{1,\alpha}$ $\mathcal{H}^k$-rectifiable provided that

$$\int_0^1 \left(\frac{\beta_2(x,r)}{r^\alpha} \right)^2 \frac{dr}{r} < \infty \quad \text{for } \mathcal{H}^k\text{-a.e. } x \in E.$$

As a Corollary of Theorem 2 we obtain a weaker sufficient condition:

Theorem 3 ($C^{1,\alpha}$-rectifiability $\Longleftarrow$ boundedness of β numbers [3])

If $\mathcal{H}^k(E) < \infty$, then E is $C^{1,\alpha}$ $\mathcal{H}^k$-rectifiable provided that

$$\limsup_{r\to 0} \frac{\beta_p(x,r)}{r^\alpha} < \infty \quad \text{for } \mathcal{H}^k\text{-a.e. } x \in E \text{ and for some } p \in [1,\infty].$$

7. Possible future directions

- Criteria for $C^{m,\alpha}$-rectifiability for $m \geq 2$ in terms of the distance to polynomial surfaces.
- Criteria for $C^{1,\alpha}$-rectifiability of doubling measures μ without density assumptions.
- Criteria using weak notions of curvature such as Menger curvature.