A short introduction to the Gan–Gross–Prasad conjectures

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Abstract

The Gan–Gross–Prasad conjectures are a series of conjectures in the theory of automorphic representations. When given a linear algebraic group G and a subgroup of it H, a classical question to ask is how do the G-irreducible representations decompose when restricted to H. An answer to this question is called a branching law and in the case of the classical groups it is known that this restriction problem is multiplicity free. The GGP conjectures give an explicit description of this multiplicity.

Contents

1 From Hecke to Waldspurger

This section is following Chao Li's notes of Beuzart-Plessis' course at Columbia in Spring 2019. We have made very small alterations, since the theory was described beautifully there.

1.1 Hecke's central value formula

Our story begins with the classical work of Hecke, who described the integral representation of L-functions of modular forms. Let $f \in S_k(\Gamma(1))$ be a cusp form of level k and consider its Fourier expansion

$$f = \sum_{n \geq 1} a_n q^n.$$

Let

$$L(s,f) = \sum \frac{a_n}{n^s}, \quad \Lambda(s,f) = (2\pi)^{-s}\Gamma(s)L(s,f)$$

be its L-function and the completed L-function respectively. Hecke showed that the completed L-function is equal to the Mellin transform of the modular form

$$\Lambda(s,f)=\int_0^\infty f(iy)y^{s-1}dy,$$

and use it to show the analytic continuation and functional equation of $\Lambda(s,f)$. Evaluating at the center of functional equation $s=k/2$ (which lies outside the range of convergence of $\Lambda(s,f)$), we obtain a central value formula

$$\Lambda(k/2,f)=\int_0^\infty f(iy)y^{\frac{k-1}{2}}dy.$$

This is the identity which the Gan-Gross-Prasad conjectures aim to generalize. In order to pass to a more general setting, first we need to interpret Hecke's central value formula in the adelic setting. Consider the bijection

$$SO(2)(\mathbb{R})\backslash SL_2(\mathbb{R})\to\mathbb{H}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\mapsto\frac{ai+b}{ci+d}$$

which maps the torus $\left\{\left(t,t^{-1}\right),t\in\mathbb{R}^\times\right\}\subset SL_2(\mathbb{R})$ to the vertical line $\{iy\mid y>0\}\subset\mathbb{H}$. We can now consider $\Lambda(f,s)$ as an integral along the image of the split torus in $SL_2(\mathbb{R})$. To make this precise, recall that f gives rise to a vector $\phi_f\in\pi$, where π is the cuspidal automorphic representation of GL_2 attached to f. The representation π has an L-function $L(\pi,s)$, which equals (up to a linear change of variables) $\Lambda(f,s)$. By Hecke's work, we can consider the following integral representation of this L-function

$$L(\pi,s)=\int_{\mathbb{A}^\times/\mathbb{Q}^\times}\phi_f\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}|t|^{s-\frac{1}{2}}\ \mathrm{d}t.$$

Note that the extra $1/2$ comes from normalization of measures. The RHS is known as an automorphic period $\mathscr{P}_A\left(\phi_f\right)$ along the subgroup $A=\mathbb{G}_m\hookrightarrow G$ (a real number since f is a normalized eigenform).

Recall also that Rankin-Selberg expressed the Petersson inner product

$$\langle f,f\rangle=\int_{\Gamma(1)\backslash\mathcal{H}}|f(z)|^2y^k\frac{dxdy}{y^2}$$

in terms of the Rankin-Selberg L-function

$$\langle f,f\rangle=2\pi^{k-1}\operatorname{res}_{s=k}\Lambda(s,f\times f)$$

The RHS is equal to an adjoint L-value at the edge of critical strip

$$2^{-k}L(1,\pi,\mathrm{Ad}),$$

while the LHS is also equal to $\frac{\pi}{6}$ (again due to normalization of measures) times

$$\langle \phi_f, \phi_f \rangle = \int_{[G]} |\phi_f(g)|^2 \, d_{\mathrm{tam}} g.$$

Squaring Hecke's central value formula and dividing the Rankin-Selberg identity, we obtain

$$\frac{\mathscr{P}_A(\phi_f)^2}{\langle \phi_f, \phi_f \rangle} = 2^{k-2} \xi(2) \frac{L(1/2,\pi)^2}{L(1,\pi,\ \mathrm{Ad}\ )},$$

where $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completed Riemann zeta function. This new identity generalizes to any vector $\phi \in \pi$ of any cuspidal automorphic representation of $G(\mathbb{A})$ over any number field F. Let $\phi = \otimes_v \phi_v \in \pi = \otimes_v' \pi_v$. Then

$$\frac{|\mathscr{P}_A(\phi)|^2}{\langle \phi, \phi \rangle} = \frac{\zeta^S(2)}{2} \frac{L^S(1/2,\pi)^2}{\mathrm{res}_{s=1}\, \zeta^S(s) L^S(1,\pi,\mathrm{Ad})} \cdot \prod_{v \in S} \alpha_v(\phi_v, \phi_v),$$

for S a sufficiently large finite set of places including the archimedean places, where the local period

$$\alpha_v(\phi_v, \phi_v) = \frac{\int_{A(F_v)} \langle \pi_v(a_v)(\phi_v), \phi_v \rangle \, da_v}{\langle \phi_v, \phi_v \rangle_v}.$$

Here we choose the Tamagawa measures on $A(\mathbb{A})$ and $G(\mathbb{A})$, and the local measures are chosen such that

$$\prod_v da_v = d_{\mathrm{tam}\, a}$$

and the local period

$$\alpha_v(\phi_v, \phi_v) = \frac{\zeta_v(2)}{\zeta_v(1)} \frac{L(1/2, \pi_v)^2}{L(1, \pi_v, \mathrm{Ad})}$$

for almost all v 's. One can show directly that α_v is not identically zero and $L(s,\pi,\ \mathrm{Ad})$ has no pole or zero at $s = 1$, hence the new identity implies the equivalence that there exists $\phi \in \pi$ such that $\mathscr{P}_A(\phi) \neq 0 \iff L(1/2,\pi) \neq 0$.

1.2 Waldspurger's theorem

Waldspurger in [5] and [6] proved a remarkable generalization by replacing A with any nontrivial torus T in G. Such a torus is isomorphic to $\mathrm{Res}_{F'/F}\, \mathbb{G}_m/\mathbb{G}_m$ for a quadratic extension of number fields F'/F (and the embedding $A \hookrightarrow G$ is unique up to conjugation). Let $\eta : \mathbb{A}^\times/F^\times \to \{\pm 1\}$ be the quadratic character associated to F'/F by global class field theory.

Theorem 1.1 (Waldspurger)**.** *For every $\phi = \otimes_v \phi_v \in \pi$, we have*

$$\frac{|\mathscr{P}_T(\phi)|^2}{\langle \phi, \phi \rangle} = \frac{\zeta_F^S(2)}{4} \frac{L^S(1/2,\pi) L^S(1/2,\pi \times \eta)}{L^S(1,\eta) L^S(1,\pi,\ Ad\ )} \prod_{v \in S} \alpha_v(\phi_v, \phi_v)$$

where the local periods are defined similarly via integration over T_v instead of A_v.

Waldspurger's formula looks exactly the same as the split torus case, but it is much harder to prove.

This is due to the fact that there is no direct relation between the toric period $\mathscr{P}_T(\phi)$ with integral representations of $L(s,\pi)L(s,\pi\times\eta)$. Moreover, unlike the split torus case, the local periods can be identically zero. It turns out that

$$\alpha_v \neq 0 \Longleftrightarrow \operatorname{Hom}_{T_v}(\pi_v, \mathbb{C}) \neq 0.$$

This implies that

$$\mathscr{P}_T|_\pi \neq 0 \Longleftrightarrow L(1/2,\pi)L(1/2,\pi\times\eta) \neq 0 \text{ and } \operatorname{Hom}_{T_v}(\pi_v, \mathbb{C}) \neq 0 \text{ for all } v.$$

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2 The GGP conjectures

2.1 Basics on automorphic forms

In order to understand a more general version of Waldspurger's formula, we will need to understand what an automorphic form looks like on a more general algebraic group. This section is a basic introduction to the theory of automorphic forms.

Let F be a number field with local completions F_v for all places v of F. Let G be a connected reductive linear algebraic group over F. Fixing any faithful algebraic representation $\rho : G \hookrightarrow GL_n$ over F, one obtains a system of open compact subgroups $K_v = \rho^{-1}(GL_n(\mathcal{O}_v)) \subset G(F_v)$, for almost all v, where $\mathcal{O}_v$ is the ring of integers of F_v; for almost all v, K_v is hyperspecial. $G(\mathbb{A}) = \prod_v' G(F_v)$ be the adelic group, which is a restricted direct product of $G(F_v)$ relative to the family $\{K_v\}$ of open compact subgroups for almost all v; $[G] = G(F)\backslash G(\mathbb{A})$ be the automorphic quotient; the locally compact group $G(\mathbb{A})$ acts on $[G]$ by right translation and there is a $G(\mathbb{A})$-invariant measure (unique up to scaling).

Definition 2.1. *An automoprhic form on G is a function*

$$f : [G] \longrightarrow \mathbb{C}$$

such that

1. *f is smooth*
2. *f is right K_f-finite, where $K_f = \prod_{v<\infty} K_v$*
3. *f is of uniform moderate growth*
4. *f is $Z(\mathfrak{g})$-finite, where $\mathfrak{g}$ is the Lie algebra of G.*

We denote the space of automorphic forms on G by $\mathcal{A}(G)$.

Note that $G(\mathbb{A})$ acts on the vector space $\mathcal{A}(G)$ by right translation, giving it a $G(\mathbb{A})$-module structure.

Definition 2.2. *An automorphic representation* π *of* G *is an irreducible subquotient of the* $G(\mathbb{A})$*-module* $\mathcal{A}(G)$*. As an irreducible abstract representation of* $G(\mathbb{A}) = \prod'_v G(F_v)$*,* π *is of the form:*

$$\pi = \otimes'_v \pi_v,$$

a restricted tensor product of irreducible smooth representations π_v *of* $G(F_v)$*.*

Definition 2.3. *An automorphic form* f *on* G *is called a cusp form if, for any parabolic subgroup* $P = MN$ *of* G*, the* N*-constant term*

$$f_N(g) = \int_{N(F)\backslash N(\mathbb{A})} f(ng)dn$$

is zero as a function on $G(\mathbb{A})$*. We denote the space of cuspidal automorphic forms on* G *by* $\mathcal{A}_0(G) \subset \mathcal{A}(G)$*.*

It turns out that, if G is semisimple or has anisotropic center, a cusp form f is rapidly decreasing as a function on $[G]$ (like a Schwarz function) and hence is square-integrable on $[G]$, i.e

$$\int_{[G]} |f(g)|^2 dg < \infty.$$

Moreover, it is known that $\mathcal{A}_0(G)$ decomposes as a direct sum $\mathcal{A}_0(G) = \bigoplus_\pi m_0(\pi) \cdot \pi$ with finite multiplicities. An irreducible summand π of $\mathcal{A}_0(G)$ is called a cuspidal automorphic representation.

Let H be a subgroup of G. As we saw in the previous lecture, we are interested in studying the automorphic H-period integral

$$\begin{aligned} \mathscr{P}_H : & \mathcal{A}_0(G) \longrightarrow \mathbb{C} \\ & \phi \longrightarrow \int_{Z(\mathbb{A})\backslash[H]} \phi(h)dh. \end{aligned}$$

It is also natural to consider a twisted version. Let χ be a character of $Z(\mathbb{A})H(F)\backslash H(\mathbb{A})$. We define the automorphic (H, χ)-period integral $\mathscr{P}_{H,\chi}$ in a similar manner,

$$\mathscr{P}_{H,\chi}(\phi) = \int_{Z(\mathbb{A})\backslash[H]} \phi(h)\chi(h)dh.$$

Let π be a unitary cuspidal automorphic representation of $G(\mathbb{A})$. We consider the restriction of $\mathscr{P}_H$ to π, which defines an element in $\mathscr{P}_{H,\pi} \in \mathrm{Hom}_{H(\mathbb{A})}(\pi, \mathbb{C})$.

Definition 2.4. *A cuspidal automorphic representation* π *is (globally) distinguished by* H *if the linear functional* $\mathscr{P}_{H,\pi} \in \mathrm{Hom}_{H(\mathbb{A})}(\pi, \mathbb{C})$ *does not vanish, i.e., there exists some* $\phi \in \pi$ *such that* $\mathscr{P}_H(\phi) \neq 0$*. We say that* π_v *is (locally) distinguished by* $H(F_v)$ *if* $\mathrm{Hom}_{H(F_v)}(\pi_v, \mathbb{C}) \neq 0$*.*

If π is distinguished by H, then $\mathrm{Hom}_{H(\mathbb{A})}(\pi, \mathbb{C}) \neq 0$, and in particular, $\mathrm{Hom}_{H(F_v)}(\pi_v, \mathbb{C}) \neq 0$ for every place v. However, the converse statement is not always true, and this observation is the key part of the GGP conjectures.

It is expected that the automorphic period integral $\mathscr{P}_H$ behaves nicely only when the pair (H, G) satisfies certain nice properties, such as

1. the multiplicity-one property $\dim \mathrm{Hom}_{H(F_v)}(\pi_v, \mathbb{C}) \leq 1$ holds for all v, or the multiplicity can be described in a certain nice way, and

2. the locally distinguished representations can be characterized in terms of L-parameters.

A large class of (H, G) called spherical pairs are expected to have the above properties. If F is an algebraically closed field, we say that the pair (H, G) is spherical if a Borel subgroup B of G has an open dense orbit on G/H. Over a general field F, the pair (H, G) is said to be spherical if its base change to an algebraic closure of F is spherical. We then call H a spherical subgroup of G. Here are some examples of spherical pairs:

Example 2.5.

1. *The Whittaker pair (N, G), where G is quasi-split and N is a maximal nilpotent subgroup of G.*

2. *The pair $(G, G \times G)$, where we view G as a subgroup of $G \times G$ via its image under the diagonal embedding.*

3. *The Rankin–Selberg pair $(GL_{n-1}, GL_n \times GL_{n-1})$.*

4. *The GGP pairs $(SO_{n-1}, SO_n \times SO_{n-1})$ and $(U_{n-1}, U_n \times U_{n-1})$.*

2.2 Statement of the conjectures

We will now proceed to state the global Gan-Gross-Prasad conjectures in the orthogonal and Hermitian cases. We follow [1]. Consider F and G as in the previous section and let the subgroup H of G be reductive. For the number field F consider and extension F'/F such that $F' = F$ in the orthogonal case and F' a quadratic extension of F in the Hermitian case. Let W_n be a non-degenerate orthogonal space or Hermitian space with F'-dimension n. Let $W_{n-1} \subset W_n$ be a non-degenerate subspace of codimension one. Let G_i be $SO\left(W_i\right)$ or $U\left(W_i\right)$ for $i = n-1, n$. Let

$$G = G_{n-1} \times G_n, \quad H = G_{n-1}$$

where we view H as a subgroup of G via the diagonal embedding $\Delta : H \hookrightarrow G$. Note that the pair (H, G) is spherical.

Let $\pi = \pi_{n-1} \otimes \pi_n$ be a tempered cuspidal automorphic representation of $G(\mathbb{A})$. The central L-values of certain automorphic L-functions $L(s, \pi, R)$ show up in their conjecture, where R is a finite dimensional representation of the L-group ${}^LG$. We can describe the L-function as the Rankin–Selberg convolution of suitable automorphic representations on general linear groups.

For $i \in \{n-1, n\}$, let $\Pi_{i,F'}$ be the endoscopic functoriality transfer of π_i from G_i to suitable $GL_N\left(\mathbb{A}_{F'}\right)$. In the Hermitian case, this is the base change of π_i to $GL_i\left(\mathbb{A}_{F'}\right)$; and in the orthogonal case, this is the endoscopic transfer from $G_i(\mathbb{A})$ to $GL_i(\mathbb{A})$ (resp.$GL_{i-1}(\mathbb{A})$) if i is even (resp. odd). The L-function $L(s, \pi, R)$ can be defined as the Rankin–Selberg convolution L-function $L\left(s, \Pi_{n-1,F'} \times \Pi_{n,F'}\right)$.

We are ready to state the global Gan-Gross-Prasad conjecture.

Conjecture 2.6 (Gan–Gross–Prasad)**.** *Let π be a tempered cuspidal automorphic representation of $G(\mathbb{A})$. The following statements are equivalent.*

1. *The automorphic H-period integral does not vanish on π, i.e. $\mathscr{P}_H(\phi) \neq 0$ for some $\phi \in \pi$.*

2. *The space* $\mathrm{Hom}_{H(\mathrm{A})}(\pi, \mathbb{C}) \neq 0$ *and the central value* $L\left(\frac{1}{2}, \pi, R\right) \neq 0$.

There is a refinment of the conjecture, due to Ichino and Ikeda [2], which resembles Waldspurger's formula in this case. In order to state it, define the local periods:

$$\mathscr{P}_{H,v} : (\phi_v, \phi'_v) \mapsto \int_{H_v} \langle \pi_v(h_v)\phi_v, \phi'_v\rangle_v dh_v.$$

The Ichino–Ikeda refinement of the GGP conjecture links the square of the automorphic H-period with central L-values. In particular, for $\phi = \bigotimes_v \phi_v \in \pi$, the conjecture states:

$$\frac{|\mathscr{P}_H(\phi)|^2}{\langle \phi, \phi\rangle} = \frac{\Delta}{S_\pi} \frac{L^S(1/2, \mathrm{BC}(\pi))}{L^S(1, \pi, \mathrm{Ad})} \prod_{v \in S} \frac{\mathscr{P}_{H,v}(\phi_v, \phi_v)}{\langle \phi_v, \phi_v\rangle_v},$$

where Δ is a product of L-values attached to the quadratic character η, $\Delta = \prod_{i=1}^{n+1} L(i, \eta^i)$, and S_π is the centralizer of the Langlands parameter of π.

Due to the incredible work of many authors, including Beuzart-Plessis, Gordon, Xue, Yun, Zhang and others, the GGP conjecture and its Ichino–Ikeda refinement are now theorems in the unitary case.

Theorem 2.7. *Let* $G = U(W_{n-1}) \times U(W_n)$ *for Hermitian spaces* $W_{n-1} \subset W_n$ *over a quadratic extension* F' *of* F*. Let* π *be a tempered cuspidal automorphic representation of* $G(\mathbb{A})$*. Then Conjecture 2.6 and its Ichino–Ikeda refinement hold.*

3 An RTF approach to the GGP conjectures

In this final section, we will present a very brief introduction to the main ideas that led to the proof of the unitary GGP conjecture. In 1985, when Waldspurger first proved his theorem, he used a direct approach via the Shimura correspondance. A year later, Jacquet [3] published a proof that was using a different method, the trace formula. Since then, trace formuli have been extremely useful in the study of automorphic forms. In [4], Jacquet and Rallis proposed a relative trace formula (RTF) approach to the conjecture. It was indeed this recipe that led to the proof of the conjecture.

Recall that in section 1.1 we observed that even though Hecke's central value formula was easy to understand, it was much harder to find a connection between the non-trivial toral automprphic period integrals and the central L-value in Waldspurger's formula. A similar issue arises in the GGP setting. In particular, due to the work of Jacquet–Pietski-Shapiro–Shalika, we know that the Rankin-Selberg period satisfies:

$$\mathscr{P}_{GL_{n,K}}(\phi) = L^S(1/2, \pi_K) \prod_{v \in S} \mathscr{P}_{GL_{n,K_v}}(\phi_v, \phi_v)$$

for some local periods $\mathscr{P}_{GL_{n,K_v}}$. The rough idea now is that if we can relate the period on the left hand side with the unitary periods we care about, we would have a relation simlar to the conjecture's.

We will know proceed to explain what a relative trace formula is.

Fix a number field K and a linear reductive algebraic group G over K. We will study the space $L^2([G])$ with its $G(\mathbb{A}_K)$-module structure.

We consider the action of the Hecke algebra $C_c^\infty(G(\mathbb{A}_K))$ on $L^2([G])$. Take $f \in C_c^\infty(G(\mathbb{A}_K))$ and $\phi \in L^2([G])$ and define the integral operator

$$R(f)(\phi)(x) = \int_{G(\mathbb{A}_K)} f(y)\phi(xy)dy.$$

This operator satisfies

$$\begin{aligned} R(f)(\phi)(x) &= \int_{G(\mathbb{A}_K)} f(y)\phi(xy)dy = \int_{G(\mathbb{A}_K)} f(x^{-1}y)\phi(y)dy \\ &= \int_{[G]} \sum_{\gamma\in G(K)} f(x^{-1}\gamma y)\phi(y) \\ &= \int_{[G]} K_f(x,y)\phi(y) \end{aligned} \tag{1}$$

where $K_f(x,y) = \sum_{\gamma\in G(K)} f(x^{-1}\gamma y)$, the kernel of $R(f)$.

We will now make an oversimplifying assumption in order to be able to give an elementary treatment of the relative trace formula. We will assume that we can write our automorphic form ϕ as $\phi = \sum_\pi \sum_{\psi\in OB(\pi)} \langle \phi, \psi\rangle \psi$, where $\langle , \rangle$ is the Peterson inner product, the first sum is over irreducible automorphic representations of G and the second one is over an orthonormal basis of π. It is very crucial to note that this sum doesn't make sense unless we take care of the continuous spectrum of $L^2([G])$ (and not just the discrete part which appears here). However, assuming that this expansion makes sense we have:

$$\begin{aligned} R(f)(\phi)(x) &= \sum_\pi \sum_{\psi\in OB(\pi)} \langle \phi, \psi\rangle R(f)(\psi)(x) \\ &= \sum_\pi \sum_{\psi\in OB(\pi)} \langle \phi, \psi\rangle \pi(f)\psi(x) \\ &= \sum_\pi \sum_{\psi\in OB(\pi)} \int_{[G]} \phi(y)\overline{\psi(y)}\pi(f)\psi(x)dy \\ &= \int_{[G]} \sum_\pi \left(\sum_{\psi\in OB(\pi)} \pi(f)\psi(x)\overline{\psi(y)} \right) \phi(y)dy \\ &= \int_{[G]} \sum_\pi K_{f,\pi}(x,y)\phi(y), \end{aligned} \tag{2}$$

where we have set $K_{f,\pi}(x,y) = \sum_{\psi\in OB(\pi)} \pi(f)\psi(x)\overline{\psi(y)}$ and so we have obtained by (1) and (2) that $K_f(x,y) = \sum_\pi K_{f,\pi}(x,y)$ which is called the spectral expansion of the kernel.

We will now fix two G-subgroups H_1, H_2 and a character $\eta : [H_2] \to \mathbb{C}^\times$.

Define for $f \in C_c^\infty(G(\mathbb{A}_K))$ a distribution

$$\mathbb{I}(f) = \int_{[H_1]\times[H_2]} K_f\left(h_1, h_2\right) \eta\left(h_2\right) dh_1 dh_2.$$

Applying the spectral expansion of the kernel on this distribution we get

$$\begin{aligned}
\mathbb{I}(f) &= \int_{[H_1]\times[H_2]} \sum_{\pi} \sum_{\psi\in OB(\pi)} \pi(f)\psi(h_1)\overline{\psi(h_2)}\eta\left(h_2\right) dh_1 dh_2 \\
&= \sum_{\pi} \sum_{\psi\in OB(\pi)} \int_{[H_1]} \pi(f)\psi(h_1) dh_1 \int_{[H_2]} \eta(h_2)\overline{\psi(h_2)} dh_2 \\
&= \sum_{\pi} \sum_{\psi\in OB(\pi)} \mathscr{P}_{H_1}(\pi(f)\psi)\overline{\mathscr{P}_{H_2,\overline{\eta}}(\psi)}.
\end{aligned}$$

We can start understanding now why the spectral expansion is useful, since the automorphic period integrals we are interested in have appeared.

The distribution $\mathbb{I}(f)$ also has a different expansion, called the geometric expansion. It stems from the fact that we have an action of $H_1 \times H_2$ on G given by $(h_1, h_2) \cdot \gamma = h_1^{-1}\gamma h_2$. We will denote the stabilizer of an element $\gamma \in G$ by $(H_1, H_2)_\gamma$. We now have

$$\begin{aligned}
\mathbb{I}(f) &= \int_{[H_1]\times[H_2]} \left(\sum_{\gamma\in G(K)} f\left(h_1^{-1}\gamma h_2\right) \right) \eta\left(h_2\right) dh_1 dh_2 \\
&= \sum_{\gamma\in H_1(K)\backslash G(K)/H_2(K)} \int_{(H_1\times H_2)(\mathbb{A}_K)/(H_1\times H_2)_\gamma(K)} f\left(h_1^{-1}\gamma h_2\right) \eta\left(h_2\right) dh_1 dh_2 \\
&= \sum_{\gamma\in H_1(K)\backslash G(K)/H_2(K)} \operatorname{vol}\left(\left[(H_1\times H_2)_\gamma\right]\right) \int_{(H_1\times H_2)(\mathbb{A}_K)/(H_1\times H_2)_\gamma(\mathbb{A}_K)} f\left(h_1^{-1}\gamma h_2\right) \eta\left(h_2\right) dh_1 dh_2 \\
&= \sum_{\gamma\in H_1(K)\backslash G(K)/H_2(K)} \operatorname{vol}\left(\left[(H_1\times H_2)_\gamma\right]\right) O(\gamma, f),
\end{aligned}$$

where we have defined $O(\gamma, f) = \int_{(H_1\times H_2)(\mathbb{A}_K)/(H_1\times H_2)_\gamma(\mathbb{A}_K)} f\left(h_1^{-1}\gamma h_2\right) \eta\left(h_2\right) dh_1 dh_2$, the orbital integral of f at γ.

The relative trace formula (RTF) for (G, H_1, H_2) is

$$\sum_{\gamma\in H_1(K)\backslash G(K)/H_2(K)} \operatorname{vol}\left(\left[(H_1\times H_2)_\gamma\right]\right) O(\gamma, f) = \sum_{\pi} \sum_{\psi\in OB(\pi)} \mathscr{P}_{H_1}(\pi(f)\psi)\overline{\mathscr{P}_{H_2,\overline{\eta}}(\psi)}.$$

To prove the unitary GGP conjecture, we need to consider two different RTFs:

1. Set $H = U(W_n)\backslash G = U(W_n) \times U\left(W_{n+1}\right)/H$ and let η be trivial. Then, for a sufficiently nice test function $f \in C_c^\infty(G(\mathbb{A}))$ we have

$$\sum_{\phi\in\mathcal{A}_0([G])} \mathscr{P}_H(R(f)\phi)\overline{\mathscr{P}_H(\phi)} = \sum_{\delta\in H(F)\backslash G_{\mathrm{rs}}(F)/H(F)} O(\delta, f)$$

2. Set $H_1 := GL_{n,K} \hookrightarrow G' = GL_{n,K} \times GL_{n+1,K} \hookleftarrow H_2 = GL_{n,F} \times GL_{n+1,F}$ and

$$\eta := \left(\eta_{K/F} \circ \det\right)^{n+1} \otimes \left(\eta_{K/F} \circ \det\right)^{n} : [H_2] \to \{\pm 1\}.$$

Then, for a sufficiently nice test function $f' \in C_c^{\infty}(G'(\mathbb{A}))$ we have

$$\sum_{\phi \in \mathcal{A}_0([G'])} \mathscr{P}_{H_1}\left(R\left(f'\right)\phi\right) \overline{\mathscr{P}_{H_2,\eta}(\phi)} = \sum_{\gamma \in H_1(F)\backslash G'_{rs}(F)/H_2(F)} O\left(\gamma, f'\right)$$

Note that both sums in the RTFs run over regular semisimple elements, hence the notation G_{rs}, G'_{rs}. The RHS of (2) is related with specific L-values. In particular, $\mathscr{P}_{H_1}\left(R\left(f'\right)\phi\right)$ detects the vanishing of the Rankin–Selberg central L-value and $\mathscr{P}_{H_2,\eta}(\phi)$ detects the same for the image of base change. In order to prove GGP, one must match the orbits on the geometric sides of the two RTFs.

The recipe for matching the geometric sides uses the isomorphism

$$\bigsqcup_{V} H^V(k)\backslash G^V_{rs}(k)/H^V(k) \simeq H_1(k)\backslash G'_{rs}(k)/H_2(k),$$

where $k = F$ or F_v and V runs over isomorphism classes of n-dimensional Hermitian spaces. Noting that the orbital integrals are local, that is if $f = \prod_v f_v, f' = \prod_v f'_v$ then $O(\delta, f) = \prod_v O\left(\delta, f_v\right)$, $O\left(\gamma, f'\right) = \prod_v O\left(\gamma, f'_v\right)$, we aim to perform a local matching. We say that $f_v = \left(f_v^V\right) \in \bigoplus_V C_c^{\infty}\left(G^V\left(F_v\right)\right)$ matches $f'_v \in C_c^{\infty}\left(G'_v\right)\left(f_v \leftrightarrow f'_v\right)$ if

$$O\left(\delta, f_v^V\right) = \Omega_v(\gamma) O\left(\gamma, f'_v\right), \text{ for } \delta \in G^V_{rs}\left(F_v\right) \leftrightarrow \gamma \in G'_{rs}\left(F_v\right)$$

where $\left(\Omega_v\right)_v$ are local transfer factors such that $\prod_v \Omega_v(\gamma) = 1$ for $\gamma \in G'_{rs}(F)$.

To make this matching work, there are two key steps:

1. (Smooth transfer) For every f_v there exists f'_v st $f_v \leftrightarrow f'_v$ and conversely
2. (Fundamental lemma) For $\mathbf{1}_{G(O_y)} \leftrightarrow \mathbf{1}_{G'(O_y)}$ for almost all places v.

Smooth transfer was shown by Zhang in the p-adic case and by Xue in the Archimedean case. The Fundamental lemma was proved by Beuzart-Plessis, Gordon and Yun.

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