INVITATION TO DESCENT

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Consider the following general situation: given a category $\mathcal{C}$ and an object X, we have some notion of an an 'object over X' given by the comma category $(\mathcal{C} \downarrow X)$. A map $X \xrightarrow{f} Y$ in $\mathcal{C}$ induces a functor

$$f^* : (\mathcal{C} \downarrow Y) \to (\mathcal{C} \downarrow X)$$

given by pulling back along f, i.e. sending $E \to Y$ to $E \times_Y X \to X$. One very general formulation of **descent** is then: given a map $E' \to X$, can we find some $E \to Y$ making the diagram below a pullback, i.e. with $E' \simeq E \times_Y X$?

$$\begin{tikzcd} E' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ \exists E? \arrow[r] & Y \end{tikzcd}$$

A familiar setting is when Y is a scheme, and $(U_\alpha)_\alpha$ some Zariski open cover of Y. Set $X := \coprod_\alpha U_\alpha$, and suppose we have a family of maps $E_\alpha \to U_\alpha$ indexed by α. The descent question in this setting asks whether this local information 'glues' to a global map $\pi : E \to Y$, such that we can recover $E_\alpha \to U_\alpha$ (up to isomorphism) by restricting to $\pi^{-1}(U_\alpha)$:

$$\begin{tikzcd} \coprod_\alpha E_\alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \coprod_\alpha U_\alpha \arrow[d] \\ E \arrow[r] & Y \end{tikzcd}$$

The classical answer to this is that such a map $E \to Y$ exists precisely when we are given isomorphisms $\varphi_{\alpha\beta} : E_\alpha|_{U_\alpha \cap U_\beta} \to E_\beta|_{U_\alpha \cap U_\beta}$ for each pair of indices, satisfying some compatibility constraint (the cocycle condition). We'll see how this is a particular instance of **descent data**, and that the existence of E says precisely that this is **effective**.

So the E_α tell us **what to glue**, and the $\varphi_{\alpha\beta}$ tell us **how to glue**; for Zariski covers as above, we are happy to glue many things: maps of schemes[1], sheaves, vector bundles, torsors. Zariski covers are an instance of a **Grothendieck topology** on the category of schemes, and a Grothendieck topology gives us a notion of a **sheaf** and accordingly **sheaf cohomology**. We'll see in general that things we can glue form the objects of a **fibred category**; gluable objects that satisfy descent are known as **(pre-)stacks**[2].

The Zariski topology is quite coarse (not many maps are nice enough to be open immersions), and cannot 'see' many things, or calculate cohomology groups that are morally correct: for instance, the sequence of group schemes over a field of characteristic $p > 0$

$$1 \to \mu_p \to \mathbb{G}_m \to \mathbb{G}_m \to 1$$

is not exact in the Zariski topology, and so Kummer theory fails here.

So we are often led to consider finer topologies: the étale topology is useful in number theory, Nisnevich for motivic things, and flat topologies (fppf, fpqc) for Kummer theory. We'll see how to work with these, and prove some descent results in these topologies analogous to the classical Zariski case.

[1] Good motivation is to note that schemes themselves are glued from affine schemes.

[2] This is a big topic and we won't have time to investigate their geometry in this seminar.

DESCENT SEMINAR WINTER TERM 22/23 LEITFADEN

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Unless otherwise specified, references below are to [Vis].

Week 1 *(03.10-07.10)* Introduce representability (Def 2.1) and state Yoneda's lemma, emphasising the functor of points perspective. Give examples (from §2.1.3) of representability. Discuss group object (Def 2.11), and equivalent formulation (Prop 2.12), alongside actions (Def 2.15 and Prop 2.16). Discuss discrete (group) objects (Defs 2.19, 2.20), and give examples.

Week 2 *(10.10-14.10)* Motivate and introduce Grothendieck (pre)topologies (Def 2.24), giving examples (from Ex 2.26-2.32). Discuss pullbacks/ equalisers, and define sheaves in a Grothendieck topology. State and prove Prop 2.39 and Cor 2.40.
Motivate fpqc topology with example of wild flat topology non-subcanonicity. State equivalent characterisations of fpqc morphisms (Prop 2.33).

Week 3 *(17.10-21.10)* Discuss subcanonicity (Def 2.57), stating classical or Zariski topology as example ([Oss, §2] gives a good treatment of this). Define comma categories and the comma topology (Def 2.58).
State Lemma 2.60, and use this to sketch a proof of Thm 2.55 (Lemma 2.61 is key). Briefly mention sheafification as a right adjoint to $\boldsymbol{\mathcal{P}re}(\mathcal{C}) \hookrightarrow \boldsymbol{\mathcal{S}hv}(\mathcal{C})$.
Define p-cartesian morphisms (Def 3.1) ([cat22] may be useful here), discuss properties, and define fibred categories and their morphisms (Defs 3.5, 3.6). Up to Def 3.8.

Week 4 *(24.10-28.10)* Discuss equivalent formulation of fibred categories as pseudofunctors (Def 3.10), define cleavings (Def 3.12), and state Prop 3.11. Discuss the converse case of the fibred category associated to a pseudofunctor.
Give examples of fibred categories, focusing on Ex 3.20 (but leave fibred category of quasi-coherent to the next session). Motivate and define the Grothendieck construction, using [nca22] and [Hol08, §1].

Week 5 *(31.10-04.11)* Sketch construction of fibred category of quasi-coherent sheaves, with reference to pseudofunctors. Briefly discuss categories fibred in sets/ groupoids (§§3.3-3.4)0, and state and prove the 2-Yoneda lemma (§3.6.2). Carefully define functor of arrows $\underline{\mathrm{Hom}}_S(\xi, \eta)$ of a fibred category (§3.7).

Week 6 *(07.11-11.11)* Beginning of §4: define the category of descent data (Def 4.2), and give the alternative definition ($\mathcal{F}_{desc}(\mathcal{U})$ – cf. discussion after Rmk 4.6). Set up situation of Prop 4.3 (discussion after Rmk 4.4), and state 4.3.
Give the definition of a (pre)stack, and state and prove (or sketch) Prop 4.7. Discuss effective descent data (Def 4.8). Work through example of $(\boldsymbol{\mathcal{S}hv}/\mathcal{C}) \to \mathcal{C})$ (Ex. 4.11). If time permits, state Prop 4.9, emphasising how this shows that stacks are a generalisation of sheaves, or categories fibred in sets.

Week 7 *(14.11-18.11:)* Work through fpqc descent for quasi-coherent sheaves (Thm 4.23). Can assume Lma 4.25 and other results as needed.

Week 8 *(21.11-25.11)* Descent for morphisms of schemes (tentaive).

Week 9 *(28.11-02.12)* fppf descent for group schemes.

Week 10 *(05.12-09.12)* Departmental hide and seek.

References

[Vis] A. Vistoli. *Notes on Grothendieck topologies, fibered categories and descent theory*. Version 4. URL: `https://arxiv.org/pdf/math/0412512.pdf`.

[Oss] B. Osserman. *Fiber products and Zariski sheaves*. URL: `http://math.stanford.edu/~vakil/d/FOAG/BOfiber-prods.pdf`.

[cat22] catlab. *Grothendieck fibrations*. Sept. 22, 2022. URL: `https://ncatlab.org/joyalscatlab/published/Grothendieck+fibrations`.

[nca22] ncatlab. *Grothendieck construction*. Sept. 22, 2022. URL: `https://ncatlab.org/nlab/show/Grothendieck+construction`.

[Hol08] S. Hollander. “Diagrams indexed by Grothendieck constructions”. In: *Homology Homotopy Appl.* 10.3 (2008), pp. 193–221.