

A Relative H-principle Via Cobordism-like Categories

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ABSTRACT. We prove an h-principle with boundary condition for a certain class of sheaves $\Psi: Emb_d^{op} \rightarrow Top$. The techniques used in the proof come from the study of the homotopy type of cobordism categories, and they are of simplicial and categorical nature. Applying the main result of this paper we recover the homotopy equivalence $BC_{k,d} \simeq \Omega^{d-1}Th(\gamma_{k,d}^\perp)$ of Galatius, Madsen, Tillmann, Weiss.

Introduction

Given a continuous sheaf $\Psi: Emb_d^{op} \rightarrow Top$ on the category of smooth d -dimensional manifolds without boundary with embeddings as morphisms, one can define a new sheaf $\Psi^*: Emb_d^{op} \rightarrow Top$ together with an h-principle (or scanning) map $h: \Psi \rightarrow \Psi^*$. Gromov shows in [5] that if Ψ is “microflexible” for an open manifold M , the map $h: \Psi(M) \rightarrow \Psi^*(M)$ is a weak homotopy equivalence.

In this paper we prove a relative result. One can define Ψ for a manifold with boundary M by gluing an external collar, obtaining a restriction map $\Psi(M) \rightarrow \Psi(\partial M \times \mathbb{R})$. Given an element $g_0 \in \Psi(\partial M \times \mathbb{R})$ consider the space $\Psi(M; g_0)$ of elements of $\Psi(M)$ that restrict to g_0 in a neighborhood of $\partial M \times [0, \infty)$. We find conditions on Ψ so that the restriction of the h-principle map $h: \Psi(M; g_0) \rightarrow \Psi^*(M; h(g_0))$ is a weak equivalence.

Our approach to this problem is motivated by the study of the weak homotopy type of cobordism categories. Let $\mathcal{C}_{k,d}$ be the cobordism category of [4], having k -dimensional compact submanifolds of $\mathbb{R}^{d-1} \times [a, b]$ as morphisms. The Madsen-Weiss theorem of [2] and [4] shows the existence of a weak equivalence

$$BC_{k,d} \simeq \Omega^{d-1}Th(\gamma_{k,d}^\perp)$$

where $\gamma_{k,d}^\perp = \{(V, v) \in \mathcal{G}_{k,d} \times \mathbb{R}^d \mid v \perp V\}$ is the total space of the complement of the tautological bundle over the Grassmannians $\mathcal{G}_{k,d}$ of k -vector subspaces of \mathbb{R}^d . This equivalence can be rephrased as a relative h-principle for the sheaf $\Psi_k: Emb_d^{op} \rightarrow$

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Top defined by the set

$$\Psi_k(M) = \{W \subset M \mid W \text{ is a } k\text{-submanifold, closed as a subset}\}$$

suitably topologized (cf. [2],[3]). The boundary condition needed for the Madsen-Weiss theorem is the empty submanifold $\emptyset \in \Psi_k(\partial M \times \mathbb{R})$.

In this paper we modify the proof of [1] of the Madsen-Weiss theorem using a Quillen B argument. Then we reformulate the structural properties of Ψ_k used in the proof to obtain conditions on a general sheaf Ψ , and we obtain the following.

MAIN THEOREM. *Suppose that*

- (1) *The h-principle maps $\Psi(M) \longrightarrow \Psi^*(M)$ and $\Psi(\partial M \times \mathbb{R}) \longrightarrow \Psi^*(\partial M \times \mathbb{R})$ are weak equivalences,*
- (2) *The orthogonal elements are almost open in $\Psi(M)$ (see 1.1 and 2.5),*
- (3) *Ψ is group like at M (see 3.7),*
- (4) *Ψ is damping at M (see 3.4).*

Then for any $g_0 \in \Psi(\partial M \times \mathbb{R})$ orthogonal to ∂M , the relative h-principle map $h: \Psi(M; g_0) \longrightarrow \Psi^(M; h(g_0))$ is a weak equivalence.*

For the sheaf Ψ_k of the Madsen-Weiss theorem, the properties above reduce essentially to the following. Orthogonal elements of $\Psi_k(\partial M \times \mathbb{R})$ are submanifolds that intersect $\partial M \times 0$ orthogonally in the classical sense. These elements are “almost open”, in the sense that the inclusion into elements that intersect $\partial M \times 0$ transversally is an equivalence, and transversality is an open condition. The group-like condition for Ψ_k is the existence of a “dual manifold” for every submanifold $N \subset \partial M \times [0, 1]$ that intersects the boundary orthogonally. It is the submanifold of $\partial M \times [1, 2]$ defined by $2 - N$. The damping condition is a technical condition needed for proving that structural maps for a Quillen theorem B are weak equivalences. It holds for Ψ_k by the smooth approximation theorem of [3].

In order to prove the main theorem we build a model for the restriction map $\Psi(M) \longrightarrow \Psi(\partial M)$ using a functor of simplicial categories $\mathcal{C}_\Psi(M) \longrightarrow \partial\mathcal{C}_\Psi(M)$. Here $\mathcal{C}_\Psi(M)$ and $\partial\mathcal{C}_\Psi(M)$ are defined imitating the construction of cobordism categories. The conditions above allow us to build weak equivalences

$$\begin{array}{ccc} BC_\Psi(M) & \simeq & \Psi(M) \\ \downarrow & & \downarrow \\ B\partial\mathcal{C}_\Psi(M) & \simeq & \Psi(\partial M) \end{array}$$

analogous to [1] and [3]. Then we identify the homotopy fiber of the left-hand map using Quillen theorem B' of [10].

In the last section of the paper we show that the sheaf of submanifolds Ψ_k of [2] and [3] satisfies the conditions above, giving a “Quillen B proof” of the Madsen-Weiss theorem. We can moreover replace the empty boundary condition with any manifold g that is orthogonal near the boundary of $D^{d-1} \times \mathbb{R}$. This gives a description of the homotopy type of the cobordism category of bordisms that agree with g around $S^{d-2} \times \mathbb{R}$.

1. Relative h-principles and strategy

Let Emb_d be the category with objects boundaryless d -dimensional manifolds and embeddings as morphisms. This category is enriched in topological spaces endowing the set of morphisms between two manifolds with the weak C^∞ -topology.

We will consider continuous sheaves $\Psi: Emb_d^{op} \rightarrow Top$, where Emb_d has the standard Grothendieck topology. Given such a sheaf, we extend it to Emb_{d-1}^{op} by

$$\Psi(N) := \Psi(N \times \mathbb{R})$$

We extend it to collared d -dimensional manifolds with boundary as follows. Let M be a d -manifold with a collar $e: \partial M \times (-\infty, 0] \xrightarrow{\cong} U \subset M$ of the boundary ∂M . Here U is an open neighborhood of ∂M , and e identifies $\partial M \times 0$ with ∂M . The collar induces a smooth structure on

$$M_{<\infty} := M \coprod_{\partial M} \partial M \times [0, \infty)$$

and we define

$$\Psi(M) := \Psi(M_{<\infty})$$

Different collars on M define diffeomorphic manifolds $M_{<\infty}$, and therefore homeomorphic $\Psi(M)$. The collar e also defines a restriction map

$$res: \Psi(M) \rightarrow \Psi(\partial M)$$

that sends an element $G \in \Psi(M_{<\infty})$ to

$$res(G) = G|_{\partial M} = \tilde{e}^*(G|_{U \coprod_{\partial M} \partial M \times [0, \infty)})$$

where $\tilde{e}: \partial M \times \mathbb{R} \xrightarrow{\cong} U \coprod_{\partial M} \partial M \times [0, \infty)$ is the extension of e by the identity on $\partial M \times [0, \infty)$.

An element $g_0 \in \Psi(\partial M)$ is called a ‘‘boundary condition’’. Given a boundary condition g_0 , define

$$\Psi(M; g_0) = \text{colim}_{\epsilon > 0} \{G \in \Psi(M) : G|_{\partial M \times (-\epsilon, \infty)} = g_0|_{\partial M \times (-\epsilon, \infty)}\}$$

the space of elements of $\Psi(M)$ that restrict to g_0 in a neighborhood of $\partial M \times [0, \infty)$. Here we used the notation $G|_{\partial M \times I} := res(G)|_{\partial M \times I}$ for an open $I \subset \mathbb{R}$. Notice that the space $\Psi(M; g_0)$ depends on the collar of ∂M only up to homeomorphism.

There is a new sheaf $\Psi^*: Emb_d^{op} \rightarrow Top$ defined by

$$\Psi^*(M) = \Gamma(Fr(M) \times_{GL_d} \Psi(\mathbb{R}^d)) \rightarrow M$$

where $Fr(M)$ is the principal GL_d -bundle of framings of TM , and Γ denotes the space of smooth sections of a bundle. We extend Ψ^* in the same way to lower dimensional manifolds and to collared manifolds with boundary.

The sheaf Ψ^* comes equipped with maps $\Psi(M) \xrightarrow{h} \Psi^*(M)$ defined as follows. Let $\rho: TM \rightarrow M$ be a fiberwise embedding induced by an exponential map. It can be defined rescaling a ball of $T_x M$ for all $x \in M$ with a radius varying continuously with x . We associate to a $G \in \Psi(M)$ the section $s_G: M \rightarrow Fr(M) \times_{GL_d} \Psi(\mathbb{R}^d)$ defined by

$$s_G(x) = [\phi_x, \phi_x^* \rho_x^*(G)]$$

for a choice of framing $\phi_x: \mathbb{R}^d \xrightarrow{\cong} T_x M$.

The restriction of the h-principle map h induces a map

$$\Psi(M; g_0) \rightarrow \Psi^*(M; h(g_0))$$

Our goal is to see when this map is a weak equivalence. We consider the following family of “orthogonal” boundary conditions $g_0 \in \Psi(\partial M)$.

DEFINITION 1.1. Let $U \subset M_{<\infty}$ and $I \subset \mathbb{R}$ be open subsets with $\partial M \times I \subset U$. An element $G \in \Psi(U)$ is **orthogonal to ∂M on I** , denoted $G \perp_I$, if for every open intervals $J, J' \subset I$ and any orientation preserving diffeomorphism $\alpha: J' \rightarrow J$

$$G|_{J'} = \alpha^*(G|_J)$$

Here $\alpha^*: \Psi(\partial M \times J) \rightarrow \Psi(\partial M \times J')$ is the map induced by $id \times \alpha: \partial M \times J' \rightarrow \partial M \times J$. Define $\Psi(U)^{\perp_I} \subset \Psi(U)$ to be the subspace of elements that are orthogonal to ∂M on I .

MAIN THEOREM. Let $\Psi: Emb_d^{op} \rightarrow Top$ be a continuous sheaf, and M a d -dimensional manifold with boundary. Suppose that the following conditions hold:

- (1) The h -principle maps $\Psi(M) \rightarrow \Psi^*(M)$ and $\Psi(\partial M) \rightarrow \Psi^*(\partial M)$ are weak equivalences,
- (2) The orthogonal elements are almost open in $\Psi(M)$ (see 1.1 and 2.5),
- (3) Ψ is group like at M (see 3.7),
- (4) Ψ is damping at M (see 3.4).

Then for any boundary condition $g_0 \in \Psi(\partial M)^{\perp_{\mathbb{R}}}$ the relative h -principle map $\Psi(M; g_0) \rightarrow \Psi^*(M; h(g_0))$ is a weak equivalence.

REMARK 1.2. By a theorem of Gromov [5] the first condition is satisfied if M is an open manifold and the sheaf Ψ is microflexible at M (that is, if the inclusions of compact pairs $K \subset K' \subset M$ induce microfibrations $\Psi(K') \rightarrow \Psi(K)$ of quasi-topological spaces. See [8] for a formulation in terms of lifting properties for topological spaces).

PROOF SKETCH. Consider the commutative diagram

$$\begin{array}{ccc} \Psi(M; g_0) & \xrightarrow{h} & \Psi^*(M; h(g_0)) \\ \downarrow & & \downarrow \\ \Psi(M) & \xrightarrow[\simeq]{h} & \Psi^*(M) \\ \text{res} \downarrow & & \downarrow \text{res}^* \\ \Psi(\partial M) & \xrightarrow[\simeq]{h} & \Psi^*(\partial M) \end{array}$$

where res^* denotes the restriction map for the sheaf Ψ^* . The map h induces a weak equivalence between the homotopy fiber $hF(res, g_0)$ of res over g_0 and the homotopy fiber $hF(res^*, h(g_0))$ of res^* over $h(g_0)$. We prove the theorem by finding weak equivalences $hF(res, g_0) \simeq \Psi(M; g_0)$ and $hF(res^*, h(g_0)) \simeq \Psi^*(M; h(g_0))$ compatible with the map h .

We start by describing this equivalence for Ψ^* . We show in 1.5 below that the canonical map

$$(res^*)^{-1}(h(g_0)) \rightarrow \Psi^*(M; h(g_0))$$

from the fiber over $h(g_0)$ is a homotopy equivalence. Let us show that the inclusion of the fiber into the homotopy fiber

$$(res^*)^{-1}(h(g_0)) \rightarrow hF(res^*, h(g_0))$$

is a weak equivalence. We denote $E := Fr(M_{<\infty}) \times_{GL_d} \Psi(\mathbb{R}^d)$ so that the map res^* is by definition the map

$$res^*: \Gamma(E) \longrightarrow \Gamma(E|_{\partial M \times \mathbb{R}})$$

that restricts the bundle sections. The standard inclusion $\partial M \xrightarrow{\cong} \partial M \times \mathbb{R}$ induces a commutative diagram

$$\begin{array}{ccc} (res^*)^{-1}(h(g_0)) & \longrightarrow & \pi^{-1}(h(g_0)|_{\partial M}) \\ \downarrow & & \downarrow \\ \Gamma(E) & \xlongequal{\quad\quad\quad} & \Gamma(E) \\ \downarrow res^* & & \downarrow \pi \\ \Gamma(E|_{\partial M \times \mathbb{R}}) & \xrightarrow{\cong} & \Gamma(E|_{\partial M}) \end{array}$$

where the right vertical map π restricts a section on the actual boundary (not just on a neighborhood). Since the inclusion $\partial M \rightarrow M_{<\infty}$ is a cofibration, π is a fibration. Since g_0 is orthogonal over \mathbb{R} , the map $h(g_0): \partial M \times \mathbb{R} \rightarrow E|_{\partial M \times \mathbb{R}}$ is constant in the \mathbb{R} direction, and therefore the top horizontal map in the diagram is a homotopy equivalence as well. All together we obtain a weak equivalence

$$hF(res^*, h(g_0)) \xleftarrow{\cong} (res^*)^{-1}h(g_0) \xrightarrow{\cong} \Psi^*(M; h(g_0))$$

It remains to identify $hF(res, g_0)$ with $\Psi(M; g_0)$. The restriction map for Ψ is in general not a quasi-fibration, and we need to find a different argument than the one used for Ψ^* . In §2 we define a simplicial functor $res: \mathcal{C}_\Psi(M) \rightarrow \partial\mathcal{C}_\Psi(M)$ and using the second condition we define weak equivalences

$$\begin{array}{ccc} BC_\Psi(M) & \simeq & \Psi(M) \\ \downarrow & & \downarrow \\ B\partial\mathcal{C}_\Psi(M) & \simeq & \Psi(\partial M) \end{array}$$

(see 2.1). In §3 we use the third and the fourth conditions to prove that the simplicial functor $res: \mathcal{C}_\Psi(M) \rightarrow \partial\mathcal{C}_\Psi(M)$ satisfies the hypothesis of a Quillen theorem B for simplicial categories (cf. 3.9 and 3.10). This theorem B gives a description of the homotopy fiber of the realization of our functor as the realization of a certain category. We identify the classifying space of this category with $\Psi(M; g_0)$ in 3.1 and 3.3. \square

REMARK 1.3. The following observation has been pointed out by the referee. One of the main steps in the proof of Gromov's theorem given in [6] for sheaves of open partial differential relations consists of proving that the restriction map $\Psi(M) \rightarrow \Psi(\partial M)$ is in fact a Serre fibration, provided the pair $(M; \partial M)$ is open (that is M is open and $M \setminus \partial M$ has no relative compact components). It is possible to generalize this proof to microflexible sheaves $\Psi: Emb_d^{op} \rightarrow Top$. The argument above is then an easy proof of the relative h-principle for an open pair $(M; \partial M)$ without having to assume conditions 2-4 of the main theorem.

EXAMPLE 1.4. The main example of a sheaf that satisfies the hypotheses above is $\Psi_k: Emb_d^{op} \rightarrow Top$ from [2] and [3]. It associates to a d -manifold M the set $\Psi_k(M)$ of closed subsets $W \subset M$ which are smooth k -dimensional submanifolds

without boundary, suitably topologized. Given an embedding $e: M \rightarrow M'$, the induced map $e^*: \Psi_k(M') \rightarrow \Psi_k(M)$ sends a submanifold $W \subset M'$ to $e^{-1}(W)$. The condition for a submanifold $W \in \Psi_k(\partial M)$ to be orthogonal to ∂M over I corresponds to orthogonality in the classical sense. That is, $W \perp_I$ exactly when the intersection $W \cap \partial M \times I$ is a product manifold $N \times I$ for some $(k-1)$ -dimensional submanifold $N \subset \partial M$. In section 4 we show that Ψ_k satisfies the conditions of our main theorem, and we relate it to the Madsen-Weiss theorem of [2].

Let $e: \partial M \times (-1, 0] \xrightarrow{\cong} U \subset M$ be a collar small enough to admit an extension

$$e': \partial M \times (-2, 0] \xrightarrow{\cong} U' \subset M$$

with $U \subset U'$. We implicitly rescaled the collar using a diffeomorphism $(-1, 0] \cong (-\infty, 0]$.

LEMMA 1.5. *Let M be a d -dimensional manifold with boundary, e a choice of collar like above, and $\Psi: Emb_d^{op} \rightarrow Top$ a continuous sheaf with induced restriction map $res: \Psi(M) \rightarrow \Psi(\partial M)$. For any $g \in \Psi(\partial M)^{\perp \mathbb{R}}$ the natural map*

$$res^{-1}(g) \rightarrow \Psi(M; g)$$

is a homotopy equivalence.

PROOF. Define a homotopy inverse for the canonical map as follows. For every $\epsilon > 0$ denote

$$\Psi(M; g)_\epsilon := \{G \in \Psi(M) : G|_{\partial M \times (-\epsilon, \infty)} = g|_{\partial M \times (-\epsilon, \infty)}\}$$

so that by definition $\Psi(M; g) = \text{colim}_{\epsilon > 0} \Psi(M; g)_\epsilon$. Let $\alpha: (-2, \infty) \rightarrow (-2, \infty)$ be a diffeomorphism that sends -1 to 0 and that is the identity in a neighborhood of -2 . This induces a diffeomorphism $\tilde{\alpha}: M_{< \infty} \rightarrow M_{< \infty}$, and for every $\epsilon > 0$ a map

$$\tilde{\alpha}^*: \Psi(M; g)_\epsilon \rightarrow res^{-1}(g)$$

commuting with the structure maps of the colimit. The restriction of $\tilde{\alpha}^*G$ agrees indeed with g on $(-1, \infty)$

$$(\tilde{\alpha}^*G)|_{(-1, \infty)} = \tilde{\alpha}^*(G|_{(0, \infty)}) = \tilde{\alpha}^*(g|_{(0, \infty)}) = g|_{(-1, \infty)}$$

since we are taking $g \in \Psi(\partial M)^{\perp \mathbb{R}}$. This induces a continuous map on the colimit

$$\tilde{\alpha}^*: \Psi(M; g) \rightarrow res^{-1}(g)$$

which is a homotopy inverse for the canonical map. The homotopies between the two composites and the identities are given by choosing a diffeotopy $\alpha_t: (-2, \infty) \rightarrow (-2, \infty)$ between α and the identity. \square

2. Categorical model for $\Psi(M) \rightarrow \Psi(\partial M)$

In this section we prove the following.

PROPOSITION 2.1. *Let M be a collared manifold of dimension d . There is a commutative diagram*

$$\begin{array}{ccccc} BC_\Psi(M) & \xleftarrow[\vartheta]{\cong} & BD_\Psi(M) & \xrightarrow{pr} & \Psi(M) \\ res \downarrow & & res \downarrow & & res \downarrow \\ B\partial C_\Psi(M) & \xleftarrow[\partial\vartheta]{\cong} & B\partial D_\Psi(M) & \xrightarrow{pr} & \Psi(\partial M) \end{array}$$

where the left vertical map is the realization of a functor of topological categories and the maps ϑ and $\partial\vartheta$ are weak equivalences.

If the orthogonal elements of Ψ are almost open at M (cf. 2.5 below) all the horizontal maps are weak equivalences.

By a topological category we mean a category internal to topological spaces. Define a topological category $\partial\mathcal{C}_\Psi(M)$ with objects space

$$Ob\partial\mathcal{C}_\Psi(M) = \coprod_{a \in \mathbb{R}} \Psi(\partial M)^{\perp_{\mathbb{R}}}$$

The disjoint union is meant to indicate the fact that we are taking the discrete topology on the real coordinate. For real numbers $0 < \epsilon$ and $a \leq b$ define

$$Mor_\epsilon(a, b) = \Psi(\partial M)^{\perp_{(-\infty, a+\epsilon)}} \cap \Psi(\partial M)^{\perp_{(b-\epsilon, \infty)}}$$

and let the morphism space of $\partial\mathcal{C}_\Psi(M)$ be

$$Mor\partial\mathcal{C}_\Psi(M) = \coprod_{a \leq b} colim_{0 < \epsilon} Mor_\epsilon(a, b)$$

The source map of $\partial\mathcal{C}_\Psi(M)$ is induced by the maps

$$s_\epsilon: Mor_\epsilon(a, b) \longrightarrow Ob\partial\mathcal{C}_\Psi(M)$$

defined by sending G to

$$s_\epsilon(G) = G|_{(-\infty, a+\epsilon)} \cup \alpha^*(G|_{(a-\epsilon, a+\epsilon)})$$

in the a -component, for a choice of diffeomorphism $\alpha: (a-\epsilon, \infty) \longrightarrow (a-\epsilon, a+\epsilon)$ preserving the orientation. By the orthogonality condition on the elements of $Mor_\epsilon(a, b)$ the map s_ϵ is independent of the choice of α , and it is continuous by continuity of the sheaf. Similarly one defines the target map as the colimit of the maps

$$t_\epsilon(G) = \beta^*(G|_{(b-\epsilon, b+\epsilon)}) \cup G|_{(b-\epsilon, \infty)}$$

for a choice of orientation preserving diffeomorphism $\beta: (-\infty, b+\epsilon) \longrightarrow (b-\epsilon, b+\epsilon)$. Composition in $\partial\mathcal{C}_\Psi(M)$ is induced by the maps

$$\circ: Mor_\epsilon(b, c) \times_{Ob\partial\mathcal{C}_\Psi(M)} Mor_\epsilon(a, b) \longrightarrow Mor_\epsilon(a, c)$$

defined by

$$G \circ H = H|_{(-\infty, b-\epsilon)} \cup G|_{(b-\epsilon, \infty)}$$

REMARK 2.2. This is an “embedded cobordism category” for the sheaf Ψ , with objects embedded in ∂M and cobordism direction given by the collar. For $\Psi = \Psi_k$ the sheaf of submanifolds of [3], and $M = \mathbb{R}^d \times (-\infty, 0]$, the category $\partial\mathcal{C}_{\Psi_k}(\mathbb{R}^d \times (-\infty, 0])$ is almost the same as the cobordism category $\mathcal{C}_{k,d}$ of [4]. The difference is that our bordisms are not necessarily compact.

Consider now the topological category $\mathcal{C}_\Psi(M)$ with objects

$$Ob\mathcal{C}_\Psi(M) = \coprod_{a \in \mathbb{R}} colim_{0 < \epsilon} \Psi(M)^{\perp_{(a-\epsilon, \infty)}}$$

and morphisms

$$Mor\mathcal{C}_\Psi(M) = \coprod_{a \leq b} colim_{0 < \epsilon} \Psi(M)^{\perp_{(a-\epsilon, a+\epsilon)}} \cap \Psi(M)^{\perp_{(b-\epsilon, \infty)}}$$

The source is induced by sending G in the $a \leq b$ component to

$$s_\epsilon(G) = G|_{M_{<a+\epsilon}} \cup \alpha^*(G|_{(a-\epsilon, a+\epsilon)})$$

in the a -component, for an orientation preserving diffeomorphism $\alpha: (a-\epsilon, \infty) \longrightarrow (a-\epsilon, a+\epsilon)$. Here $M_{<u} \subset M_{<\infty}$ is the submanifold

$$M_{<u} = M_{<\infty} \setminus (\partial M \times [u, \infty))$$

The target projects off the a -coordinate, i.e. it sends G in the $a \leq b$ component of the morphisms space to G in the b -component of the objects space. Composition of (b, c, G) and (a, b, H) is defined to be just (a, c, G) (notice that if (b, c, G) and (a, b, H) are composable then $H|_{M_{<b}} = G|_{M_{<b}}$).

The restriction map $\Psi(M) \longrightarrow \Psi(\partial M)$ induces a continuous functor

$$\mathcal{C}_\Psi(M) \longrightarrow \partial\mathcal{C}_\Psi(M)$$

REMARK 2.3. Forgetting the topologies, there is a description of $\mathcal{C}_\Psi(M)$ as a wreath construction. Consider the functor $F: \partial\mathcal{C}_\Psi(M) \longrightarrow Top$ that sends an object $(a, g) \in Ob\partial\mathcal{C}_\Psi(M)$ to the set of extensions of g to M

$$F(a, g) := colim_{0 < \epsilon} \{G \in \Psi(M) \mid G|_{(a-\epsilon, \infty)} = g|_{(a-\epsilon, \infty)}\}$$

and a morphism $(a \leq b, G) \in Mor\partial\mathcal{C}_\Psi(M)$ to the map

$$(-)|_{M_{<a+\epsilon}} \cup G|_{(a-\epsilon, \infty)}: F(a, s(G)) \longrightarrow F(b, t(G))$$

As a discrete category $\mathcal{C}_\Psi(M)$ is the Grothendieck category $\partial\mathcal{C}_\Psi(M) \wr F$, and the restriction functor $\mathcal{C}_\Psi(M) \longrightarrow \partial\mathcal{C}_\Psi(M)$ corresponds to the projection functor $\partial\mathcal{C}_\Psi(M) \wr F \longrightarrow \partial\mathcal{C}_\Psi(M)$.

The topological categories $D_\Psi(M)$ and $\partial D_\Psi(M)$ are associated to topological posets, defined as the spaces

$$D_\Psi(M) = \coprod_{a \in \mathbb{R}} colim_{0 < \epsilon} \Psi(M)^{\perp(a-\epsilon, a+\epsilon)}$$

and

$$\partial D_\Psi(M) = \coprod_{a \in \mathbb{R}} colim_{0 < \epsilon} \Psi(\partial M)^{\perp(a-\epsilon, a+\epsilon)}$$

with partial orders defined by $(a, G) \leq (b, H)$ if $a \leq b$ and $G = H$, on both spaces. The restriction map induces a map of topological posets

$$res: D_\Psi(M) \longrightarrow \partial D_\Psi(M)$$

There is a functor $\partial\vartheta: \partial D_\Psi(M) \longrightarrow \partial\mathcal{C}_\Psi(M)$ that sends a morphism $(a \leq b, G)$ with $G \in \Psi(\partial M)^{\perp(a-\epsilon, a+\epsilon)} \cap \Psi(\partial M)^{\perp(b-\epsilon, b+\epsilon)}$ to

$$\partial\vartheta(a \leq b, G) = (a \leq b, \alpha^*(G|_{(a-\epsilon, a+\epsilon)}) \cup G|_{(a-\epsilon, b+\epsilon)} \cup \beta^*(G|_{(b-\epsilon, b+\epsilon)}))$$

for diffeomorphisms $\alpha: (-\infty, a+\epsilon) \longrightarrow (a-\epsilon, a+\epsilon)$ and $\beta: (b-\epsilon, \infty) \longrightarrow (b-\epsilon, b+\epsilon)$ preserving the orientation. Similarly, there is a functor

$$\vartheta: D_\Psi(M) \longrightarrow \mathcal{C}_\Psi(M)$$

that sends a morphism $(a \leq b, G)$ with $G \in \Psi(M)^{\perp(a-\epsilon, a+\epsilon)} \cap \Psi(M)^{\perp(b-\epsilon, b+\epsilon)}$ to

$$\vartheta(a \leq b, G) = (a \leq b, G|_{(-\infty, b+\epsilon)} \cup \beta^*(G|_{(b-\epsilon, b+\epsilon)}))$$

PROPOSITION 2.4. *The square*

$$\begin{array}{ccc} BC_{\Psi}(M) & \xleftarrow{\vartheta} & BD_{\Psi}(M) \\ \text{res} \downarrow & & \downarrow \text{res} \\ B\partial C_{\Psi}(M) & \xleftarrow{\partial\vartheta} & B\partial D_{\Psi}(M) \end{array}$$

commutes and the maps ϑ and $\partial\vartheta$ are weak homotopy equivalences.

PROOF. The square clearly commutes. We show that ϑ is a weak equivalence, the same result for $\partial\vartheta$ is completely analogous. We show that the nerve of ϑ is a levelwise homotopy equivalence of simplicial spaces

$$\mathcal{N}_k\vartheta: \mathcal{N}_kD_{\Psi}(M) \longrightarrow \mathcal{N}_kC_{\Psi}(M)$$

An element of $\mathcal{N}_kC_{\Psi}(M)$ is represented by a sequence of real numbers $a_0 \leq \dots \leq a_k$, a $0 < \epsilon$ and a $G \in \Psi(M)$ with $G \perp_{(a_i - \epsilon, a_i + \epsilon)}$ for all $0 \leq i \leq k-1$ and $G \perp_{(a_k - \epsilon, \infty)}$. This clearly defines an element of $\mathcal{N}_kD_{\Psi}(M)$, and actually a natural (non-simplicial) inclusion

$$\iota_k: \mathcal{N}_kC_{\Psi}(M) \longrightarrow \mathcal{N}_kD_{\Psi}(M)$$

which is a section for $\mathcal{N}_k\vartheta$. The composite $\iota_k \circ \mathcal{N}_k\vartheta$ sends an element $(a_0 \leq \dots \leq a_k, G)$ for $G \in \Psi(M)$ with $G \perp_{(a_i - \epsilon, a_i + \epsilon)}$ to

$$(a_0 \leq \dots \leq a_k, G|_{(-\infty, a_k + \epsilon)} \cup \alpha^*(G|_{(a_k - \epsilon, a_k + \epsilon)}))$$

for a diffeomorphism $\alpha: (a_k - \epsilon, \infty) \longrightarrow (a_k - \epsilon, a_k + \epsilon)$ that preserves the orientation. Take a continuous family of embeddings $\alpha_t: (a_k - \epsilon, \infty) \longrightarrow (a_k - \epsilon, \infty)$, for $t \in [0, 1]$, satisfying

- (1) $Im(\alpha_t) = (0, a_k + \epsilon + \frac{t}{1-t})$
- (2) $\alpha_0 = \alpha$
- (3) $\alpha_1 = id$

The embedding α_t induces a continuous map $\mathcal{N}_kD_{\Psi}(M) \longrightarrow \mathcal{N}_kD_{\Psi}(M)$ by sending $(a_0 \leq \dots \leq a_k, G)$ to

$$(a_0 \leq \dots \leq a_k, G|_{(-\infty, a_k + \epsilon)} \cup \alpha_t^*(G|_{(a_k - \epsilon, a_k + \epsilon)}))$$

These maps assemble into a homotopy

$$\mathcal{N}_kD_{\Psi}(M) \times [0, 1] \longrightarrow \mathcal{N}_kD_{\Psi}(M)$$

from $\iota_k \circ \mathcal{N}_k\vartheta$ to the identity. \square

For a real number $a \in \mathbb{R}$ we denote

$$\Psi(\partial M)^{\perp a} := \text{colim}_{0 < \epsilon} \Psi(\partial M)^{\perp (a - \epsilon, a + \epsilon)}$$

and define $\Psi(M)^{\perp a}$ in a similar way.

DEFINITION 2.5. We say that **orthogonal elements of Ψ are almost open at M** if there is an open cover $\{U_a\}_{a \in \mathbb{R}}$ of $\Psi(M)$ such that

- (1) $\Psi(\partial M)^{\perp (a - \epsilon, a + \epsilon)} \subset U_a$ for every $\epsilon > 0$,
- (2) The image of U_a by the restriction map $U_a|_{\partial M} \subset \Psi(\partial M)$ is open,

(3) For every finite sequence $a_0 \leq \dots \leq a_k$ the maps induced by the inclusions

$$\cap_{i=0}^k \Psi(M)^{\perp a_i} \longrightarrow \cap_{i=0}^k U_{a_i}$$

and

$$\cap_{i=0}^k \Psi(\partial M)^{\perp a_i} \longrightarrow \cap_{i=0}^k U_{a_i}|_{\partial M}$$

are weak equivalences.

EXAMPLE 2.6. In the case of the sheaf Ψ_k , one can essentially choose U_a to be the subspace of $\Psi(M)$ of submanifolds that intersect $\partial M \times a$ transversally. These sets U_a are open only when M is compact. For non-compact M one needs to modify these sets slightly (cf. 4.5).

PROPOSITION 2.7. *Suppose that orthogonal elements of Ψ are almost open at M . Then the projections from $D_\Psi(M)$ to $\Psi(M)$ and from $\partial D_\Psi(M)$ to $\Psi(\partial M)$ define a commutative square*

$$\begin{array}{ccc} BD_\Psi(M) & \xrightarrow{\simeq} & \Psi(M) \\ \text{res} \downarrow & & \downarrow \text{res} \\ B\partial D_\Psi(M) & \xrightarrow{\simeq} & \Psi(\partial M) \end{array}$$

where the horizontal maps are weak equivalences.

PROOF. The projections $D_\Psi(M) \longrightarrow \Psi(M)$ and $\partial D_\Psi(M) \longrightarrow \Psi(\partial M)$ factor through the topological posets $\mathcal{E}(M) = \coprod_{a \in \mathbb{R}} U_a$ and $\partial \mathcal{E}(M) = \coprod_{a \in \mathbb{R}} U_a|_{\partial M}$ with partial orders again defined as $(a, G) \leq (b, H)$ if $a \leq b$ and $G = H$. The inclusions induce weak equivalences $BD_\Psi(M) \simeq B\mathcal{E}(M)$ and $B\partial D_\Psi(M) \simeq B\partial \mathcal{E}(M)$ by our assumptions. Therefore it is enough to show that the inclusions $U_a \subset \Psi(M)$ induce weak equivalences

$$B\mathcal{E}(M) \xrightarrow{\simeq} \Psi(M) \quad \text{and} \quad B\partial \mathcal{E}(M) \xrightarrow{\simeq} \Psi(\partial M)$$

The nerves of $\Psi(M)$ and $\Psi(\partial M)$ are constant simplicial spaces. By [2, §3.4], which is a corollary of [9, §A1], it is enough to show that the projections $\mathcal{N}_k \mathcal{E}(M) \longrightarrow \Psi(M)$ and $\mathcal{N}_k \partial \mathcal{E}(M) \longrightarrow \Psi(\partial M)$ are levelwise étale (that is, are open maps and local homeomorphisms) and that their fibers have contractible realizations. We show the result only for $\mathcal{E}(M) \longrightarrow \Psi(M)$, the other case being completely analogous.

The fiber F_\bullet over a fixed element $G \in \Psi(M)$ is the simplicial space with k -simplicies

$$F_k = \{(a_0 \leq \dots \leq a_k) | G \in U_{a_i} \text{ for all } 0 \leq i \leq k\}$$

Notice that since the topology on the real coordinates is discrete, this is a discrete space. Since the sets U_a cover $\Psi(M)$, the spaces F_k are non-empty. Moreover F_\bullet is the nerve of the totally ordered set $\{a \in \mathbb{R} | G \in U_a\}$, and therefore its realization is contractible.

It remains to show that the projection is levelwise étale. Let us show that it is an open map. Since the real coordinates are discrete, an open subset of $\mathcal{N}_k \mathcal{E}(M)$ is a union of sets of the form $\{(a_0 \leq \dots \leq a_k)\} \times V$, with V open in $\cap_{i=0}^k U_{a_i}$. Its image by the projection map is the union of the sets V . Since V is open in $\cap_{i=0}^k U_{a_i}$, and $\cap_{i=0}^k U_{a_i}$ is open in $\Psi(M)$, the image V is open in $\Psi(M)$. It remains to show that the projection is locally injective. Since the real coordinate is discrete, the set $\{(a_0 \leq \dots \leq a_k)\} \times \cap_{i=0}^k U_{a_i}$ is an open neighborhood of a

general element $((a_0 \leq \cdots \leq a_k), G) \in \mathcal{N}_k \mathcal{E}(M)$. The restriction of the projection to $\{(a_0 \leq \cdots \leq a_k)\} \times \cap_{i=0}^k U_{a_i}$ is injective. \square

3. The homotopy fiber of $\Psi(M) \rightarrow \Psi(\partial M)$

In this section we use the categorical model for the restriction map $\Psi(M) \rightarrow \Psi(\partial M)$ described in the previous section to show the following.

PROPOSITION 3.1. *Let M be a collared d -dimensional manifold. Suppose that the elements of $\Psi(M)$ are almost open, and Ψ is group-like and damping at M (cf. 3.7 and 3.4 below). Then the homotopy fiber of $\Psi(M) \rightarrow \Psi(\partial M)$ over an element $g \in \Psi(\partial M)^{\perp_{\mathbb{R}}}$ is naturally weakly equivalent to*

$$\Psi(M; g) := \operatorname{colim}_{0 < \epsilon} \{G \in \Psi(M) \mid G|_{(-\epsilon, \infty)} = g|_{(-\epsilon, \infty)}\}$$

The conditions on the sheaf Ψ of the proposition allow us to identify the homotopy fiber of the restriction functor $\mathcal{C}_{\Psi}(M) \rightarrow \partial \mathcal{C}_{\Psi}(M)$ over an object $(g, 0)$ as the space $\Psi(M; g)$. This is carried out by means of Quillen theorem B for simplicial categories, proved by Waldhausen in [10]. We recall its content.

Let $F: C \rightarrow D$ be a simplicial functor. Given an object $Y_q \in \operatorname{Ob} D_q$, define the (right) fiber of F over Y_q by the simplicial category Y_q/F , whose level n category has objects

$$\operatorname{Ob}(Y_q/F)_n = \coprod_{\nu: [n] \rightarrow [q]} \{(X \in \operatorname{Ob} C_n, b \in \operatorname{Mor} D_n) \mid b: F(X) \rightarrow \nu^* Y_q\}$$

The morphisms from (X, b) to (X', b') are morphisms $a: X \rightarrow X'$ of C_n making the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{b} & \nu^* Y_q \\ F(a) \downarrow & \nearrow b' & \\ F(X') & & \end{array}$$

commutative.

There are two kind of maps between the fibers over the objects of D . A map of the first kind is a map $\beta_*: Y_q/F \rightarrow Y'_q/F$ induced by a morphism $\beta: Y_q \rightarrow Y'_q$ of D_q . It maps a morphism $a: (X, b) \rightarrow (X', b')$ of $(Y_q/F)_n$ to the morphism $a: (X, \nu^*(\beta) \circ b) \rightarrow (X', \nu^*(\beta) \circ b')$ of $(Y'_q/F)_n$

$$\begin{array}{ccc} F(X) & \xrightarrow{b} & \nu^* Y_q \xrightarrow{\nu^* \beta} \nu^* Y'_q \\ F(a) \downarrow & \nearrow b' & \\ F(X') & & \end{array}$$

A map of the second kind is a map $\alpha_*: \alpha^*(Y_q)/F \rightarrow Y_q/F$ induced by a morphism $\alpha: [p] \rightarrow [q]$ in Δ . It maps a morphism $a: (X, b) \rightarrow (X', b')$ in the $\nu: [n] \rightarrow [p]$ component of $(\alpha^*(Y_q)/F)_n$ to $a: (X, b) \rightarrow (X', b')$ in the $\alpha \circ \nu$ component of $(Y_q/F)_n$.

PROPOSITION 3.2 ([10]). *Let $F: C \rightarrow D$ be a simplicial functor for which all the maps of the first and the second kind induce weak equivalences on the classifying*

space. The for every object $Y \in D_0$ the diagram

$$\begin{array}{ccc} B(Y/F) & \longrightarrow & BC \\ \downarrow & & \downarrow BF \\ * & \xrightarrow{Y} & BD \end{array}$$

is homotopy cartesian.

We briefly recall that the classifying space of a simplicial category is defined as the realization of the bisimplicial set obtained by taking the nerve levelwise. This realization is defined as the realization of the simplicial space obtained by levelwise realizing one of the simplicial directions. This is the same as realizing the diagonal simplicial set.

We want to apply this Quillen B theorem to describe the homotopy fibers of the restriction functor $BC_\Psi(M) \longrightarrow B\partial C_\Psi(M)$. The singular chains functor $Sing_\bullet$ from topological spaces to simplicial sets induces a functor from topological categories to simplicial categories. Given a topological category C , define a simplicial category $Sing_\bullet C$ in simplicial degree k by $ObSing_k C := Sing_k ObC$ and morphisms $MorSing_k C := Sing_k MorC$. Source, target and composition maps are defined pointwise. In our situation, this construction induces a simplicial functor

$$res: Sing_\bullet C_\Psi(M) \longrightarrow Sing_\bullet \partial C_\Psi(M)$$

Before showing that under appropriate conditions this functor satisfies theorem B, let us identify the classifying space of the fiber over $(b, g) \in Sing_0 \partial C_\Psi(M) = \partial C$. Let $F_{(b,g)}$ be the topological poset with underlying space

$$F_{(b,g)} = \coprod_{a \leq b} colim_{0 < \epsilon} \{G \in \Psi(M)^{\perp a} \mid G|_{(b-\epsilon, \infty)} = g|_{(b-\epsilon, \infty)}\}$$

and partial order defined by $(a, G) \leq (a', H)$ if $a \leq a'$ and $G = H$. Notice that by definition $(b, g)/res = Sing_\bullet F_{(b,g)}$.

PROPOSITION 3.3. *If orthogonal elements of $\Psi(M)$ are almost open, for every $g \in \Psi(\partial M)^{\perp \mathbb{R}}$ there is a natural zig-zag of weak equivalences*

$$B((0, g)/res) \xrightarrow{\simeq} BF_{(0,g)} \xrightarrow[\simeq]{pr} \Psi(\partial M; g)$$

where the right-hand map is defined by projecting off the real coordinate.

PROOF. As we noticed, $(b, g)/res = Sing_\bullet F_{(b,g)}$ and therefore the evaluation map induces a topological functor

$$|(b, g)/res| \longrightarrow F_{(b,g)}$$

that is a levelwise equivalence on the nerve. The projection map $pr: F_{(0,g)} \longrightarrow \Psi(\partial M; g)$ is an equivalence by the same argument of 2.7. \square

We now see under which assumptions on Ψ the restriction functor satisfies the conditions of Quillen's theorem 3.2.

Given a sequence $a_0 \leq \dots \leq a_k$ denote $\Psi(\partial M)^{\perp a_0, \dots, a_k} := \cap_{i=0}^k \Psi(\partial M)^{\perp a_i}$, and define $\Psi(M)^{\perp a_0, \dots, a_k}$ analogously. For every real number u denote $t_u: \mathbb{R} \longrightarrow \mathbb{R}$ the diffeomorphism $t_u(x) = x + u$.

DEFINITION 3.4. The sheaf Ψ is **damping at** M if given any $f: \Delta^q \rightarrow \Psi(\partial M)^{\perp_{\mathbb{R}}}$ there is a family $\{B^{\phi, \psi}: \Delta^n \rightarrow \Psi(\partial M) \mid \phi, \psi: [n] \rightarrow [q]\}$ such that

- i) $B^{\phi, \phi} = \phi^* f$
- ii) $\omega^* B^{\phi, \psi} = B^{\omega^* \phi, \omega^* \psi}$ for all $\omega: [m] \rightarrow [n]$
- iii) $B^{\phi, \psi}(\sigma)|_{(-\infty, \epsilon)} = \phi^* f(\sigma)|_{(-\infty, \epsilon)}$
- iv) $t_1^*(B^{\phi, \psi}(\sigma)|_{(1-\epsilon, \infty)}) = \psi^* f(\sigma)|_{(-\epsilon, \infty)}$

REMARK 3.5. Here is an illustration of this strange damping condition. Given $f: \Delta^q \rightarrow \Psi(\partial M)^{\perp_{\mathbb{R}}}$ and $\phi, \psi: [n] \rightarrow [q]$, consider the map $H^{\phi, \psi}: \Delta^n \times [0, 1] \rightarrow \Psi(\partial M)^{\perp_{\mathbb{R}}}$ defined by

$$H^{\phi, \psi}(x, t) := f(\phi_*(x) \cdot t + \psi_*(x) \cdot (1 - t))$$

This is a homotopy from $\phi^* f$ to $\psi^* f$, and this family satisfies

$$\omega^* H_t^{\phi, \psi} = H_t^{\omega^* \phi, \omega^* \psi}$$

for all $\omega: [m] \rightarrow [n]$ and fixed $t \in [0, 1]$. A family $B^{\phi, \psi}: \Delta^n \rightarrow \Psi(\partial M)$ for the damping condition exists if each element $H^{\phi, \psi}(x, t) \in \Psi(\partial M)^{\perp_{\mathbb{R}}}$ is determined by the restriction to a neighborhood of $\partial M \times t$ of $B^{\phi, \psi}(x)$. We show that the sheaf of submanifolds Ψ_k satisfies this property in 4.6, by means of a smooth approximation theorem.

PROPOSITION 3.6. *Suppose that Ψ is damping at M . Then for every object Y_q of $\text{Sing}_q \partial \mathcal{C}_{\Psi}(M)$ the map of the second kind*

$$(\eta_0)_*: \eta_0^* Y_q / \text{res} \rightarrow Y_q / \text{res}$$

induced by the value zero map $\eta_0: [0] \rightarrow [q]$ is a weak equivalence on classifying spaces.

PROOF. We show that for a fixed k the map of simplicial sets

$$\mathcal{N}_k(\eta_0)_*: \mathcal{N}_k(\eta_0^* Y_q / \text{res})_{\bullet} \rightarrow \mathcal{N}_k(Y_q / \text{res})_{\bullet}$$

is a simplicial homotopy equivalence. Let us assume for simplicity that $Y_q = (0, f)$ for some $f: \Delta^k \rightarrow \Psi(\partial M)^{\perp_{\mathbb{R}}}$. The argument for a general $a \neq 0$ is completely analogous. An element of $\mathcal{N}_k(Y_q / \text{res})_n$ is a family $(\phi, \underline{a}; G)$ of a sequence of real numbers $\underline{a} = (a_0 \leq \dots \leq a_k \leq 0)$, a map $\phi: [n] \rightarrow [q]$ and a continuous $G: \Delta^n \rightarrow \Psi(\partial M)^{\perp_{a_0, \dots, a_k}}$ such that

$$G(\sigma)|_{(-\epsilon, \infty)} = (\phi^* f)(\sigma)|_{(-\epsilon, \infty)}$$

for all $\sigma \in \Delta^n$ and some $\epsilon > 0$ depending on σ .

Pick a family $\{B^{\phi, \psi} \mid \phi, \psi: [n] \rightarrow [q]\}$ for f given by the damping condition 3.4. Define a simplicial map

$$\xi: \mathcal{N}_k(Y_q / \text{res})_{\bullet} \rightarrow \mathcal{N}_k(\eta_0^* Y_q / \text{res})_{\bullet}$$

by sending an element $(\phi, \underline{a}; G)$ as above to

$$\xi(\phi, \underline{a}; G) = (p: [n] \rightarrow [0], \underline{a}; \xi(G))$$

where $p: [n] \rightarrow [0]$ is the unique map, and $\xi(G): \Delta^n \rightarrow \Psi(\partial M)^{\perp_{a_0, \dots, a_k}}$ is defined by

$$\xi(G)(\sigma) := s_{\underline{a}}^*(G(\sigma)|_{(-\infty, \epsilon)}) \cup B^{\phi, \eta_0 \circ p}(\sigma)|_{(-\epsilon, 1+\epsilon)}$$

Here $s_{\underline{a}}: \mathbb{R} \rightarrow \mathbb{R}$ is a diffeomorphism that sends 0 to 1, and with the following properties. If $\underline{a} = (a_0 = \dots = a_k = 0)$ simply take $s_{\underline{a}}$ to be $t_1 = (-) + 1$. If there is a coordinate $a_i < 0$, let a_m be the biggest among them, and choose $s_{\underline{a}}$ so that it agrees with the identity in a neighborhood of $(-\infty, a_m]$, and with t_1 on a neighborhood of $[0, \infty)$. Conditions iii) and iv) of the damping property insure that the gluing is well defined, and that $\xi(G)$ agrees with $p^*(\eta_0^* f)$ around 0 as required for belonging to $\mathcal{N}_k(\eta_0^* Y_q / res)_n$. Our choice of $s_{\underline{a}}$ guarantees that $\xi(G)(\sigma)$ is still orthogonal at a_0, \dots, a_k . The map ξ is simplicial by condition ii) for the family $\{B^{\phi, \psi}\}$.

The Composite $\eta_0 \circ \xi$ sends $(\phi, \underline{a}; G)$ to $(\eta_0 \circ p, \underline{a}; \xi(G))$. Define a simplicial homotopy

$$H: \mathcal{N}_k Y_q / res_{\bullet} \times \Delta[1] \rightarrow \mathcal{N}_k Y_q / res_{\bullet}$$

by sending $(\phi, \underline{a}; G)$ and $b: [n] \rightarrow [1]$ to

$$H((\phi, \underline{a}; G), b) = (\phi \cdot b, \underline{a}; H_b(G))$$

where $\phi \cdot b$ is pointwise multiplication of maps $[n] \rightarrow [1]$, and

$$H_b(G)(\sigma) = s_{\underline{a}}^*(G(\sigma)|_{(-\infty, \epsilon)} \cup B^{\phi, \phi \cdot b}(\sigma)|_{(-\epsilon, 1+\epsilon)})$$

This is a simplicial homotopy from $\eta_0 \circ \xi$ to a map H_1 that sends $(\phi, \underline{a}; G)$ to $(\phi, \underline{a}; H_1(G))$ with

$$H_1(G)(\sigma) = s_{\underline{a}}^*(G(\sigma)|_{(-\infty, \epsilon)} \cup (\phi^* f)(\sigma)|_{(-\epsilon, 1+\epsilon)})$$

Here we used that $B^{\phi, \phi} = \phi^* f$. Thus $H_1(G)$ is a scaling G . We define a homotopy that inflates this back to G . For every \underline{a} , take a family of diffeomorphisms $s_{\underline{a}}^u: \mathbb{R} \rightarrow \mathbb{R}$ continuous in $u \in [0, 1]$ such that $s_{\underline{a}}^0 = s_{\underline{a}}$, $s_{\underline{a}}^1 = id$ and such that each $s_{\underline{a}}^u$ agrees with the identity on a neighborhood of $(-\infty, a_m]$, and with $t_u = (-) + u$ on a neighborhood of $[0, \infty)$. Define $K: \mathcal{N}_k Y_q / res_{\bullet} \times \Delta[1] \rightarrow \mathcal{N}_k Y_q / res_{\bullet}$ by sending $((\phi, \underline{a}; G), b: [n] \rightarrow [1])$ to $(\phi, \underline{a}; K_b(G))$ with

$$K_b(G)(\sigma) = (s_{\underline{a}}^{b_*}(\sigma))^*(G(\sigma)|_{(-\infty, \epsilon)} \cup (\phi^* f)(\sigma)|_{(-\epsilon, 1+\epsilon)})$$

where $b_*: \Delta^n \rightarrow \Delta^1 \cong [0, 1]$ is the induced map. This is a simplicial homotopy from H_1 to the identity.

The other composite $\xi \circ (\eta_0)_*$ sends $(p, \underline{a}; G)$ to $(p, \underline{a}; \xi(G))$ with

$$\xi(G)(\sigma) = s_{\underline{a}}^*(G(\sigma)|_{(-\infty, \epsilon)} \cup p^*(\eta_0^* f)(\sigma)|_{(-\epsilon, 1+\epsilon)})$$

There is a simplicial homotopy similar to K from this map to the identity. \square

DEFINITION 3.7. A **right inverse** for an element $G \in \Psi(\partial M)^{\perp_{0,1}}$ is an element $R \in \Psi(\partial M)$ such that

- i) $R|_{(1-\epsilon, 1+\epsilon)} = G|_{(1-\epsilon, 1+\epsilon)}$
- ii) $t_2^*(R)|_{(-\epsilon, \epsilon)} = G|_{(-\epsilon, \epsilon)}$
- iii) There is a path $\gamma: [0, 1] \rightarrow \Psi(\partial M)$ with
 - $\gamma(0) = G|_{(-\infty, 1-\epsilon)} \cup R|_{(1-\epsilon, \infty)}$
 - $\gamma(1) \in \Psi(\partial M)^{\perp_{(-\epsilon, 2+\epsilon)}}$
 - $\gamma(t)|_{(-\infty, +\epsilon)} = G|_{(-\infty, +\epsilon)}$ and $\gamma(t)|_{(2-\epsilon, \infty)} = R|_{(2-\epsilon, \infty)}$

A **left inverse** for $G \in \Psi(\partial M)^{\perp_{0,1}}$ is a $L \in \Psi(\partial M)$ satisfying

- i) $L|_{(-\epsilon, \epsilon)} = G|_{(-\epsilon, \epsilon)}$
- ii) $t_{-2}^*(L)|_{(1-\epsilon, 1+\epsilon)} = G|_{(1-\epsilon, 1+\epsilon)}$
- iii) There is a path $\delta: [0, 1] \rightarrow \Psi(\partial M)$ with

- $\delta(0) = L|_{(-\infty, \epsilon)} \cup G|_{(-\epsilon, \infty)}$
- $\delta(1) \in \Psi(\partial M)^{\perp(-1-\epsilon, 1+\epsilon)}$
- $\delta(t)|_{(-\infty, -1+\epsilon)} = L|_{(-\infty, -1+\epsilon)}$ and $\gamma(t)|_{(1-\epsilon, \infty)} = G|_{(1-\epsilon, \infty)}$

A sheaf $\Psi: Emb_d^{op} \rightarrow Top$ is called **group like at M** if every element $G \in \Psi(\partial M)^{\perp(-\infty, \epsilon)} \cap \Psi(\partial M)^{\perp(1-\epsilon, \infty)}$ has both a right and a left inverse.

Notice that the gluing conditions force a right inverse R to belong to $\Psi(\partial M)^{\perp 1, 2}$, and similarly for a left inverse $L \in \Psi(\partial M)^{\perp -1, 0}$. Also, if R is a right inverse for G then G is a left inverse for R .

EXAMPLE 3.8. We show in 4.7 that the sheaf Ψ_k is group like at M if $\partial M = N \times \mathbb{R}$ for some manifold without boundary N of dimension $d - 2$. The idea is to define a left inverse of a submanifold $G \in \Psi_k(\partial M)^{\perp 0, 1}$ as the flipped manifold $(-id)^*G$. Similarly $(2 - id)^*G$ is going to be a right inverse for G .

PROPOSITION 3.9. *Suppose that Ψ is group like and damping at M , and that orthogonal elements of $\Psi(M)$ are almost open. Then all the maps of the first kind for $res: Sing \mathcal{C}_\Psi(M) \rightarrow Sing \partial \mathcal{C}_\Psi(M)$ are weak equivalences.*

PROOF. Let β be a morphism in $\partial \mathcal{C}_\Psi(M)_q$ from Y_q to Y'_q . The value zero map $\eta_0: [0] \rightarrow [q]$ induces a commutative diagram of simplicial functors

$$\begin{array}{ccc} Y_q/res & \xrightarrow{\beta_*} & Y'_q/res \\ (\eta_0)_* \uparrow \simeq & & (\eta_0)_* \uparrow \simeq \\ \eta_0^* Y_q & \xrightarrow{(\eta_0^* \beta)_*} & \eta_0^* Y'_q \end{array}$$

where $\eta_0^* \beta$ is a morphism in $\partial \mathcal{C}_\Psi(M)_0$, and the vertical maps are equivalences by proposition 3.6 above. It is therefore enough to show that the maps of the first kind induced by morphisms in degree zero are equivalences.

Let $(b, g \in \Psi(\partial M)^{\perp \mathbb{R}})$ be an object of $Sing_0 \partial \mathcal{C}_\Psi(M) = \partial \mathcal{C}_\Psi(M)$. Recall the topological poset

$$F_{(b, g)} = \coprod_{a \leq b} colim_{0 < \epsilon} \{H \in \Psi(M)^{\perp a} \mid G|_{(b-\epsilon, \infty)} = g|_{(b-\epsilon, \infty)}\}$$

A morphism $\beta = (b \leq c, G)$ of $\partial \mathcal{C}_\Psi(M)$ from (b, g) to (c, h) induces a commutative diagram

$$\begin{array}{ccc} |(b, g)/res| & \xrightarrow{|\beta_*|} & |(c, h)/res| \\ \downarrow \simeq & & \downarrow \simeq \\ F_{(b, g)} & \xrightarrow{G_*} & F_{(c, h)} \end{array}$$

where G_* sends a morphism $(a \leq a', H)$ of $F_{(b, g)}$ to $(a \leq a', H|_{(-\infty, b+\epsilon)} \cup G|_{(b-\epsilon, \infty)})$. The vertical maps are induced by evaluation, and are levelwise weak equivalences on the nerve as showed in 3.3. Now let $F_{(b, g)}^t$ be the topological poset with same underlying set as $F_{(b, g)}$, but topologized using the standard topology of \mathbb{R} on the real coordinate $a \leq b$. The inclusion of \mathbb{R} with the discrete topology into \mathbb{R} with

the standard topology induces a commutative diagram

$$\begin{array}{ccc} F_{(b,g)} & \xrightarrow{G_*} & F_{(c,h)} \\ \simeq \downarrow & & \downarrow \simeq \\ F_{(b,g)}^t & \xrightarrow{G_*} & F_{(c,h)}^t \end{array}$$

We are going to show in lemma 3.11 below that the vertical maps are equivalences. Therefore it is enough to show that on the nerve

$$G_* : \mathcal{N}_k F_{(b,g)}^t \longrightarrow \mathcal{N}_k F_{(c,h)}^t$$

is a homotopy equivalence for all k . We assume for simplicity that $b = 0$ and $c = 1$, the general argument being a translation of this case.

Take a right inverse R for G in the sense of definition 3.7. Define a map

$$G^* : \mathcal{N}_k F_{(1,h)}^t \longrightarrow \mathcal{N}_k F_{(0,g)}^t$$

by sending $(a_0 \leq \dots \leq a_k \leq 1; H)$ to $(\underline{a} - 2 = (a_0 - 2 \leq \dots \leq a_k - 2 \leq 0); G^*(H))$ where

$$G^*(H) = t_2^*(H|_{(-\infty, 1+\epsilon)} \cup R|_{(1-\epsilon, 2+\epsilon)} \cup g|_{(2-\epsilon, \infty)})$$

The map $G^* \circ G_*$ sends a $(\underline{a} \leq 0; H)$ to $(\underline{a} - 2 \leq 0, G^*G_*H)$ with

$$G^*G_*H = t_2^*(H|_{(-\infty, +\epsilon)} \cup G|_{(-\epsilon, 1+\epsilon)} \cup R|_{(1-\epsilon, 2+\epsilon)} \cup g|_{(2-\epsilon, \infty)})$$

Define a map $K : \mathcal{N}_k F_{(0,g)}^t \times I \longrightarrow \mathcal{N}_k F_{(0,g)}^t$ by sending $(\underline{a}; H)$ and $u \in [0, 1]$ to $(\underline{a} - 2; K_u(H))$ with

$$K_u(H) = t_2^*(H|_{(-\infty, +\epsilon)} \cup \gamma(u)|_{(-\epsilon, \infty)})$$

where γ is a path for the right inverse (cf. 3.7). This defines a homotopy from $G^* \circ G_*$ to a map K_1 given by

$$K_1(H) = t_2^*(H|_{(-\infty, +\epsilon)} \cup g|_{(-\epsilon, \infty)})$$

This is because $\gamma(1)$ is orthogonal on $(-\epsilon, 2 + \epsilon)$, and it agrees with G on $(-\epsilon, \epsilon)$ which is equal to $g|_{(-\epsilon, \epsilon)}$. Now g is orthogonal on all \mathbb{R} , and therefore $\gamma(1)$ must agree with g on $(-\epsilon, 2 + \epsilon)$. Finally define a homotopy K' that sends $(\underline{a}; H)$ and $u \in [0, 1]$ to $(\underline{a} - 2 + 2u, K'_u(H))$ with

$$K'_u(H) = t_{2-2u}^*(H|_{(-\infty, +\epsilon)} \cup g|_{(-\epsilon, \infty)})$$

This shows that G^* is a left homotopy inverse for G_* . Similarly a left inverse L for G defines a right homotopy inverse for G_* . \square

PROPOSITION 3.10. *Suppose that Ψ is group like and damping at M , and that orthogonal elements of $\Psi(M)$ are almost open. Then all the maps of the second kind for $\text{res} : \text{Sing}\mathcal{C}_\Psi(M) \longrightarrow \text{Sing}\partial\mathcal{C}_\Psi(M)$ are weak equivalences.*

PROOF. Notice that for a general map $\alpha : [p] \longrightarrow [q]$ there is a commutative diagram

$$\begin{array}{ccc} \alpha^* Y_q / \text{res} & \xrightarrow{\alpha_*} & Y_q / \text{res} \\ \eta_* \uparrow & \nearrow (\alpha \circ \eta)_* & \\ \eta^*(\alpha^* Y_q) / \text{res} & & \end{array}$$

for any choice of $\nu: [0] \longrightarrow [p]$, and $\alpha \circ \nu$ has also source $[0]$. Therefore it is enough to show that every map with source zero $\eta: [0] \longrightarrow [q]$ induces a weak equivalence η_* . Suppose that $Y_q = (0, f)$ for a map $f: \Delta^n \longrightarrow \Psi(\partial M)^{\perp_{\mathbb{R}}}$, and consider the diagram

$$\begin{array}{ccc} \eta^*(0, f)/res & \xrightarrow{\eta_*} & (0, f)/res \\ B_*^{\eta \circ p, \eta_0 \circ p} \downarrow & \nearrow (\eta_0)_* & \\ \eta_0^*(0, f)/res & & \end{array}$$

where we recall that $p: [n] \longrightarrow [0]$ is the unique map, and $\{B^{\phi, \psi}\}$ is a family for f given by the damping condition. The functor $B_*^{\eta \circ p, \eta_0 \circ p}$ sends $(p, \underline{a} \leq 0; G)$ to $(p, \underline{a} \leq 0; B_*(G))$ with

$$B_*(G) = s_{\underline{a}}^*(G|_{(-\infty, \epsilon)} \cup B^{\eta \circ p, \eta_0 \circ p}|_{(-\epsilon, \infty)})$$

where $s_{\underline{a}}: \mathbb{R} \longrightarrow \mathbb{R}$ is a diffeomorphism that sends 0 to 1 with the same properties as in the proof of 3.6. The diagram does not commute strictly, but there is a homotopy between $(\eta_0)_* \circ B_*^{\eta \circ p, \eta_0 \circ p}$ defined in a completely analogous way as the homotopy of the proof of 3.6. The map η_0 is a homotopy equivalence by 3.6, and $B_*^{\eta \circ p, \eta_0 \circ p}$ is the scaling of a map of the first kind. Choosing right and left inverses for $B^{\eta \circ p, \eta_0 \circ p}$ defines left and right homotopy inverses for $B_*^{\eta \circ p, \eta_0 \circ p}$ analogously as in the proof of 3.9 above. \square

Given $(b, g) \in \text{Ob} \partial \mathcal{C}_{\Psi}(M)$, let $F_{(b, g)}^t$ be the topological poset with the same underlying set as $F_{(b, g)}$ but topologized using the standard topology of \mathbb{R} on the real coordinate.

LEMMA 3.11. *Suppose that orthogonal elements of $\Psi(M)$ are almost open. The map of topological posets $F_{(b, g)} \longrightarrow F_{(b, g)}^t$ induced by the inclusion of \mathbb{R} with the discrete topology into \mathbb{R} with the standard topology is a weak equivalence on classifying spaces*

$$BF_{(b, g)} \xrightarrow{\simeq} BF_{(b, g)}^t$$

PROOF. Again assume for simplicity that $b = 0$. The projection off the real coordinate induces a commutative diagram

$$\begin{array}{ccc} BF_{(0, g)} & \longrightarrow & BF_{(0, g)}^t \\ & \searrow \simeq & \downarrow \\ & & \Psi(M; g) \end{array}$$

and the diagonal map is an equivalence by 3.3. Therefore it is enough to show that the projection

$$BF_{(0, g)}^t \longrightarrow \Psi(M; g)$$

is a weak equivalence. Let \mathcal{E}_g be the topological poset

$$\mathcal{E}_g = \coprod_{a \leq 0} U_a^g$$

where

$$U_a^g := U_a \cap \Psi(M; g) = \text{colim}_{\epsilon > 0} \{G \in U_a \mid G|_{(-\epsilon, \infty)} = g|_{(-\epsilon, \infty)}\}$$

Here the real coordinate $a \leq 0$ has the standard topology, and U_a is the open subset of $\Psi(M)$ weakly equivalent to $\Psi(M)^{\perp_a}$ given by the almost open condition (see 2.5). Notice that since U_a is open in $\Psi(M)$ the colimit topology on U_a^g is the same as the topology as a subspace of $\Psi(M; g)$. The poset structure is defined as usual by $(a, G) \leq (a', H)$ if $a \leq a'$ and $G = H$. Since the maps $\Psi(M)^{\perp_a} \rightarrow U_a$ are weak equivalences, the map above factors as

$$BF_{(0,g)}^t \xrightarrow{\cong} B\mathcal{E}_g \xrightarrow{pr} \Psi(M; g)$$

We define an inverse in homotopy groups for $pr: B\mathcal{E}_g \rightarrow \Psi(M; g)$. The author learned the following trick from Søren Galatius and Oscar Randall-Williams. Let $f: S^n \rightarrow \Psi(M; g)$ be a continuous map representing an element of $\pi_n \Psi(M; g)$. Since the family $\{U_a\}$ covers $\Psi(M)$, for every $x \in S^n$ there is a $a_x \leq 0$ with $f(x) \in U_{a_x}^g$. Since every $U_{a_x}^g$ is open in $\Psi(M; g)$, the family $V_{a_x} := f^{-1}(U_{a_x}^g)$ is an open cover of S^n . Pick a finite sequence $a_0 < \dots < a_k$ such that V_{a_0}, \dots, V_{a_k} covers the sphere, and choose a partition of unity ϕ_0, \dots, ϕ_k subordinate to the covering $\{V_{a_i}\}$. Such a choice of coordinates for every representative $f: S^n \rightarrow \Psi(M; g)$ of homotopy class defines a map

$$s: \pi_n \Psi(M; g) \rightarrow \pi_n B\mathcal{E}_g$$

given by

$$s(f)(x) = [(a_0 \leq \dots \leq a_k; f(x)), (\phi_0(x), \dots, \phi_k(x))]$$

where $(\phi_0(x), \dots, \phi_k(x)) \in \Delta^k$. The map s is a section for the projection map $\pi_n B\mathcal{E}_g \rightarrow \pi_n \Psi(M; g)$.

Given a map $h: S^n \rightarrow B\mathcal{E}_g$ we define a homotopy between $s(pr \circ h)$ and h . At every point $x \in S^n$ the element $h(x)$ is an equivalence class of the form

$$h(x) = [(b_0(x) \leq \dots \leq b_{m_x}(x) \leq 0; G_x), (t_0(x), \dots, t_{m_x}(x))]$$

for a $G_x \in U_{b_0(x)}^g \cap \dots \cap U_{b_{m_x}(x)}^g \subset \Psi(M; g)$ and a $\underline{t}(x) = (t_0(x), \dots, t_{m_x}(x)) \in \Delta^{m_x}$. Notice that we can assume $b_i(x) \neq b_j(x)$ for $i \neq j$. Let $\underline{a} = (a_1 < \dots < a_k)$ be the sequence chosen for the map $pr \circ h$ for the definition of $s(pr \circ h)$, and $\underline{\phi}(x) = (\phi_1(x), \dots, \phi_k(x))$ the associated partition of unity. Denote $\underline{c}(x)$ the ordered union of the sequences $\underline{b}(x) = (b_0(x) \leq \dots \leq b_{m_x}(x))$ and \underline{a} . Let $\omega_x: [k] \rightarrow [m_x + k + 1]$ be the unique map such that $\omega_x^* \underline{c}(x) = \underline{a}$, and $\rho_x: [m_x] \rightarrow [m_x + k + 1]$ the one satisfying $\rho_x^* \underline{c}(x) = \underline{b}(x)$. Define a map $H: S^n \times [0, 1] \rightarrow B\mathcal{E}_g$ by

$$H(x, u) = [(\underline{c}(x); G_x), u \cdot (\omega_x)_* \underline{\phi}(x) + (1 - u) \cdot (\rho_x)_* \underline{t}(x)]$$

This is a homotopy from h to $s(pr \circ h)$. □

REMARK 3.12. This last lemma essentially shows that we can endow all our constructions with either the standard or the discrete topology on the real coordinate without changing the weak homotopy type. However, it is easier to deal with the hypothesis of Quillen's theorem B if one uses the discrete topology.

4. The Madsen-Weiss theorem as a relative h-principle

Let $\Psi_k: Emb_d \rightarrow Top$ be the sheaf of [3]. It associates to a manifold M the space $\Psi_k(M)$ of k -dimensional submanifolds of M that are closed as a subset. This space is topologized by defining a neighborhood $\mathcal{V}_{K,W}(G)$ of $G \in \Psi_k(M)$ for each

pair (K, W) , where $K \subset M$ is compact and $W \subset Emb(G, M)$ is a neighborhood of the inclusion $G \subset M$ for the weak C^∞ -topology. The neighborhood is defined by

$$\mathcal{V}_{K,W}(G) = \{P \in \Psi_k(M) \mid P \cap K = j(G) \cap K \text{ for some } j \in W\}$$

The functor Ψ_k sends an embedding $e: M \rightarrow M'$ to

$$e^*(G) := e^{-1}(G)$$

The cobordism category of k -dimensional bordisms in \mathbb{R}^d of [4] is the topological category $\mathcal{C}_{k,d}$ with objects space

$$\coprod_{a \in \mathbb{R}} \{g \in \Psi_k(\mathbb{R}^{d-1}) \mid g \text{ is compact}\}$$

and morphisms space

$$\coprod_{a \leq b \in \mathbb{R}} \text{colim}_{\epsilon > 0} \{G \in \Psi_k(\mathbb{R}^{d-1} \times [a, b]) \mid G \perp_{[a, a+\epsilon]}, G \perp_{(b-\epsilon, b]}, G \text{ is compact}\}$$

Here we denoted $G \perp_J$ if $G \cap (\mathbb{R}^{d-1} \times J) = N \times J$ for some $(k-1)$ -submanifold $N \subset \mathbb{R}^{d-1}$. Source and target take intersection with $\mathbb{R}^{d-1} \times a$ and $\mathbb{R}^{d-1} \times b$ respectively, and composition of morphisms is defined by union. Notice that if we dropped the compactness conditions this category would be isomorphic to $\partial\mathcal{C}_\Psi(M)(\mathbb{R}^{d-1} \times (-\infty, 0])$ of §2.

In this section we use our main theorem to give another proof of the following theorem. Let $\mathcal{G}_{k,d}$ be the Grassmannians of k -planes in \mathbb{R}^d , and

$$\gamma_{k,d}^\perp = \{(V, v) \in \mathcal{G}_{k,d} \times \mathbb{R}^d \mid v \perp V\}$$

be the complement of the tautological bundle over $\mathcal{G}_{k,d}$.

THEOREM 4.1 ([7],[4]). *There is a weak equivalence*

$$BC_{k,d} \simeq \Omega^{d-1}Th(\gamma_{k,d}^\perp)$$

where $Th(\gamma_{k,d}^\perp)$ is the Thom space of the complement of the tautological bundle.

PROOF SKETCH. Following [1], we prove the theorem using the relative h-principle map $\Psi_k(M; g_0) \rightarrow \Psi_k^*(M; h(g_0))$ for $M = D^{d-1} \times \mathbb{R}$ and boundary condition $g_0 = \emptyset \in \Psi_k(D^{d-1} \times \mathbb{R})$. By expanding the interior of D^{d-1} to \mathbb{R}^{d-1} , we see that $\Psi_k(D^{d-1} \times \mathbb{R}; \emptyset)$ is homeomorphic to the space of k -submanifolds of \mathbb{R}^d which are bounded in the first $d-1$ components. This is the space $\mathcal{D}_{k,(1,d)}$ of [1], which is weakly equivalent to the classifying space of $\mathcal{C}_{k,d}$ via a zig zag

$$BC_{k,d} \xleftarrow{\simeq} BD \xrightarrow{\simeq} \mathcal{D}_{k,(1,d)} \cong \Psi_k(D^{d-1} \times \mathbb{R};)$$

through a topological poset D . This construction is analogous to our proof of 2.4 and we refer to [1] for the details.

Recall that $\Psi_k^*(D^{d-1} \times \mathbb{R})$ is defined in terms of sections of a certain bundle over $D^{d-1} \times \mathbb{R}$. Since $D^{d-1} \times \mathbb{R}$ is canonically contractible this bundle is trivial, and the space $\Psi_k^*(D^{d-1} \times \mathbb{R})$ is canonically homeomorphic to the space of maps $D^{d-1} \times \mathbb{R} \rightarrow \Psi_k(\mathbb{R}^d)$. Under this identification the boundary condition $h(\emptyset)$ is a constant map, and contracting the \mathbb{R} component gives a homotopy equivalence

$$\Psi_k^*(D^{d-1} \times \mathbb{R}; h(\emptyset)) \simeq \Omega^{d-1}\Psi_k(\mathbb{R}^d)$$

The map $Th(\gamma_{k,d}^\perp) \longrightarrow \Psi_k(\mathbb{R}^d)$ that sends a pair (V, v) to $V + v \subset \mathbb{R}^d$, and the basepoint ∞ to \emptyset is a weak equivalence (see [1]). All together we get a diagram

$$BC_{k,d} \simeq \Psi_k(D^{d-1} \times \mathbb{R}; \emptyset) \xrightarrow{h} \Psi_k^*(D^{d-1} \times \mathbb{R}; h(\emptyset)) \simeq \Omega^{d-1}Th(\gamma_{k,d}^\perp)$$

We prove in 4.4, 4.5, 4.6 and 4.7 below that the sheaf Ψ_k satisfies the conditions of the main theorem for $M = D^{d-1} \times \mathbb{R}$, and therefore the relative h-principle map

$$\Psi_k(D^{d-1} \times \mathbb{R}; \emptyset) \xrightarrow{h} \Psi_k^*(D^{d-1} \times \mathbb{R}; h(\emptyset))$$

is a weak equivalence. \square

REMARK 4.2. Our main theorem gives a relative h-principle for any boundary condition $g \in \Psi_k(\partial(D^{d-1} \times \mathbb{R})) = \Psi_k((S^{d-2} \times \mathbb{R}) \times \mathbb{R})^{\perp \mathbb{R}}$ that is not necessarily the empty manifold. One can define a category $\mathcal{C}_{k,d}^g$ whose morphisms are bordisms in $D^{d-1} \times [a, b]$ that agree with g in a neighborhood of $(\partial D^{d-1}) \times [a, b]$. Our proof gives an equivalence

$$BC_{k,d}^g \simeq Map_{h(g)}(D^{d-1} \times \mathbb{R}, \Psi_k(\mathbb{R}^d))$$

where $Map_{h(g)}(D^{d-1} \times \mathbb{R}, \Psi_k(\mathbb{R}^d))$ is the space of maps $D^{d-1} \times \mathbb{R} \longrightarrow \Psi_k(\mathbb{R}^d)$ that agree with $h(g): (S^{d-2} \times \mathbb{R}) \times \mathbb{R} \longrightarrow \Psi_k(\mathbb{R}^d)$ near the boundary. Here $h(g)$ is the scanning of the boundary condition g , and it is in general not a constant map.

REMARK 4.3. If $\theta: B \longrightarrow BGL_d$ is a fibration, a tangential θ -structure on a d -manifold $W \subset \mathbb{R}^k$ is a lift along θ for the classifying map $W \longrightarrow \mathcal{G}_{k,d}$ of the tangent bundle. There are analogues Ψ_k^θ and $\mathcal{C}_{k,d}^\theta$ where the manifolds are equipped with a tangential θ -structure, and a Madsen-Weiss theorem

$$BC_{k,d}^\theta \simeq \Omega^{d-1}Th(\theta^* \gamma_{k,d}^\perp)$$

(see [4], [2], [1], [3]). All the steps of our proof go through in this more general situation, except the group-like condition that might fail for a general θ . It does however hold in the case of oriented manifolds, when $\theta: \mathcal{G}_{k,d}^+ \longrightarrow \mathcal{G}_{k,d}$ is the projection map from the oriented Grassmannians.

Condition (1) of the main theorem holds for Ψ_k for deep results of [8] and [5]. Recall that we are considering Ψ_k as a sheaf on d -manifolds, and therefore we define $\Psi_k(\partial M) = \Psi_k(\partial M \times \mathbb{R})$.

PROPOSITION 4.4 ([8],[5]). *If M is an open manifold, the h-principle maps $\Psi_k(M) \longrightarrow \Psi_k^*(M)$ and $\Psi_k(\partial M) \longrightarrow \Psi_k^*(\partial M)$ are weak equivalences.*

PROOF. In [8] the author proves that the sheaf

$$\mathcal{O}^{op}(W) \longrightarrow Top$$

defined on the category of open subsets of any d -manifold W by restricting Ψ_k is microflexible. By Gromov's theorem [5] the h-principle maps $\Psi_k(M) \longrightarrow \Psi_k^*(M)$ and $\Psi_k(\partial M) \longrightarrow \Psi_k^*(\partial M)$ are weak equivalences for M open. \square

Notice that for $M = D^{d-1} \times \mathbb{R}$ the space $\Psi^*(D^{d-1} \times \mathbb{R})$ is canonically equivalent to $\Psi(\mathbb{R}^d)$ since $D^{d-1} \times \mathbb{R}$ is contractible. The h-principle map $\Psi(D^{d-1} \times \mathbb{R}) \longrightarrow \Psi^*(D^{d-1} \times \mathbb{R})$ is then always an equivalence (this is however not true in general for $\partial(D^{d-1} \times \mathbb{R})$).

The main ingredient of the following is a result of [4], where the authors show that the inclusion of orthogonal elements into transverse elements is a weak equivalence.

PROPOSITION 4.5 ([4]). *The orthogonal elements of the sheaf Ψ_k are almost open at M , if $\partial M = N \times \mathbb{R}^n$ for some compact manifold N and $0 \leq n \leq d$.*

PROOF. Given a real number a , define $\Psi_k(M)^{\pitchfork a} \subset \Psi_k(M)$ to be the subspace of manifolds that intersect $\partial M \times a$ transversally. Clearly every submanifold that intersects $\partial M \times a$ orthogonally belongs to $\Psi_k(M)^{\pitchfork a}$, and this defines a map

$$\Psi_k(M)^{\perp a} \longrightarrow \Psi_k(M)^{\pitchfork a}$$

This is showed to be a weak equivalence in [1] and [3]. Unfortunately, the space $\Psi_k(M)^{\pitchfork a}$ is not open in $\Psi_k(M)$ unless M is compact. If $\partial M = N \times \mathbb{R}^n$, let $B^\delta \subset \mathbb{R}^n$ be the open ball of radius $1 + \delta$ centered at the origin. Define U_a as the subspace of $\Psi_k(M)$ of submanifolds that are transverse to $\partial M \times a$ in a neighborhood of the unit compact ball

$$U_a := \{G \in \Psi_k(M) \mid (G \cap N \times B^\delta) \pitchfork \partial M \times a \text{ for some } \delta > 0\}$$

Clearly $\Psi_k(M)^{\pitchfork a} \subset U_a$. The subspace U_a is open, since given a $G \in U_a$ one can find a neighborhood of the inclusion $W \subset \text{Emb}(G, M)$ small enough so that $\mathcal{V}_{K, N \times K} \subset \Psi_k(M)^{\pitchfork a}$ for every compact subset $K \subset \mathbb{R}^n$ containing the compact unit ball.

It remains to show that the inclusion $\Psi_k(M)^{\pitchfork a} \longrightarrow U_a$ is a weak equivalence. Given a map $f: S^n \longrightarrow U_a$ one can choose $\delta > 0$ such that $f(x) \cap N \times B^\delta$ is transverse to $\partial M \times a$ for all x . Pick a diffeomorphism $\phi: \mathbb{R}^n \longrightarrow B^\delta$, and define a map $r(f): S^n \longrightarrow \Psi_k(M)^{\pitchfork a}$ by

$$r(f)(x) = (id_N \times \phi)^* f(x)$$

This construction defines an inverse for the inclusion map on the n -th homotopy group. \square

The main ingredient for the next result is the smooth approximation theorem for Ψ_k of [3].

PROPOSITION 4.6. *The sheaf Ψ_k is damping at any manifold M .*

PROOF. For any d -manifold W , define a map $F: [0, 1] \longrightarrow \Psi_k(W)$ to be smooth if the subset

$$E = \{(p, t) \in W \times [0, 1] \mid p \in F(t)\}$$

is a submanifold of $W \times [0, 1]$. By a result of [3], one can homotope compact families of continuous maps $[0, 1] \longrightarrow \Psi_k(W)$ to compact families of smooth maps keeping fixed subintervals of $[0, 1]$ where the family was already smooth, just as for usual manifolds.

Let $f: \Delta^q \longrightarrow \Psi_k(\partial M)^{\perp \mathbb{R}}$ be a continuous map. Since the values of f are orthogonal on \mathbb{R} , the submanifolds $f(x) \subset \partial M \times (0, \infty)$ are of the form

$$f(x) = N_x \times (0, \infty)$$

for some $(k-1)$ -dimensional submanifold N_x of ∂M . This defines a continuous map $\bar{f}: \Delta^q \longrightarrow \Psi_{k-1}(\partial M)$ by $\bar{f}(x) = N_x$. Here we consider Ψ_{k-1} as a sheaf on $(d-1)$ -dimensional manifolds

$$\Psi_{k-1}: \text{Emb}_{d-1}^{op} \longrightarrow \text{Top}$$

so that $\Psi_{k-1}(\partial M)$ consists of actual submanifolds of ∂M and not of $\partial M \times \mathbb{R}$. Let $\alpha: [0, 1] \rightarrow [0, 1]$ be a monotone smooth map with constant value 0 in a neighborhood of 0, with constant value 1 in a neighborhood of 1, and that agrees with the identity away from zero and one. Consider the continuous map $H: \Delta^q \times \Delta^q \times [0, 1] \rightarrow \Psi_{k-1}(\partial M)$ defined by

$$H_{x,y}(t) = \bar{f}(x \cdot (1 - \alpha(t)) + y \cdot \alpha(t))$$

For every fixed $x, y \in \Delta^q$, the map

$$H_{x,y}: [0, 1] \rightarrow \Psi_{k-1}(\partial M)$$

is smooth near 0 and 1 by our choice of α . Approximate H to a map $K: \Delta^q \times \Delta^q \times [0, 1] \rightarrow \Psi_{k-1}(\partial M)$ such that $K_{x,y}: [0, 1] \rightarrow \Psi_{k-1}(\partial M)$ is smooth for all $x, y \in \Delta^q$ and that agrees with $H_{x,y}$ around 0 and 1. Since $K_{x,y}$ is smooth, the space

$$B_{x,y} := \{(p, t) \in \partial M \times [0, 1] \mid p \in K_{x,y}(t)\}$$

is a smooth k -dimensional submanifold of $\partial M \times [0, 1]$. Since $H_{x,y}$ is constant near 0 and 1, the submanifold $B_{x,y}$ agrees with $f(x)$ near $\partial M \times 0$ and with $f(y)$ near $\partial M \times 1$. Given $\phi, \psi: [n] \rightarrow [q]$ define a continuous map $B^{\phi, \psi}: \Delta^n \rightarrow \Psi_k(\partial M)$ for the damping condition by

$$B^{\phi, \psi}(x) = B_{\phi_* x, \psi_* x}$$

as a submanifold of $\partial M \times \mathbb{R}$ extended orthogonally outside $\partial M \times [0, 1]$. \square

PROPOSITION 4.7. *If $\partial M = N \times \mathbb{R}$ for some $(d-2)$ -manifold N without boundary, Ψ_k is group like at M .*

PROOF. Let us define a right inverse for $G \in \Psi_k(\partial M)^{\perp(-\infty, \epsilon)} \cap \Psi_k(\partial M)^{\perp(1-\epsilon, \infty)}$. This is the same as a submanifold of $\partial M \times [0, 1]$ that intersects the boundary components orthogonally. Define

$$\bar{G} = (2 - id)^* G$$

to be the flip of G along $\partial M \times 1$. Since G intersects the boundary orthogonally $\bar{G}|_{(1-\epsilon, 1+\epsilon)} = G|_{(1-\epsilon, 1+\epsilon)}$. By abuse of notation we write

$$G \cup \bar{G} := G|_{(-\infty, 1+\epsilon)} \cup \bar{G}|_{(1-\epsilon, \infty)}$$

Let us define the path $\gamma: I \rightarrow \Psi_k(\partial M)$ from $G \cup \bar{G}$ to an element orthogonal over \mathbb{R} . Recall that we are assuming $\partial M = N \times \mathbb{R}$. For every $t \in [0, 1]$ let us define a diffeomorphism ϕ_t of $\mathbb{R} \times \mathbb{R}$ by

$$\phi_t(u, v) := (u + \frac{t}{1-t}, v)$$

The topology on Ψ_k is defined so that the path $\omega: [0, 1] \rightarrow \Psi_k(\partial M)$ defined for $0 \leq t < 1$ by

$$\omega(t) = (id_N \times \phi_t)^*(G \cup \bar{G})$$

and $\omega(1) = \emptyset$ is continuous. Similarly, the path $\omega': [0, 1] \rightarrow \Psi_k(\partial M)$ defined for $0 \leq t < 1$ by

$$\omega'(t) = (id_N \times \phi_t)^*(\alpha^* G|_{(-\epsilon, \epsilon)})$$

and by $\omega'(1) = \emptyset$ is also continuous. Here $\alpha: \mathbb{R} \rightarrow (-\epsilon, \epsilon)$ is an orientation preserving diffeomorphism. The concatenation γ' of ω with the inverse path of ω' is a path between $G \cup \bar{G}$ and the manifold $\alpha^* G|_{(-\epsilon, \epsilon)}$ which is orthogonal on \mathbb{R} . The path γ' is however not constant on a neighborhood of $(-\infty, 0] \cup [1, \infty)$, but

notice that $\gamma'(0)$ and $\gamma'(1)$ agree on $(-\infty, \epsilon) \cup (1 - \epsilon, \infty)$. In order to obtain a path constant outside $[0, 1]$ replace the family of diffeomorphisms $\phi_t: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with a family of diffeomorphisms $\psi_t: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with

- $\psi_t = \phi_t$ on $\mathbb{R} \times (\frac{2\epsilon}{3}, 1 - \frac{2\epsilon}{3})$
- $\psi_t = id$ on $\mathbb{R} \times ((-\infty, \frac{\epsilon}{3}) \cup (1 - \frac{\epsilon}{3}, \infty))$
- $\psi_0 = id$

A similar construction using the flip around 0 defines a left inverse for G . \square

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