Contemporary Mathematics
Volume **708**, 2018
http://dx.doi.org/10.1090/conm/708/14262

A Dundas-Goodwillie-McCarthy theorem for split square-zero extensions of exact categories

Emanuele Dotto

Abstract. Given a bimodule M over an exact category C, we define an exact category $C \ltimes M$ with a projection to C. This construction classifies certain split square-zero extensions of exact categories. We show that the trace map induces an equivalence between the relative K-theory of $C \ltimes M$ and its relative topological cyclic homology, after p-completion. When applied to the bimodule $\hom(-, - \otimes_A M)$ on the category of finitely generated projective modules over a ring A we recover the Dundas-Goodwillie-McCarthy theorem for split square-zero extensions of rings.

Introduction

The celebrated Dundas-Goodwillie-McCarthy theorem of [**McC97**], [**Dun97**], [**DGM13**] establishes a relationship between the K-theory and the topological cyclic homology of a nilpotent extension of ring spectra, and it led to several fundamental calculations of algebraic K-theory. The proof builds on the base case of split square-zero extensions of rings: if A is a ring and M is an A-bimodule, the cyclotomic trace map induces an equivalence between the reduced K-theory of the semi-direct product $A \ltimes M$ and its reduced topological cyclic homology. The methods employed in the proof involve objects which are defined in the more general framework of bimodules over linear categories. The aim of this paper is to introduce the notion of split square-zero extension of exact categories, and to show that the arguments of [**DGM13**] in fact prove a stronger result, which compares the K-theory and the TC of these more general extensions.

A bimodule over a category enriched in Abelian groups C is an enriched functor $M \colon C^{op} \otimes C \to Ab$ to the category of Abelian groups. We define the semi-direct product of C and M to be the category $C \ltimes M$ with the same objects as C, morphism groups

$$(C \ltimes M)(a,b) = C(a,b) \oplus M(a,b),$$

and where the composition of $(f,m)\colon b \to c$ and $(g,n)\colon a \to b$ is defined by the formula

$$\big(f,m\big) \circ \big(g,n\big) = \big(f \circ g, f_* n + g^* m \in M(a,c)\big)$$

(see §1). When the category C has one object this construction recovers the standard semi-direct product of rings and bimodules.

2010 *Mathematics Subject Classification.* Primary 19D55, 18E10.

If C has an exact structure and M is exact in both variables, we show in 1.8 that $C \ltimes M$ has an induced exact structure. Projecting off the module summand on morphisms gives an exact functor $p\colon C \ltimes M \to C$. The aim of this paper is to compare the relative K-theory of $C \ltimes M$ and its relative topological cyclic homology.

Main Theorem. *Let C be an exact category and $M\colon C^{op} \otimes C \to Ab$ an additive functor which is exact in both variables. Then the square of spectra*

$$\begin{tikzcd} K(C \ltimes M) \arrow[r, "\mathrm{Tr}"] \arrow[d, "p_*"'] & \mathrm{TC}(C \ltimes M) \arrow[d, "p_*"] \\ K(C) \arrow[r, "\mathrm{Tr}"'] & \mathrm{TC}(C) \end{tikzcd}$$

induced by the cyclotomic trace map is homotopy cartesian after completion at any prime.

Example. Suppose that $C = \mathcal{P}_A$ is the category of finitely generated projective modules over a ring A, and that M is an A-bimodule. There is an equivalence of exact categories

$$\mathcal{P}_A \ltimes \hom_A(-, - \otimes_A M) \simeq \mathcal{P}_{A \ltimes M}$$

over $\mathcal{P}_A$ (see 1.11), and our main theorem recovers the theorem of [**McC97**] for the split square-zero extension of rings $A \ltimes M \to A$.

Example. The exactness condition on the bimodule $M\colon C^{op} \otimes C \to Ab$ is automatically satisfied if the exact structure on C is split-exact. Thus our theorem always applies to direct-sum K-theory. One can for instance consider the category $C = \mathrm{Vect}_X$ of vector bundles over a scheme X and the bimodule $\hom(-, - \otimes V)$ for a fixed vector bundle V, and obtain the corresponding homotopy cartesian square for direct-sum K-theory.

Example. The theorem above applies also to categories that are not split-exact. Consider for example the category $\mathcal{M}_A$ of finitely generated modules over a ring A. Given a ring homomorphism $\phi\colon R \to A$, we equip $\mathcal{M}_A$ with the exact structure defined by the exact sequences in $\mathcal{M}_A$ which become split-exact after restricting scalars, that is whose image under the functor $\phi^*\colon \mathcal{M}_A \to Mod_R$ to the category of R-modules is split-exact (see [**Hel58**, §7]). Then any R-bimodule M defines a bimodule on the category $\mathcal{M}_A$ by sending two finitely generated A-modules P and Q to

$$\hom_R(\phi^* P, (\phi^* Q) \otimes_R M).$$

This bimodule is exact in each variable.

In §1 we show that the construction $C \ltimes M \to C$ classifies certain split square-zero extensions of exact categories: the exact functors $p\colon D \to C$ which admit an essentially surjective exact splitting, and whose kernel has zero composition and defines a bi-exact bimodule on C. In §2 we define an exact structure on $C \ltimes M$ and examine its K-theory. We employ the techniques of [**DGM13**] to determine a decomposition

$$K(C \ltimes M) \simeq \coprod_{c \in S_\bullet C} S_\bullet M(c, c),$$

where $S_\bullet M$ is an extension of the bimodule M to the $S_\bullet$-construction of C. The main ingredient of this decomposition is that the zero section induces a levelwise essentially surjective functor $S_\bullet C \to S_\bullet(C \ltimes M)$, which might be surprising when C is not split-exact. Finally in §3 we run the argument of [**McC97**] and use functor calculus on the decomposition of $K(C \ltimes M)$ above to prove the main theorem.

1. Split square-zero extensions of exact categories and bimodules

Let Ab be the symmetric monoidal category of Abelian groups with respect to the tensor product.

Definition 1.1. A split extension of Ab-enriched categories is a triple (p, s, U), where $p\colon D \to C$ and $s\colon C \to D$ are Ab-enriched functors, $U\colon p \circ s \Rightarrow \mathrm{id}_C$ is a natural isomorphism, and the section s is essentially surjective.

We call (p, s, U) square-zero if for every pair of composable morphisms f and g in D with $p(f) = 0$ and $p(g) = 0$ in C, the composite $g \circ f$ is zero in D.

If C and D are exact categories we say that (p, s, U) is a split extension of exact categories if it is a split extension on the underlying Ab-enriched categories and p and s exact functors.

We observe that the isomorphism $U\colon p \circ s \Rightarrow \mathrm{id}_C$ insures that s is essentially injective. Since it is assumed to be essentially surjective, it follows that D and C have essentially the same objects.

Example 1.2. Split square-zero extensions of Ab-enriched categories with one object are precisely the split square-zero extensions of rings $f\colon R \to A$. These are all of the form $A \ltimes M \to A$ for some A-bimodule M, where $A \ltimes M$ is the Abelian group $A \oplus M$ with multiplication

$$(a, m) \cdot (b, n) = (ab, an + mb).$$

What follows is the generalization of this classification for exact categories.

Definition 1.3. A bimodule over an Ab-enriched category is an Ab-enriched functor $M\colon C^{op} \otimes C \to Ab$. For a morphism $f\colon a \to b$ and an object c in C we denote

$$f_* = M(\mathrm{id}_c \otimes f)\colon M(c, a) \to M(c, b) \quad \text{and} \quad f^* = M(f \otimes \mathrm{id}_c)\colon M(b, c) \to M(a, c)$$

the induced maps. If C has an exact structure we require $M\colon C^{op} \otimes C \to Ab$ to be exact in both variables. In particular $M(0, c) = M(c, 0) = 0$.

Given a bimodule M over an Ab-enriched category C we define an Ab-enriched category $C \ltimes M$ with the same objects of C, and morphism groups $(C \ltimes M)(a, b) = C(a, b) \oplus M(a, b)$. Composition is defined by

$$\big(f\colon b \to c, m \in M(b, c)\big) \circ \big(g\colon a \to b, n \in M(a, b)\big) = \big(f \circ g, f_* n + g^* m \in M(a, c)\big).$$

The projection onto the morphisms of C defines a functor $p\colon C \ltimes M \to C$. This functor is split (strictly) by the functor $s\colon C \to C \ltimes M$ which is the identity on objects and that sends a morphism f to $(f, 0)$.

Remark 1.4. If C is a groupoid, there is a functor $\Delta\colon C \to C^{op} \times C$ that sends $f\colon c \to d$ to $(f^{-1}, f)\colon (c, c) \to (d, d)$. The category $C \ltimes M$ is then naturally isomorphic to the Grothendieck construction of the composite $M \circ \Delta\colon C \to C^{op} \times C \to Ab$, in symbols

$$C \ltimes M \cong C \wr (M \circ \Delta).$$

In particular by denoting iC the groupoid of isomorphisms of C we have that

$$i(C \ltimes M) = (iC) \ltimes M \cong (iC) \wr (M \circ \Delta),$$

where we denoted the restriction $(iC)^{op} \times iC \to Ab$ also by M.

Example 1.5. Considering a ring A as an Ab-enriched category with one object, an Ab-enriched functor $M\colon A^{op} \otimes A \to Ab$ is precisely a bimodule over A in the classical sense. The construction $A \ltimes M$ above is the semi-direct product of rings and bimodules.

Example 1.6. Let $\mathcal{P}_A$ be the exact category of finitely generated projective modules over a ring A, and let M be an A-bimodule. The functor

$$\hom_A(-, - \otimes_A M)\colon \mathcal{P}_A^{op} \otimes \mathcal{P}_A \longrightarrow Ab$$

is a bimodule on $\mathcal{P}_A$. It is exact in both variables since $\mathcal{P}_A$ is a split-exact category.

Definition 1.7. Let C be an exact category. A sequence $a \xrightarrow{(i,m)} b \xrightarrow{(q,n)} c$ in $C \ltimes M$ is exact if the underlying sequence $a \xrightarrow{i} b \xrightarrow{q} c$ is exact in C, and if

$$q_*m + i^*n = 0 \quad \in M(a, c).$$

Proposition 1.8. *Let C be an exact category and $M\colon C^{op} \otimes C \to Ab$ a functor which is exact in both variables. The exact sequences of 1.7 define an exact structure on $C \ltimes M$, and the triple $(p\colon C \ltimes M \to C, s, \mathrm{id})$ is a split square-zero extension of exact categories.*

Proof. We verify the axioms of an exact category as presented in [**Kel90**, A1]. First we need to show that the sequences of 1.7 are closed under isomorphisms and that they are "exact". A morphism (f, m) is invertible in $C \ltimes M$ if and only if f is invertible in C. Moreover the condition $q_*m + i^*n = 0$ simply says that the second component of the composition $(q, n) \circ (i, m)$ is zero. It follows that the collection of exact sequences is closed under isomorphisms.

In the language of [**Kel90**] verifying exactness means showing that if $a \xrightarrow{(i,m)} b \xrightarrow{(q,n)} c$ is as in 1.7 then (i, m) is a kernel for (q, n), and that (q, n) is a cokernel for (i, m). The composite $(q, n) \circ (i, m) = (q \circ i, q_*m + i^*n)$ is zero by definition. We need to show that given any $(f, l)\colon d \to b$ with

$$(q, n) \circ (f, l) = (q \circ f, q_*l + f^*n) = 0$$

there is a unique map $(u, x)\colon d \to a$ with $(i, m) \circ (u, x) = (f, l)$. Since i is a kernel of q in C there is a unique $u\colon d \to b$ with the corresponding universal property in C. The element $x \in M(d, a)$ must be the unique solution of the equation

$$l = u^*m + i_*x.$$

Since $i_*\colon M(d, a) \to M(d, b)$ is injective by exactness of M, if a solution exists it must be unique. For the existence, $l - u^*m$ must belong to the image of i_* which is equal to the kernel of $q_*\colon M(d, b) \to M(d, c)$. We calculate that

$$q_*(l - u^*m) = q_*l - q_*u^*m = -f^*n - u^*q_*m = -f^*n + u^*i^*n = -f^*n + f^*n = 0$$

is indeed zero. The third equality holds because $-q_*m = i^*n$ by exactness. A dual proof shows that (q, n) is a cokernel for (i, m).

Next we show axiom Ex0 from [**Kel90**, A1]. The object 0 in C defines a zero-object in $C \ltimes M$ since the Abelian groups $C(0, c)$, $C(c, 0)$, $M(c, 0)$ and $M(0, c)$ are trivial for every object c of C. The identity map $(\mathrm{id}_0, 0)\colon 0 \to 0$ is an admissible epimorphism, since $(\mathrm{id}_0, 0) \circ (\mathrm{id}_0, 0) = (\mathrm{id}_0, 0)$ has null second component. This proves Ex0.

Now we show Ex2: pullbacks of admissible epimorphisms exist in $C \ltimes M$, and are admissible epimorphisms. Given a diagram $c \xrightarrow{(g,n)} b \overset{(f,m)}{\twoheadleftarrow} a$ in $C \ltimes M$, let

$$\begin{tikzcd} p \arrow[r, "\overline{g}"] \arrow[d, two heads, "\overline{f}"'] & a \arrow[d, two heads, "f"] \\ c \arrow[r, "g"'] & b \end{tikzcd}$$

be the pullback of the underlying diagram in C. First we find $\overline{m} \in M(p, c)$ and $\overline{n} \in M(p, a)$ such that

$$\begin{tikzcd} p \arrow[r, "(\overline{g}{,}\overline{n})"] \arrow[d, two heads, "(\overline{f}{,}\overline{m})"'] & a \arrow[d, two heads, "(f{,}m)"] \\ c \arrow[r, "(g{,}n)"'] & b \end{tikzcd}$$

commutes. Commutativity of this square is equivalent to the condition $f_*\overline{n}+\overline{g}^*m = g_*\overline{m} + \overline{f}^*n$. Choose $\overline{m} = 0$ and $\overline{n}$ in the preimage of $\overline{f}^*n - \overline{g}^*m$ by the surjective map $f_*\colon M(p, a) \twoheadrightarrow M(p, b)$. We show that p equipped with these maps satisfies the universal property

$$\begin{tikzcd} d \arrow[drr, bend left, "(k{,}x)"] \arrow[dr, dashed, "\exists!(u{,}z)"] \arrow[ddr, bend right, "(h{,}y)"'] & & \\ & p \arrow[r, "(\overline{g}{,}\overline{n})"] \arrow[d, two heads, "(\overline{f}{,}0)"] & a \arrow[d, two heads, "(f{,}m)"] \\ & c \arrow[r, "(g{,}n)"'] & b\,. \end{tikzcd}$$

Let $u\colon d \to p$ be the universal map for the corresponding diagram in C. The element $z \in M(d, p)$ must be the unique solution of the system of equations

$$\begin{cases} \overline{g}_*z = x - u^*\overline{n} \\ \overline{f}_*z = y \end{cases} .$$

We observe that $x - u^*\overline{n}$ and y lie over the same element of $M(d, b)$, since

$$f_*(x - u^*\overline{n}) = f_*x - u^*f_*\overline{n} = f_*x - u^*(-\overline{g}^*m + \overline{f}^*n) = f_*x + k^*m - h^*n = g_*y.$$

The last equality holds by the commutativity of the outer diagram. Therefore a solution z to the equations above exists and is unique precisely when the map $(\overline{f}_*, \overline{g}_*)\colon M(d, p) \to M(d, c) \times_{M(d,b)} \times M(d, a)$ is an isomorphism, that is when

$$\begin{tikzcd} M(d,p) \arrow[r, "\overline{g}_*"] \arrow[d, two heads, "\overline{f}_*"] & M(d,a) \arrow[d, two heads, "f_*"] \\ M(d,c) \arrow[r, "g_*"'] & M(d,b) \end{tikzcd}$$

is a pullback square. This follows from exactness of M in the second variable, and from the fact that kernels of f and $\overline{f}$ in C are isomorphic. To show that $(\overline{f},0)$ is an admissible epimorphism we choose an exact sequence $k \overset{i}{\rightarrowtail} p \overset{f}{\twoheadrightarrow} c$ in C. Then $k \overset{(i,0)}{\rightarrowtail} p \overset{(f,0)}{\twoheadrightarrow} c$ is clearly exact in $C \ltimes M$. A dual argument shows the analogous statement for pushouts of admissible monomorphisms, that is Ex2^{op} from [**Kel90**].

Axiom Ex1 states that admissible epimorphisms are closed under composition. That is, if

$$a \xrightarrow{(i,m)} b \xrightarrow{(q,n)} c \qquad\qquad a' \xrightarrow{(j,k)} c \xrightarrow{(p,l)} d$$

are exact, there exist an exact sequence $a'' \longrightarrow b \xrightarrow{(pq, p_*n+q^*l)} d$. By Ex2 we can form the pullback

$$\begin{tikzcd} a'' \arrow[r, "(\overline{j},\overline{k})"] \arrow[d, two heads, "(\overline{q},\overline{n})"'] & b \arrow[d, two heads, "(q,n)"] \\ a' \arrow[r, "(j,k)"'] & c \end{tikzcd}$$

in $C \ltimes M$, and the underlying square in C is also a pullback. Since C is an exact category the sequence $a'' \xrightarrow{\overline{j}} b \xrightarrow{pq} d$ is exact in C (see the third step in the proof of [**Kel90**, A1]). Thus we only need to verify that the composite $(p,l)\circ(q,n)\circ(\overline{j},\overline{k})$ is zero in $C \ltimes M$. By the commutativity of the square above this is the same as $(p,l)\circ(j,k)\circ(\overline{q},\overline{n})$, and $(p,l)\circ(j,k) = 0$ by exactness. This concludes the proof that the sequences of 1.7 form an exact structure.

The functors p and s are clearly exact. To see that (p,s,U) is square-zero, we need to show that the composition of morphisms of the form $(0,m)$ is zero. This is clear. □

Let $p\colon D \to C$ be the projection of a split extension of *Ab*-enriched categories (p,s,U). We let $\ker p\colon C^{op}\otimes C \to Ab$ be the bimodule defined by

$$\ker p(a,b) = \ker\big(D(s(a),s(b)) \xrightarrow{p} C(ps(a),ps(b))\big).$$

If the extension (p,s,U) is square-zero, there is a canonical functor $F\colon C \ltimes \ker p \to D$ that sends an object c to $s(c)$, and a morphism (f,m) to $s(f)+m$. The square-zero condition is used for showing that F preserves the composition of morphisms. Moreover the diagram

$$\begin{tikzcd} C \ltimes \ker p \arrow[r, "F"] \arrow[d] & D \arrow[d, "p"] \\ C \arrow[r, "\simeq", "p\circ s"'] & C \end{tikzcd}$$

commutes, and the functor $p\circ s$ is an equivalence of *Ab*-enriched categories. If (p,s,U) is a split square-zero extension of exact categories we observe that the bimodule $\ker p$ is automatically left exact in both variables.

Proposition 1.9. *Let (p,s,U) be a split square-zero extension of Ab-enriched categories. The functor $F\colon C \ltimes \ker p \to D$ is an equivalence of Ab-enriched categories.*

Moreover, if p and s are exact and $\ker p$ is right exact in each variable, $F\colon C \ltimes \ker p \to D$ is an equivalence of exact categories for the exact structure on $C \ltimes \ker p$ of 1.8*.*

Proof. The functor F is obviously essentially surjective, since s is by assumption. To see that it is fully faithful, we define an inverse of

$$F\colon C(a,b) \oplus \ker p(a,b) \longrightarrow D(s(a), s(b))$$

for every pair of objects a, b in C. Given a morphism $g\colon s(a) \to s(b)$ we let $U_*p(g)$ denote the composite

$$a \xrightarrow{U_a^{-1}} ps(a) \xrightarrow{p(g)} ps(b) \xrightarrow{U_b} b.$$

The inverse F^{-1} for the map above sends $g\colon s(a) \to s(b)$ to the pair

$$F^{-1}(g) = (U_*p(g), g - s(U_*p(g))).$$

For F^{-1} to be well-defined we need to verify that the second component lies in $\ker p(a,b)$, that is that $ps(U_*p(g)) = p(g)$. We start by remarking that by naturality of $U\colon ps \to \mathrm{id}$ the square

$$\begin{tikzcd} psps(c) \arrow[r, "U_{ps(c)}"] \arrow[d, "ps(U_c)"'] & ps(c) \arrow[d, "U_c"] \\ ps(c) \arrow[r, "U_c"'] & c \end{tikzcd}$$

commutes, and since U_c is an isomorphism this implies that $ps(U_c) = U_{ps(c)}$. Thus

$$ps(U_*p(g)) = ps(U_b) \circ psp(g) \circ ps(U_a^{-1}) = U_{ps(b)} \circ psp(g) \circ U_{ps(a)}^{-1} = p(g)$$

where the last equality holds by naturality of U. This shows that F^{-1} is well defined. It remains to verify that this is an inverse, namely that

$$FF^{-1}(g) = s(U_*p(g)) + g - s(U_*p(g)) = g,$$

and that

$$\begin{aligned} F^{-1}F(f,m) &= (U_*p(s(f)+m), s(f)+m-s(U_*p(s(f)+m))) \\ &= (U_*ps(f), s(f)+m-s(U_*ps(f))) \end{aligned}$$

equals (f,m). This is the case since by naturality $U_*ps(f) = f$.

Now we show that F is exact. By lemma 1.10 below one can choose an isomorphism between any exact sequence E in $C \ltimes \ker p$ and one of the form $a \xrightarrow{(i,0)} b \xrightarrow{(q,0)} c$. This gives an isomorphism between $F(E)$ and

$$F\big(a \xrightarrow{(i,0)} b \xrightarrow{(q,0)} c\big) = \big(s(a) \xrightarrow{s(i)} s(b) \xrightarrow{s(q)} s(c)\big)$$

which is exact by exactness of s.

It remains to show that if $E = \big(a \xrightarrow{(i,m)} b \xrightarrow{(q,n)} c\big)$ is any sequence of $C \ltimes \ker p$ whose image $F(E)$ is exact in D, then E is exact in $C \ltimes \ker p$. If $F(E)$ is exact, then the underlying sequence

$$\big(a \xrightarrow{i} b \xrightarrow{q} c\big) = pF(E)$$

is also exact. Moreover the composition $F(q,n) \circ F(i,m)$ is zero, and since F is faithful we must have $(q,n) \circ (i,m) = 0$ in $C \ltimes \ker p$, that is $q_*m + i^*n = 0$. $\square$

Lemma 1.10. *Any exact sequence of $C \ltimes M$ is isomorphic (non canonically) to an exact sequence of the form $a \xrightarrow{(i,0)} b \xrightarrow{(q,0)} c$.*

Proof. Let $a \xrightarrow{(i,m)} b \xrightarrow{(q,n)} c$ be an exact sequence of $C \ltimes M$. We find an isomorphism of the form

$$\begin{tikzcd}
a \arrow[r, "{(i,m)}"] \arrow[d, "{(\mathrm{id},0)}"'] & b \arrow[r, "{(q,n)}"] \arrow[d, "{(\mathrm{id},x)}"'] & c \arrow[d, "{(\mathrm{id},0)}"] \\
a \arrow[r, "{(i,0)}"'] & b \arrow[r, "{(q,0)}"'] & c
\end{tikzcd}$$

for some $x \in M(b,b)$. The vertical maps are isomorphisms for any $x \in M(b,b)$. For the squares to commute, x needs to satisfy

$$i^*x = -m \qquad \text{and} \qquad q_*x = n.$$

This is the case precisely when the element $(-m,n)$ is in the image of the map

$$(i^*, q_*)\colon M(b,b) \longrightarrow M(a,b) \times_{M(a,c)} M(b,c).$$

Consider the diagram of Abelian groups

$$\begin{tikzcd}
M(c,b) \arrow[r, two heads, "q_*"] \arrow[d, tail, "q^*"'] & M(c,c) \arrow[r, equal] \arrow[d, tail] & M(c,c) \arrow[d, tail, "q^*"] \\
M(b,b) \arrow[r, "{(i^*,q_*)}"] \arrow[d, two heads, "j^*"'] & M(a,b) \times_{M(a,c)} M(b,c) \arrow[r] \arrow[d, two heads] & M(b,c) \arrow[d, two heads, "j^*"] \\
M(a,b) \arrow[r, equal] & M(a,b) \arrow[r, two heads, "q_*"'] & M(a,c)\ .
\end{tikzcd}$$

The middle column is exact since it is the pullback of an exact sequence. Therefore (i^*, q_*) is surjective, and a lift of $(-m,n)$ exists. □

Example 1.11. Let A be a ring, M an A-bimodule, and $p\colon A \ltimes M \to A$ the projection with zero section $s\colon A \to A \ltimes M$. The induced functors $p_* = (-) \otimes_{A\ltimes M} A$ and $s_* = (-) \otimes_A (A \ltimes M)$ define a split square-zero extension of exact categories (s_* is essentially surjective by [**DGM13**, 1.2.5.4]). Therefore 1.9 provides an equivalence of exact categories

$$\mathcal{P}_{A\ltimes M} \xleftarrow{\simeq} \mathcal{P}_A \ltimes \ker p_*.$$

For $Q \in \mathcal{P}_A$ there is a natural isomorphism

$$s_*Q \cong Q \oplus (Q \otimes_A M)$$

where the right-hand side has module structure $(p', p\otimes n)\cdot(a,m) = (p'a, p\otimes na + p\otimes m)$. Under this isomorphism a module map $s_*Q \to s_*Q'$ is a matrix $\left(\begin{smallmatrix} \phi & 0 \\ f & \phi\otimes M \end{smallmatrix}\right)$ for module maps $\phi\colon Q \to Q'$ and $f\colon Q \to Q' \otimes_A M$ (see [**DGM13**, 1.2.5.1]). This gives a natural isomorphism of bimodules $\ker p_* \cong \hom_A(-, -\otimes_A M)$, and combined with 1.9 an equivalence of exact categories

$$\mathcal{P}_{A\ltimes M} \xleftarrow{\simeq} \mathcal{P}_A \ltimes \hom_A(-, -\otimes_A M).$$

2. The K-theory of $C \ltimes M$

Let C be an exact category and $M\colon C^{op} \otimes C \to Ab$ a bimodule over C which is exact in both variables. We use the techniques of [**DGM13**] to give a description of the K-theory $\mathrm{K}(C \ltimes M)$ in terms of the K-theory of bimodules of [**DGM13**, 1.3.4.2]. We recall from [**DGM13**, §1.3.3] that for every natural number p, the bimodule M extends to a bimodule $S_pM\colon S_pC^{op} \otimes S_pC \to Ab$ on S_pC, defined as the Abelian subgroup

$$S_pM(X,Y) \subset \bigoplus_{\theta \in Cat([1],[p])} M(X_\theta, Y_\theta)$$

of collections $\underline{m}$ satisfying the condition

$$X(a)^* m_\theta = Y(a)_* m_\rho$$

as elements of $M(X_\rho, Y_\theta)$, for all morphism $a\colon \rho \to \theta$ in $Cat([1],[p])$. In the terminology of §1 this condition expresses the commutativity of the diagram

$$\begin{tikzcd} X_\rho \arrow[r, "{(0,m_\rho)}"] \arrow[d, "{(X(a),0)}"'] & Y_\rho \arrow[d, "{(Y(a),0)}"] \\ X_\theta \arrow[r, "{(0,m_\theta)}"'] & Y_\theta \end{tikzcd}$$

in $C \ltimes M$. By iterating this construction we define, for all $\underline{p} = (p_1, \ldots, p_n)$, a bimodule on $S^{(n)}_{\underline{p}}C$

$$S^{(n)}_{\underline{p}}M = S_{p_1} \ldots S_{p_n} M.$$

A map $\sigma\colon \underline{p} \to \underline{q}$ induces a natural transformation $\sigma^*\colon S^{(n)}_{\underline{q}}M \Rightarrow S^{(n)}_{\underline{p}}M \circ ((\sigma^*)^{op} \otimes \sigma^*)$ defining an n-simplicial structure on $S^{(n)}_{\bullet}M$.

Example 2.1. Consider the bimodule $\hom_A(-, - \otimes_A M)$ on the category $\mathcal{P}_A$ of finitely generated projective modules over a ring A, induced by an A-bimodule M. In this case $S^{(n)}_{\underline{p}}M$ is the Abelian group of natural transformations of diagrams

$$S^{(n)}_{\underline{p}}M(X,Y) = \hom_A(X, Y \otimes_A M),$$

where both the tensor product of a diagram of A-modules with M and the sum of natural transformations are pointwise.

Let $\coprod_C M$ be the following groupoid. It has the same objects of C, and only automorphisms. The automorphism group of c is $M(c,c)$, and composition is defined by addition in $M(c,c)$. It is the disjoint union of groups

$$\coprod\nolimits_C M = \coprod_{c \in ObC} M(c,c)$$

and its nerve is the disjoint union of the classifying spaces $\coprod\limits_{c \in ObC} BM(c,c)$. This construction applied to $S^{(n)}_{\underline{p}}M$ defines a category

$$S^{(n)}(C;M)_{\underline{p}} := \coprod_{S^{(n)}_{\underline{p}}C} S^{(n)}_{\underline{p}}M.$$

The simplicial structure maps $\sigma^*\colon S^{(n)}_{\underline{q}}C \to S^{(n)}_{\underline{p}}C$ together with the natural transformations $\sigma^*\colon S^{(n)}_{\underline{q}}M \Rightarrow S^{(n)}_{\underline{p}}M \circ ((\sigma^*)^{op} \otimes \sigma^*)$ define an n-simplicial category $S^{(n)}_{\bullet}(C;M)$, whose realization is denoted

$$\mathrm{K}(C;M)_n = |\mathcal{N}_{\bullet}S^{(n)}_{\bullet}(C;M)| = |\coprod_{X \in ObS^{(n)}_{\bullet}C} S^{(n)}_{\bullet}BM(X,X)|.$$

There are structure maps

$$\mathrm{KR}(C;M)_n \wedge S^1 \longrightarrow \mathrm{K}(C;M)_{n+1}$$

induced by the inclusions of 1-simplices $S^{(n+1)}_{(1,p_1,\dots,p_n)} = S^{(n)}_{(p_1,\dots,p_n)}$, defined in a fashion analogous to the structure maps of $\mathrm{K}(C)$. Permuting the $S_{\bullet}$-constructions gives a Σ_n-action on $\mathrm{K}(C;M)_n$ compatible with the structure maps, defining a symmetric spectrum $\mathrm{K}(C;M)$.

Remark 2.2. There is a category $C[M]$ with objects $\coprod_{c\in C} M(c,c)$ and morphism from m_c in the c-summand to $m_{c'}$ in c'-summand

$$C[M](m_c, m_{c'}) = \{f \in C(c,c') | f^* m_{c'} = f_* m_c\}.$$

This has an exact structure by defining a sequence to be exact if it is exact in C, and as simplicial sets

$$ObS_{\bullet}C[M] = \hom\left(\coprod_{S_{\bullet}C} S_{\bullet}M\right).$$

This induces an equivalence in K-theory $K(C[BM]) \simeq K(C;M)$, see [**DGM13**, §5.1.4].

Remark 2.3. Our notation is off by a suspension factor compared to the notation of [**DM94**]. For $C = \mathcal{P}_A$ and $M = \hom_A(-, - \otimes_A M)$ what we denote $\mathrm{K}(\mathcal{P}_A; M)$ is $\mathrm{K}(\mathcal{P}_A; BM)$ in the notation of [**DM94**, 3.1].

Define a functor $\Psi\colon \coprod_C M \to i(C \ltimes M)$ by the identity on objects, and the inclusion at the identity $(\mathrm{id}_c, -)\colon M(c,c) \to C(c,c) \oplus M(c,c)$ on morphisms. This extends to a map of symmetric spectra $\mathrm{K}(C;M) \to \mathrm{K}(C \ltimes M)$.

Theorem 2.4. *The map $\mathrm{K}(C;M) \to \mathrm{K}(C \ltimes M)$ is a levelwise equivalence of symmetric spectra above spectrum level one.*

Proof. Since all our constructions are natural, $S_{\bullet}$ preserves equivalences, and the realization of bisimplicial sets preserves levelwise equivalences, it is enough to show that the map of the statement is an equivalence in spectrum degree one, that is that the functor

$$\Psi\colon \coprod_{S_{\bullet}C} S_{\bullet}M \longrightarrow iS_{\bullet}(C \ltimes M)$$

induces an equivalence on classifying spaces. We factor this map as

$$\Psi\colon \coprod_{S_{\bullet}C} S_{\bullet}M \longrightarrow tS_{\bullet}(C \ltimes M) \longrightarrow iS_{\bullet}(C \ltimes M),$$

where t is the set of isomorphisms of $S_{\bullet}(C \ltimes M)$ which lie in the image of Ψ. By the standard argument of [**Wal85**], the second map is an equivalence on realizations, as both spaces are equivalent to the realization of the objects of $S_{\bullet}(C \ltimes M)$ (see also [**DGM13**, §1.2.3.2]).

The first map is levelwise a fully faithful functor. We show that it is also essentially surjective, proving that it is a levelwise equivalence of categories. We show that any diagram $X \in S_p(C \ltimes M)$ is isomorphic to s_*p_*X, by a natural isomorphism $\phi\colon X \to s_*p_*X$ which at every object θ in $Cat([1],[p])$ is of the form $\phi_\theta = (\mathrm{id}, x_\theta)\colon X_\theta \to X_\theta$ for some $x_\theta \in M(X_\theta, X_\theta)$. This is a higher version of 1.10, where we proved this result for $p = 2$. For every morphism $\theta \le \rho$ in $Cat([1],[p])$ let us denote the corresponding map in the diagram X by $(f_{\theta\rho}, m_{\theta\rho})\colon X_\theta \to X_\rho$. For ϕ to define a natural transformation, the elements x_θ need to satisfy the relations

$$(1) \qquad f^*_{\theta\rho}x_\rho + m_{\theta\rho} = (f_{\theta\rho})_*x_\theta \qquad \in M(X_\theta, X_\rho)$$

for every $\theta \le \rho$ in $Cat([1],[p])$. We represent an injective map $\theta\colon [1] \to [p]$ by listing its two values in the form $\theta = (j, j+k)$, where $1 \le k \le p$ and $0 \le j \le p-k$. We construct the elements $x_{(j,j+k)}$ inductively on k.

For $k = 1$ the elements $x_{(j,j+1)}$ lie on the diagonal of X, and we define $x_{(j,j+1)} = 0$. Since all the maps $X_{(j,j+1)} \to X_{(i,i+1)}$ in the diagram X are zero, these elements satisfy condition (1) for θ and ρ of the form $(j, j+1)$.

Now suppose that $x_{(i,i+l)}$ is defined for all $1 \le l < k$ and $0 \le i \le p-l$, and let us define $x_{(j,j+k)}$. We observe that every non-identity map $X_{(i,i+l)} \to X_{(j,j+k)}$ with $l \le k$ factors through $X_{(j,j+k-1)}$, where $x_{(j,j+k-1)}$ has already been inductively defined. Similarly every non-identity map $X_{(j,j+k)} \to X_{(i,i+l)}$ with $l \le k$ factors through $X_{(j+1,j+k)}$, where $x_{(j+1,j+k)}$ has already been inductively defined. Thus for condition (1) to be satisfied among θ and ρ of the form $(j, j+l)$ with $l \le k$, we only need to guarantee that the new elements $x_{(j,j+k)}$ are compatible with $x_{(j,j+k-1)}$ and $x_{(j+1,j+k)}$. Let us denote the corresponding maps by $(i,m)\colon X_{(j,j+k-1)} \to X_{(j,j+k)}$ and $(p,n)\colon X_{(j,j+k)} \to X_{(j+1,j+k)}$. The compatibility conditions that $x_{(j,j+k)}$ needs to satisfy are

$$i^*x_{(j,j+k)} = i_*x_{(j,j+k-1)} - m \qquad \text{and} \qquad p_*x_{(j,j+k)} = n + p^*x_{(j+1,j+k)}.$$

The elements $i_*x_{(j,j+k-1)} - m$ and $n + p^*x_{(j+1,j+k)}$ lie over the same point of $M(X_{(j,j+k-1)}, X_{(j+1,j+k)})$, and therefore a solution $x_{(j,j+k)}$ exists if the map (i^*, p_*) surjects onto the pullback P in the diagram

$$\begin{tikzcd}
M(X_{(j+k,j+k-1)}, X_{(j,j+k)}) \arrow[r, two heads, "p_*"] \arrow[d, tail] & M(X_{(j+k,j+k-1)}, X_{(j+1,j+k)}) \arrow[r, equal] \arrow[d, tail] & M(X_{(j+k,j+k-1)}, X_{(j+1,j+k)}) \arrow[d, tail] \\
M(X_{(j,j+k)}, X_{(j,j+k)}) \arrow[r, "(i^*,p_*)"] \arrow[d, two heads, "i^*"'] & P \arrow[r] \arrow[d, two heads] & M(X_{(j,j+k)}, X_{(j+1,j+k)}) \arrow[d, two heads, "i^*"] \\
M(X_{(j,j+k-1)}, X_{(j,j+k)}) \arrow[r, equal] & M(X_{(j,j+k-1)}, X_{(j,j+k)}) \arrow[r, "p_*"'] & M(X_{(j,j+k-1)}, X_{(j+1,j+k)})\,.
\end{tikzcd}$$

The vertical sequences are exact by exactness of M applied to the exact sequence induced by $(j, j+k-1, j+k)\colon [2] \to [p]$. Moreover the top left map p_* is surjective since p is an admissible epimorphism. Therefore (i^*, p_*) is surjective. □

Remark 2.5. Theorem 2.4 can be interpreted in terms of Grothendieck constructions. The equivalence of the statement factors as

$$\Psi\colon \coprod_{S_\bullet C} S_\bullet M \longrightarrow (iS_\bullet C) \ltimes S_\bullet M \xrightarrow{\simeq} iS_\bullet(C \ltimes M),$$

where the second map is an equivalence by 1.9, as $S_\bullet(C \ltimes M) \to S_\bullet C$ is a levelwise split square-zero extension of exact categories (the section is essentially surjective by the proof of 2.4). We saw in 1.4 that $(iS_\bullet C) \ltimes S_\bullet M$ is levelwise the Grothendieck construction of the composite

$$iS_pC \xrightarrow{\Delta} (iS_pC)^{op} \times iS_pC \xrightarrow{S_pM} Ab.$$

Similarly the category $\coprod_{S_\bullet C} S_\bullet M$ is levelwise the Grothendieck construction of the restriction to the discrete category of objects

$$ObS_pC \xrightarrow{\Delta} (ObS_pC)^{op} \times ObS_pC \xrightarrow{S_pM} Ab.$$

Theorem 2.4 can be rephrased as saying that the inclusion of objects $ObS_\bullet C \to iS_\bullet C$ induces an equivalence after geometric realization between the Grothendieck constructions

$$(ObS_\bullet C) \wr (S_\bullet M \circ \Delta) \xrightarrow{\simeq} (iS_\bullet C) \wr (S_\bullet M \circ \Delta).$$

Theorem 2.4 provides the following description of the homotopy fiber spectrum

$$\widetilde{\mathrm{K}}(C \ltimes M) = \mathrm{hof}\left(\mathrm{K}(C \ltimes M) \longrightarrow \mathrm{K}(C)\right).$$

Let $\widetilde{\mathrm{K}}(C; M)$ be the symmetric spectrum of [**DGM13**, 1.3.4.5], defined in degree n by the space

$$\widetilde{\mathrm{K}}(C; M)_n = \bigvee_{X \in ObS_\bullet^{(n)}C} BS_\bullet^{(n)}M(X, X)$$

with structure maps and symmetric actions defined similarly to the ones of $\mathrm{K}(C; M)$.

Corollary 2.6. *There is a natural equivalence of symmetric spectra*

$$\widetilde{\mathrm{K}}(C; M) \simeq \widetilde{\mathrm{K}}(C \ltimes M).$$

Proof. Projecting off the bimodule defines maps of $(n+1)$-simplicial sets

$$p_n \colon \mathcal{N}_\bullet S_\bullet^{(n)}(C; M) \longrightarrow ObS_\bullet^{(n)}C,$$

where the target has a trivial simplicial direction. The collection of realizations of n-simplicial sets $|ObS_\bullet^{(n)}C|$ forms a symmetric spectrum in the standard way, and there is a commutative diagram

$$\begin{tikzcd} \mathrm{K}(C; M) \arrow[r, "\simeq"] \arrow[d, "p"'] & \mathrm{K}(C \ltimes M) \arrow[d] \\ \{|ObS_\bullet^{(n)}C|\} \arrow[r, "\simeq"'] & \mathrm{K}(C)\ . \end{tikzcd}$$

This gives an equivalence from the homotopy fiber of p to $\widetilde{\mathrm{K}}(C \ltimes M)$. The inclusion at zero in $M(c, c)$ induces a splitting s for p. As a general fact, the homotopy fiber of p is equivalent to the homotopy cofiber of s. Indeed if $p \colon E \to B$ is a map of spectra split by a levelwise cofibration s, the square

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] & \mathrm{hoc}(s) \arrow[d] \\ B \arrow[r] & * \end{tikzcd}$$

is cartesian since the horizontal homotopy fibers are equivalent ($B \to E \to \mathrm{hoc}(s)$ is a fiber sequence). The map on vertical homotopy fibers $\mathrm{hof}(p) \to \mathrm{hoc}(s)$ is therefore also an equivalence. In our case s is a levelwise cofibration and the homotopy cofiber of s is equivalent to its cofiber, which is precisely $\widetilde{\mathrm{K}}(C;M)$. $\square$

Remark 2.7. When $C = \mathcal{P}_A$ with bimodule $\hom_A(-,-\otimes_A M)$ we recover the equivalence

$$\widetilde{\mathrm{K}}(\mathcal{P}_{A\ltimes M}) \underset{1.11}{\simeq} \widetilde{\mathrm{K}}(\mathcal{P}_A \ltimes \hom_A(-,-\otimes_A M)) \simeq \bigvee_{X\in ObS^{(n)}_{\bullet}\mathcal{P}_A} \hom_A(X, X\otimes_A BM)$$

of [**DM94**].

3. The categorical Dundas-Goodwillie-McCarthy theorem

We refer to [**DM96**] for the definition of the topological Hochschild homology space $\mathrm{THH}(C)$ of an exact category C. We let the topological Hochschild homology spectrum of C be the spectrum defined in degree n by the space $\mathrm{THH}(S^{(n)}_{\bullet}C)$, and we write $\mathrm{TC}(C)$ for the associated topological cyclic homology. For concreteness, we choose the model of [**DM96**] for TC. For many purposes this construction does not have the right rational homotopy type, but it has the right homotopy type after p-completion. This model admits a convenient definition of the trace map $\mathrm{Tr}\colon \mathrm{K} \to \mathrm{TC}$, see [**DM96**].

Main Theorem. *Let C be an exact category, and $M\colon C^{op}\otimes C \to Ab$ an additive functor which is exact in both variables. The square of spectra*

$$\begin{tikzcd} \mathrm{K}(C\ltimes M) \arrow[r, "\mathrm{Tr}"] \arrow[d] & \mathrm{TC}(C\ltimes M) \arrow[d] \\ K(C) \arrow[r, "\mathrm{Tr}"'] & \mathrm{TC}(C) \end{tikzcd}$$

is homotopy cartesian after completion at any prime, where $C\ltimes M$ has the exact structure defined in 1.7.

Using the description of the homotopy fiber $\widetilde{K}(C\ltimes M)$ of 2.6, the proof of [**McC97**] for the functor $\mathcal{P}_{A\ltimes M}\to\mathcal{P}_A$ adapts to our situation with minor changes. We recall the argument.

Proof. For an Abelian group G and a finite pointed set X, let $G(X)$ be the Abelian group

$$G(X) = (\bigoplus_{x\in X} G\cdot x)/G\cdot *.$$

This construction is functorial in G, and precomposing it with $M\colon C^{op}\otimes C\to Ab$ defines a bimodule $M(X)\colon C^{op}\otimes C \to Ab$, with corresponding exact category $C\ltimes M(X)$. Any functor F from exact categories to the category Sp^{Σ} of symmetric spectra induces a functor $F(C\ltimes M(-))\colon Set^f_* \to \mathrm{Sp}^{\Sigma}$ on the category of finite sets. This extends to a functor from finite pointed simplicial sets to simplicial symmetric spectra. For any finite pointed simplicial set $X\in sSet^f_*$ define $F(C\ltimes M(X))$ as the realization of the simplicial spectrum

$$F(C\ltimes M(X)) = |[k]\longmapsto F(C\ltimes M(X_k))|.$$

The projection functor $C \ltimes M(X) \to C$ induces a map $F(C \ltimes M(X)) \to F(C)$, and we define

$$\widetilde{F}(C \ltimes M(X)) = \operatorname{hof}\big(F(C \ltimes M(X)) \longrightarrow F(C)\big)$$

as a functor $\widetilde{F}(C \ltimes M(-))\colon sSet_*^f \to \mathrm{Sp}^\Sigma$.

Let $\underline{\mathrm{THH}}(C)$ denote the spectrum defined by the sequence of spaces $\mathrm{THH}(S_\bullet^{(n)}C)$. In [**McC97**, 4.2] the author constructs a commutative diagram

$$\begin{tikzcd}
\widetilde{\mathrm{K}}(C \ltimes M(X)) \arrow[rr, "\widetilde{\mathrm{Tr}}"] \arrow[dr] \arrow[ddr, dashed] & & \widetilde{\mathrm{TC}}(C \ltimes M(X)) \arrow[dl] \arrow[ddl, dashed] \\
& \widetilde{\underline{\mathrm{THH}}}(C \ltimes M(X)) \arrow[d, dashed] & \\
& \underline{\mathrm{THH}}(C; M(X \wedge S^1)) &
\end{tikzcd}$$

where the dashed maps are maps in the homotopy category. We denote for simplicity $S_\bullet^{(n)}M = M$. The left dashed map corresponds under the equivalence of 2.6 to the inclusion of wedges into sums

$$\bigvee_{c \in ObS_\bullet^{(n)}C} BM(c,c)(X) \longrightarrow \bigoplus_{c \in ObS_\bullet^{(n)}C} BM(c,c)(X) \xrightarrow{\simeq} \mathrm{THH}(S_\bullet^{(n)}C; BM(X)),$$

where the last map is the inclusion of the 0-simplices, which is an equivalence by [**DGM13**, 1.3.3.1, §4.3.4]. The composite is then roughly twice as connected as the connectivity of X, and therefore it induces an equivalence on Goodwillie differentials

$$D_*\widetilde{\mathrm{K}}(C \ltimes M(-)) \simeq D_*\underline{\mathrm{THH}}(C; BM(-)).$$

Now let us analyze the right-hand dashed map. Let H denote the Eilenberg-MacLane functor from categories enriched in Abelian groups to categories enriched in symmetric spectra. The inclusion of wedges into products defines an equivalence of spectrally enriched categories $HC \vee HM \to H(C \ltimes M)$ which induces an equivalence

$$\mathrm{THH}(HS_\bullet^{(n)}C \vee HM) \xrightarrow{\simeq} \mathrm{THH}(S_\bullet^{(n)}C \ltimes M) \xrightarrow{\simeq} \mathrm{THH}(S_\bullet^{(n)}(C \ltimes M))$$

over $\mathrm{THH}(S_\bullet^{(n)}C)$. The second map is the equivalence of remark 2.5. Moreover $\mathrm{THH}(HC \vee HM)$ splits as

$$\mathrm{THH}(HC \vee HM) \simeq \bigvee_{a \geq 0} \mathrm{THH}_a(C; M),$$

where $\mathrm{THH}_a(C; M)$ is the subspace of $\mathrm{THH}(HC \vee HM)$ defined as in [**Hes94**, 2.1], by distributing the smash products defining $\mathrm{THH}(HC \vee HM)$ onto the wedges of the mapping spectra of $HC \vee HM$, and by keeping only the wedge components in which the bimodule factors appear a times. By replacing C with the iterates of $S_\bullet$ we obtain a decomposition

$$\mathrm{THH}(S_\bullet^{(n)}(C \ltimes M)) \simeq \bigvee_{a \geq 0} \mathrm{THH}_a(S_\bullet^{(n)}C; M).$$

Hesselholt's connectivity estimates from [**Hes94**, §2.2] for the fixed-points of the spectra of $\mathrm{THH}_a(L \oplus P)$ hold just as well for our spectra $\underline{\mathrm{THH}}_a(C; M)$. The only difference is that our category C is allowed to have more than one object, and our construction of THH picks up an extra wedge sum over the set of objects of C which does not affect connectivity. Thus by Proposition [**Hes94**, 2.2] there is an equivalence

$$D_*\widetilde{\mathrm{TC}}(C\ltimes M(-);p)\simeq D_*\operatorname*{holim}_r \underline{\mathrm{THH}}_1(C;M(-))^{C_{p^r}}.$$

By [**Hes94**, 3.2,3.3] the fixed-points of $\underline{\mathrm{THH}}_1$ are equivalent to

$$\begin{aligned}\underline{\mathrm{THH}}_1(C;M(-))^{C_{p^r}} &\simeq (\underline{\mathrm{THH}}(C;M(-))\wedge S^1_+)^{C_{p^r}}\\ &\simeq (\underline{\mathrm{THH}}(C;M(-))\vee \Sigma\underline{\mathrm{THH}}(C;M(-)))\end{aligned}$$

and the fixed-points inclusions of the homotopy limit correspond to the map $p\vee\mathrm{id}$. Thus after p-completion this is equivalent to $\Sigma\underline{\mathrm{THH}}(C;M(-))\simeq\underline{\mathrm{THH}}(C;BM(-))$. Combining these equivalences we have shown that the right dashed map induces an equivalence on differentials at a point

$$D_*\widetilde{\mathrm{TC}}(C\ltimes M(-))\simeq D_*\underline{\mathrm{THH}}(C;BM(-))$$

after completion at a prime p.

By the commutativity of the diagram above the trace map induces an equivalence $D_*\widetilde{\mathrm{Tr}}\colon D_*\widetilde{\mathrm{K}}(C\ltimes M(-))\xrightarrow{\simeq} D_*\widetilde{\mathrm{TC}}(C\ltimes M(-))$ after p-completion. We further observe that for any pointed set B there is an isomorphism of exact categories

$$C\ltimes M(B\vee X)\cong C\ltimes (M(B)\oplus M(X))\cong (C\ltimes M(B))\ltimes M(X),$$

where $M(X)$ is a bimodule on $C\ltimes M(B)$ via the projection $C\ltimes M(B)\to C$. This shows that $D_*\widetilde{\mathrm{K}}(C\ltimes M(B\vee -))\simeq D_*\widetilde{\mathrm{TC}}(C\ltimes M(B\vee -))$, and therefore by [**Goo90**, 1.3(iv)] that the differential of the trace at any finite pointed simplicial set B

$$D_B\widetilde{\mathrm{Tr}}\colon D_B\widetilde{\mathrm{K}}(C\ltimes M(-))\xrightarrow{\simeq} D_B\widetilde{\mathrm{TC}}(C\ltimes M(-))$$

is an equivalence after p-completion. Both functors $\widetilde{\mathrm{K}}(C\ltimes M(-))$, $\widetilde{\mathrm{TC}}(C\ltimes M(-))\colon sSet^f_*\to \mathrm{Sp}^\Sigma$ are (-1)-analytic in the sense of [**Goo90**],[**Goo92**]. The first one by the argument of [**McC97**, §3], which just uses the decomposition $\widetilde{\mathrm{K}}(C\ltimes M(-))\simeq\widetilde{\mathrm{K}}(C;M(-))$. The second one by [**McC97**, §1] using the wedge decomposition of $\underline{\mathrm{THH}}(C\ltimes M(X))$ above. By [**Goo92**, 5.3] the trace map must have already been a p-completed equivalence before taking the differential

$$\widetilde{\mathrm{Tr}}\colon \widetilde{\mathrm{K}}(C\ltimes M(X))\xrightarrow{\simeq}\widetilde{\mathrm{TC}}(C\ltimes M(X))$$

for any pointed ((-1)-connected) space X. For $X=S^0$ this shows that the square of the statement is cartesian. □

References

[DGM13] Bjørn Ian Dundas, Thomas G. Goodwillie, and Randy McCarthy, *The local structure of algebraic K-theory*, Algebra and Applications, vol. 18, Springer-Verlag London, Ltd., London, 2013. MR3013261

[DM94] Bjørn Ian Dundas and Randy McCarthy, *Stable K-theory and topological Hochschild homology*, Ann. of Math. (2) **140** (1994), no. 3, 685–701. MR1307900

[DM96] Bjørn Ian Dundas and Randy McCarthy, *Topological Hochschild homology of ring functors and exact categories*, J. Pure Appl. Algebra **109** (1996), no. 3, 231–294. MR1388700

[Dun97] Bjørn Ian Dundas, *Relative K-theory and topological cyclic homology*, Acta Math. **179** (1997), no. 2, 223–242. MR1607556

[Goo90] Thomas G. Goodwillie, *Calculus. I. The first derivative of pseudoisotopy theory*, K-Theory **4** (1990), no. 1, 1–27. MR1076523

[Goo92] Thomas G. Goodwillie, *Calculus. II. Analytic functors*, K-Theory **5** (1991/92), no. 4, 295–332. MR1162445

[Hel58] Alex Heller, *Homological algebra in abelian categories*, Ann. of Math. (2) **68** (1958), 484–525. MR0100622

[Hes94] Lars Hesselholt, *Stable topological cyclic homology is topological Hochschild homology*, Astérisque **226** (1994), 8–9, 175–192. K-theory (Strasbourg, 1992). MR1317119

[Kel90] Bernhard Keller, *Chain complexes and stable categories*, Manuscripta Math. **67** (1990), no. 4, 379–417. MR1052551

[McC97] Randy McCarthy, *Relative algebraic K-theory and topological cyclic homology*, Acta Math. **179** (1997), no. 2, 197–222. MR1607555

[Wal85] Friedhelm Waldhausen, *Algebraic K-theory of spaces*, Algebraic and geometric topology (New Brunswick, N.J., 1983), Lecture Notes in Math., vol. 1126, Springer, Berlin, 1985, pp. 318–419. MR802796

Mathematical Institute, University of Bonn, Endenicher Allee 60, D-53115 Bonn, Germany

Email address: `dotto@math.uni-bonn.de`