A GENTLE INTRODUCTION TO $(\infty,1)$-CATEGORIES

An (n,k)-category
- objects
- 1-morphisms
- 2-mor.
- ⋮
- n-morph.

i-isomorphisms are invertible for $i>k$ + Coherence data.

TRUNCATION

Def $X \in \mathrm{Top}$ $\quad \Pi_{\leq n} X$ $\rightsquigarrow$ $\tau_{\leq n} X$ reconstructed by $\Pi_{\leq n} X$
- obj = points in X
- mor = paths
- 2mor = htpy bet. paths
- 3mor = htpy bet htpy

THM [SIMPSON]
$\tau_{\leq 3} S^2$ cannot be modelled by a strict 3-groupoid

⇒ Homotopy hypothesis
n-types (spaces with $\pi_k(X) \simeq *$ for $k>n$) $\simeq$ "n-groupoid" $\rightsquigarrow$ strict n-groupoids will not work.

topological spaces $\simeq \infty$-groupoids

Idea Define an ∞-grpd to be a topological space.

1st def An $(\infty,1)$-category is a category enriched in Top.
($X, Y \in \mathcal{C}$ $Map(X,Y) \in Top$)
$\rightsquigarrow$ Not good for computations.

Def Δ is the category with:
obj: $[n] = \{0, \dots, n\}$
mor: order preserving maps.

Def $sSet := [\Delta^{op}, Set]$

$\rightsquigarrow$ $\Delta \longrightarrow Top$
$[n] \longmapsto \Delta^n$ (standard n-simplex)

This functor extends

$$|-| : sSet \rightleftarrows Top : S(-)$$

Prop There is a Quillen model structure sSet $f: X \to Y$ in sSet is a weak equivalence if $|f|: |X| \to |Y|$ weak equiv. of Top

$\rightsquigarrow Ho(sSet) \simeq Ho(Top)$

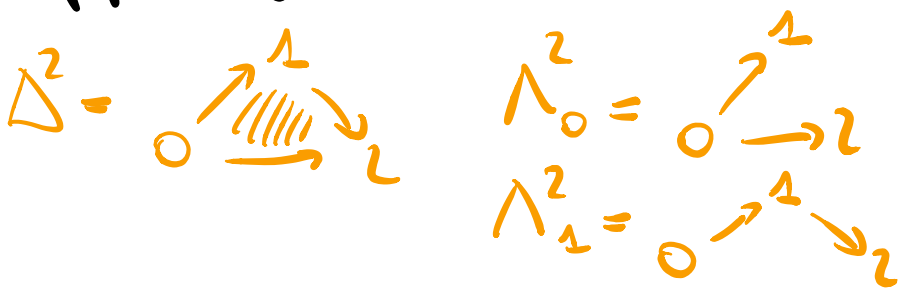
2nd def An $(\infty,1)$-category is a category enriched in sSet.

§ QUASI CATEGORIES

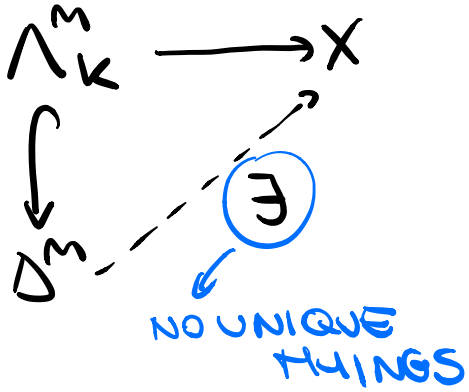
↝ for each n there is an object $\Delta^n \in sSet$

$$(\Delta^n)_k = \mathrm{Hom}_\Delta([k],[n])$$

↝ Hom_Δ Λ^n_k subsimplicial set of Δ^n (remove the interior on the face opp point vertex k).



Def $X \in sSet$, X is a **KAN COMPLEX** if



$\forall\, 0 \le k \le n$

Def X is a quasi category if just for $0 < k < n$ (inner horns)

Def Nerve functor

$N : \mathrm{Cat} \to \mathrm{Sset}$

$\mathcal{C} \longmapsto (N\mathcal{C})_k = \mathrm{Hom}_{\mathrm{CAT}}(\lceil k \rceil, \mathcal{C})$

$(N\mathcal{C})_0$ = objects

$(N\mathcal{C})_1$ = morphisms

$(N\mathcal{C})_2 = \cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot$ Pairs of composable morphisms.

$\vdots$

PROP $\mathcal{C} \in \mathrm{CAT}$, $N\mathcal{C}$ is a quasi category with unique inner horn fillers.

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr, "\exists!"'] & & 2 \end{tikzcd}$$

this statement is iff

$\rightsquigarrow$ $\mathcal{C} \in \mathrm{Grpd}$ $N\mathcal{C}$ is a Kan-complex w/ unique horn fillers for all horns.

THM [ ]

- QCat is a model for $(\infty,1)$-cat.
- Kan complexes " " $(\infty,0)$-cat.

$\rightsquigarrow$ We can define an ∞-cat. to be a simplicial set with the inner horn condition.

JOYAL: Develop category theory for quasi categories.

suppose $\mathcal{C}, \mathcal{D} \in \mathrm{Qcat}$.

$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) := \mathrm{sSet}(\mathcal{C}, \mathcal{D})$

$\Rightarrow$ PROP $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \in \mathrm{Qcat}$. (it's cartesian closed)

∞-CAT of ∞-CATEGORIES

- Using the previous proposition we can define an $(\infty,2)$-cat of $(\infty,1)$-categories.
- Form a simplicially enriched category
  $\mathrm{Cat}_\infty \subseteq \mathrm{sSet}$
  obj: small Quasi-categories
  $\mathrm{Map}(\mathcal{C}, \mathcal{D})$: Maximal Kan complex inside $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$

$N_{hc} : \mathrm{CAT}_{\mathrm{sSet}} \to \mathrm{sSet}$

$N_{hc}(\mathrm{Cat}_\infty) = (\infty,1)$-cat of $(\infty,1)$ categories.

$= \mathrm{Cat}_\infty$

PROP [LURIE]

Cat_∞ is (∞)-complete

$\rightsquigarrow$ Discussion about what is the limit and colimit of ∞-categories.

Canonical example of an ∞-cat is $\mathcal{S}$ (∞-category of spaces).

Recall A category is pointed if it has a zero object. $X \in \mathcal{C}$

$0 \to X$ contractible space of maps
$X \to 0$

Def $\mathcal{C} \in \mathrm{QCat}$ is STABLE if it has a zero object and

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \downarrow \\ c & \longrightarrow & d \end{array}$$

every homotopy pullback is a htpy pushout.

$\rightsquigarrow$ $\mathcal{C}$ has a zero object

$\Omega_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$

$$X \longmapsto \begin{array}{ccc} \Omega_{\mathcal{C}} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & X \end{array}$$

Def $\mathcal{C}$ a pointed ∞-cat. Define $\mathrm{Sp}(\mathcal{C})$ homotopy inverse limit

$$\cdots \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C} \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C}$$

Def The ∞-cat. of spectra is $\mathrm{Sp}(\mathcal{S})$