

DEFINITION OF THH

goal: for E_1 -alg. $A: ASS^{\otimes} \rightarrow Sp^{\otimes}$
 define $\text{THH}(A)$ S^1 -eq.-spectrum

1) CYCLIC OBJECTS

will define a cat. Λ , a cyclic object in a category C is then a functor

$$\Lambda^{\text{op}} \rightarrow C \quad \left(\begin{array}{l} \text{for 1-category} \\ \text{take the same} \\ \text{everywhere} \end{array} \right)$$

a) The pericyclic category Λ_{∞} : 1-category

$$\underline{\text{obj}}: [n]_{\Lambda_{\infty}} := \left(\frac{1}{n}\mathbb{Z}, \leq \right) \quad m \in M_{\geq 1}$$

poset equipped with an action of \mathbb{Z}

$$\mathbb{Z} \times \frac{1}{m}\mathbb{Z} \longrightarrow \frac{1}{m}\mathbb{Z}$$

$$(r, x) \mapsto r+x$$

Map: order preserving \mathbb{Z} -equiv. maps
 of posets.

$$\boxed{\text{map}: \frac{1}{m}\mathbb{Z} \xrightarrow{\quad} \frac{1}{m}\mathbb{Z} \quad \in \Lambda_{\infty}}$$

$i/m \mapsto o_i^{\oplus} \quad o_0 \leq o_1 \leq \dots \leq o_{m-1} \leq 1 + o_0$

Notation: G group \underline{G} : 1-category
with 1 object e and map: $G(e,e) = G$

$N(\underline{G}) = BG$ classifying space of G

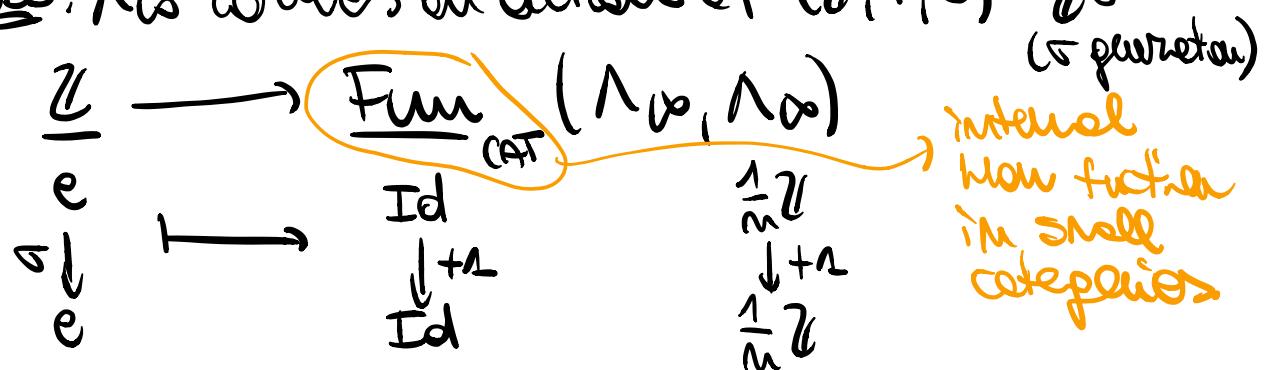
- C 1-category, a G -object is a functor $\underline{G} \rightarrow C$
($\Rightarrow BG \rightarrow NC$)

- C ∞ -cat., a G -object is a functor $BG \rightarrow C$

$$C^{BG} = \underline{\text{Hom}}_D(BG, C) = \underset{\text{∞-CAT}}{\text{Fun}}(BG, C)$$

" ∞ -cat. of G -equiv. objects in C "

OBS: Λ_∞ carries an action of $(\mathbb{Z}, +, 0) = \mathbb{Z}^\mathbb{Z}$



Def

$$\Lambda_p := \Lambda_\infty / \mathbb{Z}^\mathbb{Z} \quad \text{obj: same as } \Lambda_\infty$$

$$\text{map: } \frac{1}{n} \mathbb{Z} \xrightarrow{f} \frac{1}{m} \mathbb{Z} \quad \begin{array}{l} \text{order preserving} \\ \mathbb{Z}\text{-equiv.} \\ f \circ p + f \end{array}$$

$\Lambda := \Lambda_1$, here extraction $\Lambda_\infty \rightarrow \Lambda$

2) Grothendieck realization of cyclic objects

$$\Delta \xrightarrow{\varphi} \Lambda_\infty \longrightarrow \Lambda$$

S
 finite
 ordered
 set

$\mathbb{Z} \times \mathbb{Z}^S$
 \mathbb{Z} -eq. poset
 with lexicographic
 ordering

Simplicial
 object out
 of a
 cyclic one
 (recapitulate
 with this
 functor)

Note $\mathbb{Z} \times \underbrace{[n]}_{\{0, \dots, n\}} \cong \frac{1}{n+1} \mathbb{Z}$

$$(r, i) \longmapsto r + \frac{i}{n+1}$$

Thm $\Delta^{\text{op}} \xrightarrow{\varphi} \Lambda_\infty^{\text{op}}$ is cofinal (i.e.

$$|N(\varphi \downarrow \frac{1}{n} \mathbb{Z})| \simeq *$$

Cor if you have $F: \Lambda_\infty^{\text{op}} \rightarrow \mathcal{C}$ (\mathcal{C} infinity category)

$$\text{hocolim}_{\Delta^{\text{op}}} (F \circ \varphi) \xrightarrow{\text{?}} \text{hocolim}_{\Lambda_\infty^{\text{op}}} (F)$$

\mathcal{C}

\mathcal{C}

Ex $\mathcal{C} = \text{Top}$ $X : \Lambda_\infty^{\text{op}} \rightarrow \text{Top}$ p-acyclic space

$$\text{hocolim } X \xrightarrow{\sim} \text{hocolim } X$$

Δ^{op} $\Lambda_{\infty}^{\text{op}}$

$\underbrace{\quad}_{=}$ $\underbrace{\quad}_{=}$

$$|X|_D = X \underset{D}{\otimes} \Delta^{\text{op}}$$

$$X \underset{\Lambda_{\infty}}{\otimes} |\Lambda_{\infty}| = |X|_{\Lambda_{\infty}}$$

$$F: C \rightarrow \text{TOP} \quad G: C^{\text{OP}} \rightarrow \text{TOP} \quad F \otimes G := \coprod_{C \in C} F_C \times G_C / \begin{matrix} (x_C, y) \sim (x, y) \\ C \rightarrow D \end{matrix}$$

→ This defines an eq. of sets $\Lambda_\infty \xrightarrow{\cong} \Lambda_\infty^{\text{op}}$
 $|\Lambda_\infty^{[m]}| \cong \left\{ \frac{1}{m} \right\} \rightarrow \mathbb{R}$ order preserving } $\approx *$
 if - eq.

Note: $|\Lambda_\infty^T|$ has a canonical \mathbb{R} -action

$$\begin{array}{ccc} \mathbb{R} \times |\Lambda_\infty^T| & \longrightarrow & |\Lambda_\infty^T| \\ (r, f) & \longmapsto & r + f \end{array}$$

$$|\Lambda_\infty|: \Lambda_\infty \rightarrow \text{RTop}$$

$\Rightarrow |X|_{\Lambda_\infty}$ has an action by \mathbb{R} .

∴ every $X: \Lambda_\infty^{\text{op}} \rightarrow \text{Top}$ carries a \mathbb{U} -action

$$X_m \xrightarrow{X^{(+1)}} X_m \quad \begin{matrix} \text{id}_{\Lambda_\infty} \\ \downarrow +1 \\ \text{id}_{\Lambda_\infty} \end{matrix}$$

$$\begin{array}{ccc} \mathbb{Z} \times |X.|_{\Lambda_\infty} & \xrightarrow{|X^{(+1)}|} & |X.|_{\Lambda_\infty} \\ \downarrow & \nearrow g & \\ \mathbb{R} \times |X|_{\Lambda_\infty} & & \end{array}$$

If $\Lambda_\infty^{\text{op}} \xrightarrow{X} \text{Top}$ is an acyclic object then
the \mathbb{U} -action on $\Lambda_\infty^{\text{op}} \xrightarrow{X} \text{Top}$ is trivial
 $\Rightarrow \mathbb{R} \times |X| \rightarrow |X|$ factors through

$$\underbrace{\mathbb{R}/\mathbb{Z}}_{S^1} \times |X.| \rightarrow |X.|$$

Prop If \mathcal{C} is an ∞ -cat. w/ hocolim^s (e.g. sp)

then for every cyclic object $N(\Lambda_{\infty}^{\text{op}}) \xrightarrow{\times} \mathcal{C}$

hocolim X is a $\underbrace{\text{B}\mathbb{Z}}$ -eq. obj. in \mathcal{C}

if $\text{Fun}(N\Lambda^{\text{op}}, \mathcal{C}) \rightarrow \text{Fun}(N\Lambda_{\infty}^{\text{op}}, \mathcal{C}) \rightarrow$

hocolim
 $\Lambda_{\infty}^{\text{op}}$

$\xrightarrow{\text{B}\mathbb{Z}}$ $\text{Fun}(\text{pt}, \mathcal{C}) = \text{Fun}(\text{B}\mathbb{Z}, \mathcal{C})$

acts on the functor category
and I can take the fixed points \square

3) TMD

$\mathcal{C}^{\otimes} \rightarrow \text{Fin}_*$ the "active part" of \mathcal{C}^{\otimes} is

$$\mathcal{C}_{\text{act}}^{\otimes} = \mathcal{C}^{\otimes} \times_{\text{Fin}_*} \text{Fin}_*$$

if $\mathcal{C}^{\otimes} \rightarrow \text{Fin}_*$ symm monoidal

$$\mathcal{C}_{\text{act}}^{\otimes} \xrightarrow{\otimes} \mathcal{C}$$

$$\begin{array}{ccc} \mathcal{C}_{(n)}^{\otimes} & & \mathcal{C} \\ \downarrow \otimes^n & \nearrow \otimes & \downarrow \\ \mathcal{C}_{(1)}^{\otimes n} A_1 \otimes \dots \otimes A_m & & A_1 \otimes \dots \otimes A_m \end{array}$$

\otimes

Fun on E_1 -alg. in Sp . $\text{Ass} \xrightarrow{A} \text{Sp}^\otimes$

$$\Lambda_\infty^{\text{op}} \rightarrow \Lambda^{\text{op}} \xrightarrow{\Psi} \text{Ass}_{\text{act}}^\otimes \xrightarrow{A} \text{Sp}_{\text{act}}^\otimes \xrightarrow{\otimes} \text{Sp} \quad (*)$$

$$\text{THH}(A) = \text{hocolim}_{\Lambda_\infty^{\text{op}}} \quad) \text{ BV-equiv.}$$

↓ To the geometric realization of the composition (*)

Ψ obtained from comm. diagram

$$\begin{array}{ccc} \Lambda^{\text{op}} & \xrightarrow{\cong} & \text{Ass}^{\text{op}} \\ \downarrow \frac{1}{m} \mathbb{Z} & & \downarrow \\ \text{Fin} & \longrightarrow & \text{Fin}^* \\ \text{Mon}(\mathbb{Z}, \frac{1}{m}\mathbb{Z}) & & \end{array}$$

What is (*) on a discrete object?

$$\begin{array}{ccccccc} \Lambda_\infty^{\text{op}} & \rightarrow & \Lambda^{\text{op}} & \xrightarrow{\Psi} & \text{Ass}_{\text{act}}^\otimes & \xrightarrow{A} & \text{Sp}_{\text{act}}^\otimes \xrightarrow{\otimes} \text{Sp} \\ \downarrow \frac{1}{m} \mathbb{Z} & & & & A^{\otimes \frac{1}{m} \mathbb{Z}} & \longrightarrow & A^{\otimes \frac{1}{m} \mathbb{Z}} / \mathbb{Z} = A^{\otimes m} \\ \downarrow \frac{1}{m} \mathbb{Z} & & & & \downarrow \frac{1}{m} \mathbb{Z} & \longrightarrow & \downarrow \frac{1}{m} \mathbb{Z} \end{array}$$

m^6

$\mathbb{H}^m \rightarrow A^{(k)m}/\mathcal{U}$