

BÖKSTEDT'S THEOREM

$$MM_*(A|k) = \text{Tot}_*^{A \otimes_k A^{op}}(A, A) = \pi_* (A \otimes_{A \otimes_k A^{op}} A) =$$

$$= H_* \left(\underbrace{A \otimes A \otimes \dots \otimes A}_{M+1} \right) \quad \text{GEM. REALIZ. OF KOSZUL COMPLEX}$$

$$\begin{array}{c} \text{SPHERE} \\ \text{SPECTRUM} \\ \swarrow \\ S \end{array} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{F}_p$$

EASY: $MM_*(\mathbb{F}_p | \mathbb{F}_p) = \mathbb{F}_p$
 $MM_*(\mathbb{F}_p | \mathbb{Z}) = \text{Tot}_*^{\wedge_{\mathbb{F}_p}(\tau)}(\mathbb{F}_p, \mathbb{F}_p) = \Gamma(\mathfrak{d})$ ②

THEOREM 1 (BÖKSTEDT)

$$MM_*(\mathbb{F}_p | S) = \mathbb{F}_p[\mathbb{N}]$$

$$\text{TMH}_*(\mathbb{F}_p)$$

REM: Dualizing

$$MM^*(\mathbb{F}_p | \mathbb{Z}) = \text{Ext}^{\mathbb{N}(\tau)}(\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p[\mathbb{N}]$$
 ②

$$MM^*(\mathbb{F}_p | S) = \Gamma(\mathbb{N})$$
 ②

$$MH^{i+2}(\mathbb{F}_p | k) \text{ classifies } k\text{-algebras } \wedge_{\mathbb{F}_p}(\tau)$$

ASIDE: $MM^*(\mathbb{F}_p | \mathbb{Z}) \rightarrow MM^*(\mathbb{F}_p | \mathbb{S})$

$$\mathbb{N} \longrightarrow \mathbb{N}$$

$$\mathbb{N}^p \longrightarrow 0$$

Hence (Dwyer-Sipley)

Two distinct DGAs with $\wedge_{\mathbb{F}_p}(\tau)$
are top. equivalent

$$p=2 \quad \frac{\mathbb{Z}[e, de=2]}{e^4}$$

THM 2 (BÖKSTEDT) $\ell = \mathbb{F}_p, M_*(-) = M_*(-; \ell)$

$$M_*(TMH(\mathbb{F}_p)) = \mathcal{A}_* \otimes \mathbb{F}_p[\alpha]$$

where α is primitive.

$$\text{where } \mathcal{A}_* = \pi_*(\ell \wedge \ell) = M_*(\ell)$$

dual Steenrod algebra

$$\mathcal{A}^* = [\ell, \ell]^* = M^*(\ell)$$

more than 2 \Rightarrow Thm 1

for any X , M_*X is an \mathcal{A}_* -module

$$\text{via } \ell \wedge X = \ell \wedge \mathbb{S} \wedge X \rightarrow \ell \wedge \ell \wedge X$$

$$\mathbb{1} = \mathbb{1} \otimes \mathbb{1}$$

$\alpha \in M_*X$ primitive
if $\psi(\alpha) = 1 \otimes \alpha$

π_*

$$\begin{array}{ccc}
 U_* X & \xrightarrow{\quad \psi \quad} & \pi_*(k \wedge k \wedge X) \\
 \searrow \psi & & \parallel \\
 & & \pi_*(k \wedge k \wedge X) \\
 & & \parallel \\
 \mathcal{A}_* \otimes U_* X & = & \pi_*(k \wedge k) \otimes_{\pi_*(k)} \pi_*(k \wedge X)
 \end{array}$$

Lemma 3 if B (e.g. $\text{THH}(\mathbb{F}_p)$) is an E_{∞} k -algebra then $\pi_* B = \text{Prim}(U_* B)$

Proof

$$\begin{array}{ccccc}
 B & \xrightarrow{\eta_{L,1}} & k \wedge B & \xrightarrow{\eta_{L,1}} & k \wedge k \wedge B \\
 & & \eta_{R,1} & & \\
 & \curvearrowright & & \curvearrowleft & \\
 \lambda b & \longleftarrow & \lambda a b & \longleftarrow & \lambda a \mu b \\
 & & b \mapsto \psi(b) & &
 \end{array}$$

Apply π_* : $\pi_* B \rightarrow U_* B \rightrightarrows \mathcal{A} \otimes U_* B$
 $b \mapsto 1 \otimes b$

Lemma 4 (Bökstedt SS) R is a k -algebra. (ANGELWIST. ROBINES)

$$U_* U_* (U_* R | k) \Rightarrow U_* (\text{THH}(R))$$

This is a SS. of \mathcal{A}_* -coalgebra $U_* R$ -algebras

Pf

$$\text{THH}(R) = \left| \begin{array}{c} \cong \\ \cong R \wedge \dots \wedge R \cong \\ \cong \end{array} \right|$$

$| U_*$

$$\downarrow \\ H_*(T\mathbb{H}^n(\mathbb{R})) \cong H_*\left(\bigoplus_{\mathbb{Z}} H_*\mathbb{R} \otimes \dots \otimes H_*\mathbb{R}\right) \quad \square$$

Since $T = \text{circle group}$ acts on $T\mathbb{H}^n$

$$\begin{array}{ccc} \sigma: \mathbb{Z}\mathbb{R} & \longrightarrow & T\mathbb{H}^n(\mathbb{R}) \\ \parallel & & \swarrow \text{ACTION} \\ T\mathbb{R} & \longrightarrow & T \wedge T\mathbb{H}^n(\mathbb{R}) \\ & & \parallel \\ & & T \downarrow \wedge T\mathbb{H}^n(\mathbb{R}) \end{array}$$

induce $\sigma_*: H_k(\mathbb{R}) \longrightarrow H_{k+2}(T\mathbb{H}^n(\mathbb{R}))$

lemma σ_* is a derivation & for $\alpha \in H_*\mathbb{R}$

$$Q^k \sigma_*(\alpha) = \sigma_* Q^k(\alpha) \quad \square$$

Recall $\mathcal{A}_* = \mathbb{F}_p[\bar{e}_1, \bar{e}_2, \dots] \otimes \wedge(\bar{t}_0, \bar{t}_2, \dots)$
 (p odd) $|\bar{e}_i| = 2p^i - 2 \quad |\bar{t}_i| = 2p^i - 1$

PROP 6

$$E_{**}^2 = HH_*(\mathcal{A}_* | K) = \mathcal{A}_* \otimes \wedge(\sigma \bar{e}_1, \sigma \bar{e}_2, \dots)$$

where $\sigma: H_*\mathbb{R} \longrightarrow HH_{2i}(\mathbb{R} | \mathbb{F}_p) \otimes \Gamma(\sigma \bar{t}_0, \sigma \bar{t}_2, \dots)$
 $\alpha \longmapsto 1 \otimes \alpha$

$$\textcircled{2} (\sigma_* \bar{t}_i)^p = \sigma_* \bar{t}_{i+2}$$

$$\textcircled{3} \sigma_* \bar{\xi}_i = 0$$

$$\text{pf } \textcircled{2}: \sigma_* \bar{t}_{i+1} = \sigma_* (Q^{p^i} \bar{t}_i) = Q^{p^i} (\sigma_* \bar{t}_i) = (\sigma_* \bar{t}_i)^p$$

$|\sigma_* \bar{t}_i| = 2^{p^i}$

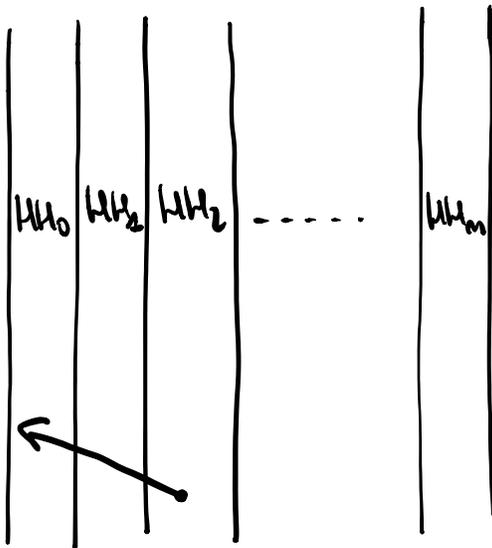
$$\textcircled{3} \sigma_* \bar{f}_i = \sigma_* \beta \bar{t}_i = \beta \sigma_* \bar{t}_i = \beta (\sigma_* \bar{t}_{i-1})^p = 0 \quad \square$$

Lemma 7

$d^i = 0$ for $i = 2, 3, \dots, p-2$.

$$d^{p-2}(\sigma \bar{t}_i^{[p]}) = \sigma \bar{f}_{i+2}, \quad d^{p-2}(\sigma \bar{t}_i^{[pk]}) = \sigma \bar{f}_i \sigma \bar{t}_i^{[k]}$$

pf SS:



Diff. satisfies Leibniz

Hence first diff. is nonzero on mult. index. Hence from a power of p .

Also we are in primitives hence d^{p-2} is the first

Cor 8

$$E^p = E^\infty = \mathcal{A}_* \otimes \mathbb{F}_p \left[\frac{\sigma \bar{t}_0, \sigma \bar{t}_2, \dots}{\sigma \bar{t}_0^p, \sigma \bar{t}_2^p, \dots} \right]$$

Resolving mult ext^m using ②

$$H_*(\text{THH}(\mathbb{F}_p)) = \mathcal{A}_* \otimes \mathbb{F}_p \left[\sigma \bar{t}_0 \right] \quad \square$$