

CYCLOTOMIC SPECTRA AND TC

§1 THE TATE CONSTRUCTION

A finite group, \mathcal{C} preadditive ∞ -category
(zero object, finite (co)product, $X \amalg Y \xrightarrow{\cong} X \times Y$)

$\cong \mathcal{C} = Sp$, $G = C_p$
 $X: BG \rightarrow \mathcal{C}$ a G -object

Lemma There is a map

$$N: X_{hG} := \underset{BG}{\text{colim}} X \rightarrow \underset{BG}{\text{lim}} X =: X^{hG}$$

For $\mathcal{C} = Sp$, $X = MA$, A abelian with G -action

$$\pi_0 N: A_G \longrightarrow A^G$$

$$[a] \longmapsto \left[\begin{array}{c} a \\ g \cdot a \end{array} \right]_{g \in G}$$

idea of const.:

$$\text{Hom}_{\mathcal{C}}(X_{hG}, X^{hG}) \xrightarrow{\text{left adjoint to const. diag}} \text{Hom}_{\mathcal{C}^{BG \times BG}}(\text{pr}_1^* X, \text{pr}_2^* X)$$

right adjoint to const. diag.

where $\text{pr}_i: BG \times BG \rightarrow BG$

$$\text{pointwise given by: } X \xrightarrow{\sim} \underset{G}{\amalg} X \xrightarrow{\cong} \underset{G}{\coprod} X \xrightarrow{\cong} X$$

$$x \longmapsto (g \mapsto g \cdot x)$$

□

Def $X^{tG} = \text{cofib } N = \text{colim} \left(\begin{array}{ccc} X_{hG} & \xrightarrow{N} & X^{hG} \\ \downarrow & & * \end{array} \right)$

Ex $X = MA$

$$\pi_i((MA)^{tG}) = \hat{H}^{-i}(G; A)$$

Rmk If $H \trianglelefteq G$ normal

$$(-)^{tH} : \mathcal{C}^{BG} \longrightarrow \mathcal{C}^{BG/H}$$

q2 CYCLOTOMIC SPECTRA AND TC

Def A cyclotomic spectrum is a spectrum X with an action of the circle π ($B\pi \xrightarrow{\times} \mathbb{S}^0$) & π -equiv. maps $\varphi_p : X \rightarrow X^{tC_p}$ where π acts on X^{tC_p} via $\pi \xleftarrow[\cong]{(-)^p} \pi/C_p$ for every prime p .

A p -cyclotomic spectrum is a spectrum X with an action of $C_{p^\infty} \subseteq S^1$ & a C_{p^∞} -equivariant map

$$(C_{p^\infty} \xleftarrow[\cong]{(-)^p} C_{p^\infty}/C_p) \quad X \longrightarrow X^{tC_p}$$

Ex. $S^{\text{triv}} = (S \text{ with trivial action},$

$$\varphi_p: \mathbb{S} \xrightarrow{\text{const.}} \text{Map}(B\mathcal{C}_p+, \mathbb{S}) = \mathbb{S}^{h\mathcal{C}_p} \downarrow \mathbb{S}^{t\mathcal{C}_p})$$

- \mathbb{Z} a connected space, $\wedge \mathbb{Z} = \text{Map}(S^1/\pi, \mathbb{Z})$ free loops.

$$\begin{aligned} \mathcal{E}^\infty(\wedge \mathbb{Z})_+ &\stackrel{\cong}{(-)^p} \mathcal{E}^\infty(\text{Map}(S^1/\mathcal{C}_p, \mathbb{Z})_+) \cong \\ &\cong \mathcal{E}^\infty(\text{Map}(S^1, \mathbb{Z})_+^{h\mathcal{C}_p}) \rightarrow (\mathcal{E}^\infty \text{Map}(\mathbb{Z})_+)^{h\mathcal{C}_p} \\ &\quad \downarrow \qquad \qquad \qquad ()^{t\mathcal{C}_p} \end{aligned}$$

- $\text{THH}(A)$ (cofibr), φ does not factors through $(\mathbb{Z})^{h\mathcal{C}_p}$

Given $F, G: \mathcal{C} \rightarrow \mathcal{D}$ define

$$\begin{array}{ccc} \text{Eq}(F, G) & \longrightarrow & \mathcal{D}^{\Delta^2} \\ \downarrow \text{---}^{\text{of SSet}} & & \downarrow (\text{ev}_0, \text{ev}_1) \\ \mathcal{C} & \xrightarrow{(F, G)} & \mathcal{D} \times \mathcal{D} \end{array}$$

so that objects are $(C, f: F(C) \rightarrow G(C))$

this is a pullback also in Cat_{∞} , ϵ stable
if C, D stable & F, G exact.

$$\text{Def } CycSp = \text{Eq} \left(Sp^{BT} \xrightarrow{\Delta} \prod_{P \text{ prime}} Sp^{BT} \right)$$

$$\pi_P(-)^{TCP}$$

$$CycSp_p = \text{Eq} \left(Sp^{BC_{p\infty}} \xrightarrow{\begin{array}{c} \text{Id} \\ (-)^{TCP} \end{array}} Sp^{BC_{p\infty}} \right)$$

$x \in CycSp$, $TC(x) = \text{Map}_{CycSp}(S^{\text{triv.}}, x)$ (looping spectrum

$x \in CycSp_p$, $TC(x; p) = \text{Map}_{CycSp_p}(S^{\text{triv.}}, x)$

and you have for free there is symmetric monoidal

"LEMMA" (MOST IMPORTANT THING OF THE PAPER)

$$TC(x) = \text{fib} \left(x^{h\prod} \xrightarrow{\pi_p^{h\prod} - \text{con}} \prod_p (x^{TCP})^{h\prod} \right)$$

where $\text{con}: x^{h\prod} = (x^{hCP})^{h\prod/CP} \xrightarrow{\sim} (x^{TCP})^{h\prod/CP} \simeq (x^{TCP})^{h\prod}$

$$TC(x; p) = \text{fib} \left(x^{hCP_p} \xrightarrow{CP^{hCP_p} - \text{con}} (x^{TCP})^{hCP_p} \right)$$

$$\begin{aligned}
 &\text{pf } \text{Mop}_{\text{Eq}(F, G)}((c, f), (d, g)) = \\
 &= \text{Eq}(\text{Mop}_C(c, d) \xrightarrow{\substack{f^*G \\ g^*F}} \text{Mop}_C(F(c), G(d))) \\
 &+ \text{Mop}_{S^{\text{BT}}}(\mathbb{S}, x) = \text{Mop}_{S^{\text{p}}}(\mathbb{S}, x)^{\text{ht}} = x^{\text{ht}}
 \end{aligned}$$

□

Point: This agrees with an earlier definition for bounded below spectra

⚠

§3 THE CYCLOTOMIC STRUCTURE ON THH

The construction of $\Psi: \text{THH}(A) \rightarrow \text{THH}(A)^{\text{tp}}$ relies on the Tate disperal $\Delta: X \rightarrow (\underbrace{X \otimes X \otimes \dots \otimes X}_p)^{\text{tp}}$ of Spectre (For all the Spectre!). It factors through $()^{\text{tp}}$ for $X = \Sigma^\infty Y$.

prop $T_p: S^p \rightarrow S^p$ is exact

pf $p=2$

let $x_0 \rightarrow x_1 \rightarrow x_2$ exact.

↪ filtration

$$\begin{array}{ccccc}
 & & \curvearrowleft & & \\
 & x_0 \otimes x_0 & \rightarrow & \omega^{\text{THH}} \left(\begin{array}{c} x_0 \otimes x_0 \rightarrow x_0 \otimes x_1 \\ \downarrow \\ x_1 \otimes x_0 \end{array} \right) & \rightarrow x_1 \otimes x_1 \\
 & \curvearrowleft & | & & | \\
 & & & &
 \end{array}$$

$$\begin{array}{ccc}
 & \downarrow & \downarrow \\
 \text{ACTION OF } G & X_0 \otimes x_1 \amalg x_2 \otimes x_0 & x_1 \otimes x_2 \\
 (-)^{tG} & \xrightarrow{\quad} & \xrightarrow{\quad} \\
 (X_0 \otimes x_0)^{tG} & \xrightarrow{\sim} \text{colim}(-) \rightarrow (x_1 \otimes x_1)^{tG} & \\
 & \downarrow & \downarrow \\
 & * & (x_2 \otimes x_2)^{tG} \quad \square
 \end{array}$$

FACT (URIE/NIKOLAEV)

$$\text{Mop}_{\text{Fun}(S^1, S^1)}^{\text{ex}}(\text{Id}, F) \xrightarrow[\text{on } S]{} \text{Mop}(S, F)$$

$$\begin{array}{c}
 \text{Def } D: \text{Id} \rightarrow T_p \text{ on } S \rightarrow T_p(S) = \underbrace{(S \otimes \dots \otimes S)}_p^{tG} \\
 \text{take diagonal in this case} \\
 \text{if for striv.} \quad \parallel \quad S^{tG}
 \end{array}$$

REMK Every natural transformation of chain complexes

$$C \rightarrow (C \otimes \dots \otimes C)^{tG} \text{ is zero (pointwise)}$$

(doesn't work for chains!)

Recall:

$$\begin{aligned} \text{THH}(A) &= | A \leftarrow A^{\otimes 2} \leftarroweq A^{\otimes 3} \leftarroweq \dots | \\ &\quad \downarrow \quad \downarrow \quad \downarrow \\ &| (A^{\otimes p})^{t(p)} \leftarroweq (A^{\otimes 2p})^{t(p)} \leftarroweq (A^{\otimes 3p})^{t(p)} \leftarroweq \dots | \end{aligned}$$

p-fold subdivision of THH.

on $(\text{sd}_p \text{THH}(A))^{t(p)}$

(*)

$$|\text{sd}_p \text{THH}(A)|^{t(p)} \stackrel{\text{THEOREM}}{\simeq} \text{THH}(A)^{t(p)}$$

(*)

$$X: \Delta^{\text{op}} \rightarrow \mathcal{C}$$

$$\text{sd}_p X: \Delta^{\text{op}} \rightarrow \Delta^{\text{op}} \xrightarrow{X} \mathcal{C}$$

$$[\alpha] \longmapsto [\alpha] * \dots * [\alpha]$$

(It's just concatenating
these things! think in
this way!!)