

## TC ( $\mathbb{F}_p$ )

RECALL: (EMANUELE TALK)

- Cyclostationary spectrum  $X$ :  $T$ -action, and  $T$ -equivalent maps  $\varphi_p: X \rightarrow X^{t\mathbb{G}_p}$  + prime  $p$ . (Frobenius)
- EX:  $\text{TMH}(R)$   $R$ : ring spectrum.

$$\text{TC}(X) = \text{fib} \left( X^{\text{ht}} \xrightarrow{\prod_p (\varphi_p^{\text{ht}} - \text{can})} \prod_p (X^{t\mathbb{G}_p})^{\text{ht}} \right)$$

$$\text{can}: X^{\text{ht}} = (X^{\text{ht}\mathbb{G}_p})^{\text{ht}/\mathbb{G}_p} \longrightarrow (X^{t\mathbb{G}_p})^{\text{ht}/\mathbb{G}_p} \simeq (X^{t\mathbb{G}_p})^{\text{ht}}$$

$$\text{TC}(X, p) = \text{fib} \left( X^{\text{ht}} \xrightarrow{\varphi_p^{\text{ht}} - \text{can}} (X^{t\mathbb{G}_p})^{\text{ht}} \right)$$

GOAL:  $\prod_* \text{TC}(\mathbb{F}_p)$

RECALL (JOHN TALK)

TMM (BÖKSTEN)

$$\prod_* (\text{TMH}(M\mathbb{F}_p)) = \mathbb{F}_p[\mu] \quad |\mu|=2$$


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NOTE:

- $M\mathbb{F}_p$  is a  $p$ -complete spectrum ( $p=0$ )
- $\text{TMH}(M\mathbb{F}_p)$  is a  $p$ -complete spectrum ( $\text{TMH}(M\mathbb{F}_p)$  is an algebra over  $M\mathbb{F}_p$ ).

$$\Rightarrow \text{TC}(M\mathbb{F}_p) = \text{TC}(M\mathbb{F}_p, p)$$

- LEMMA (NS, II.4.2)

$X$  bounded below spectrum with  $T$ -action then

$$X^{tT} \longrightarrow (X^{t(\mathbb{F}_p)})^{hT/\mathbb{F}_p} \simeq (X^{t(\mathbb{F}_p)})^{hT}$$

is a  $p$ -completion. If  $X$  is moreover  $p$ -complete then the map is an equivalence.

Rmk Tate for the circle group: cofiber sequence:

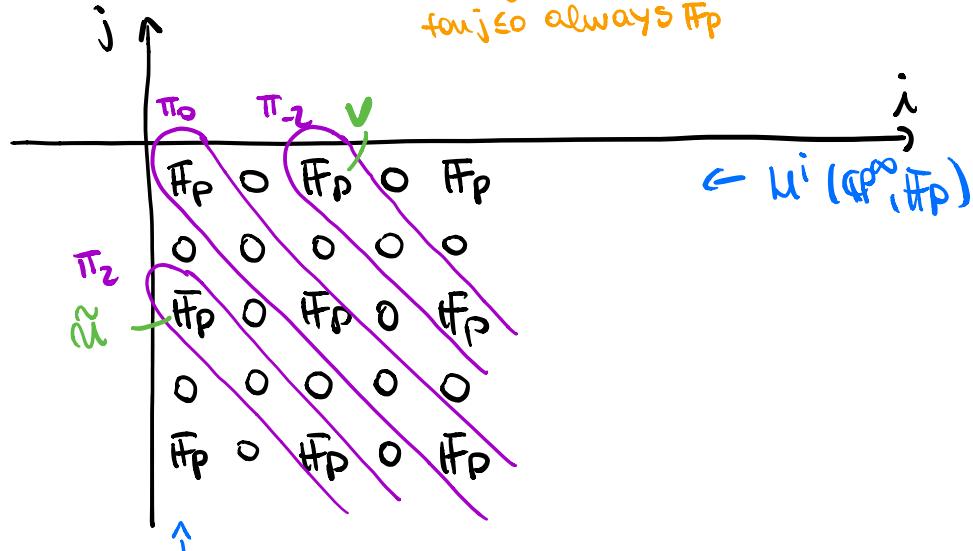
$$\{ X_{hT} \xrightarrow{N} X^{hT} \longrightarrow X^{tT}$$

MAYBE: ~~COFIBER SEQUENCE:~~

$$\text{TC}(\mathbb{F}_p) \xrightarrow{\quad} \text{TMU}(\mathbb{H}\mathbb{F}_p)^{hT} \xrightarrow{\varphi_p^{hT} - \text{com}} \text{TMU}(\mathbb{H}\mathbb{F}_p)^{tT}$$

①  $\text{TMU}(\mathbb{H}\mathbb{F}_p)^{hT}$

$\rightarrow$  a multiplicative s.s. (homotopy fixed points s.s.)

$$E_2^{ij} = \mu^i(BT; \pi_{-j} \text{TMU}(\mathbb{H}\mathbb{F}_p)) \xrightarrow{\text{for } j \leq 0 \text{ always } \mathbb{F}_p} \pi_{-i-j} \text{TMU}(\mathbb{H}\mathbb{F}_p)^{hT}$$


$$\pi_{-j}^1(\mathrm{THH}(M_{\mathbb{F}_p}))$$

- $\exists v \in \pi_2 \mathrm{THH}(M_{\mathbb{F}_p})^{ht}$  lift of the natural generation of  $M^2(\mathbb{CP}^\infty; \mathbb{F}_p)$
- $\pi_* \mathrm{THH}(M_{\mathbb{F}_p})^{ht} \rightarrow \pi_* \mathrm{THH}(M_{\mathbb{F}_p})$  surjective  
 $\exists \tilde{m} \in \pi_2 \mathrm{THH}(M_{\mathbb{F}_p})^{ht} \mapsto m \in \pi_2 \mathrm{THH}(M_{\mathbb{F}_p})$   
 $(\pi_* \mathrm{THH}(M_{\mathbb{F}_p}) = \mathbb{F}_p[m])$

LEMMA (NS IV.4.7) the image of  $p \in \pi_0 \mathrm{THH}(M_{\mathbb{F}_p})^{ht}$   
 in  $E_2^{2,2} = M^2(BT, \pi_2 \mathrm{THH}(M_{\mathbb{F}_p}))$  is  $mv$ .

- on the zeroth diagonal:

$$\begin{array}{ccccccc} \mathbb{F}_p & \xrightarrow{P} & Q_1 & \xrightarrow{P^2} & Q_2 & \rightarrow \dots & \pi_0 \mathrm{THH}(M_{\mathbb{F}_p})^{ht} = \text{colim} \\ & \downarrow & \downarrow & \downarrow & \downarrow & & \\ mv & \mathbb{F}_p & mv^2 & \mathbb{F}_p & & & \end{array}$$

$P$  to  $Q_2$ , multiplicativity of the SS.:

$$P^2 = m^2 v^2$$

They are all non-zero

$$P^3 = m^3 v^3$$

$\Rightarrow$  the extension is of ab. groups:

:

$$\Rightarrow \pi_0 \mathrm{THH}(M_{\mathbb{F}_p})^{ht} = \mathbb{Z}_p$$

- now shift the 0th diagonal mult. by powers of  $\tilde{m}$  on  $v$  obtain the same thing.

changing  $\tilde{m}$  with a unit :  $\tilde{m}v = p$

$$\pi_* \text{THH}(Mfp)^{ht} = \mathbb{Z}_p[\tilde{m}, v] / (\tilde{m}v - p)$$

$$\textcircled{2} \underline{\text{THH}(Mfp)^{tT}} \quad \text{set } X := \text{THH}(Mfp)$$

$\exists$  a multiplicative S.S. (Tate S.S.) :

$$E_2^{ij} = \pi_{-i}(\mu(\pi_{-j}X))^{tT} \Rightarrow \pi_{-i-j}X^{tT}$$

FACT let  $M$  an abelian group, then

$$\pi_{-i}(HM)^{ht} \xrightarrow[i \geq 0]{\cong} \pi_{-i}(HM)^{tT} \quad (\text{so for } i \geq 0)$$

$\cong$   
 $h^i(T, M)$   
 (group cohomology)

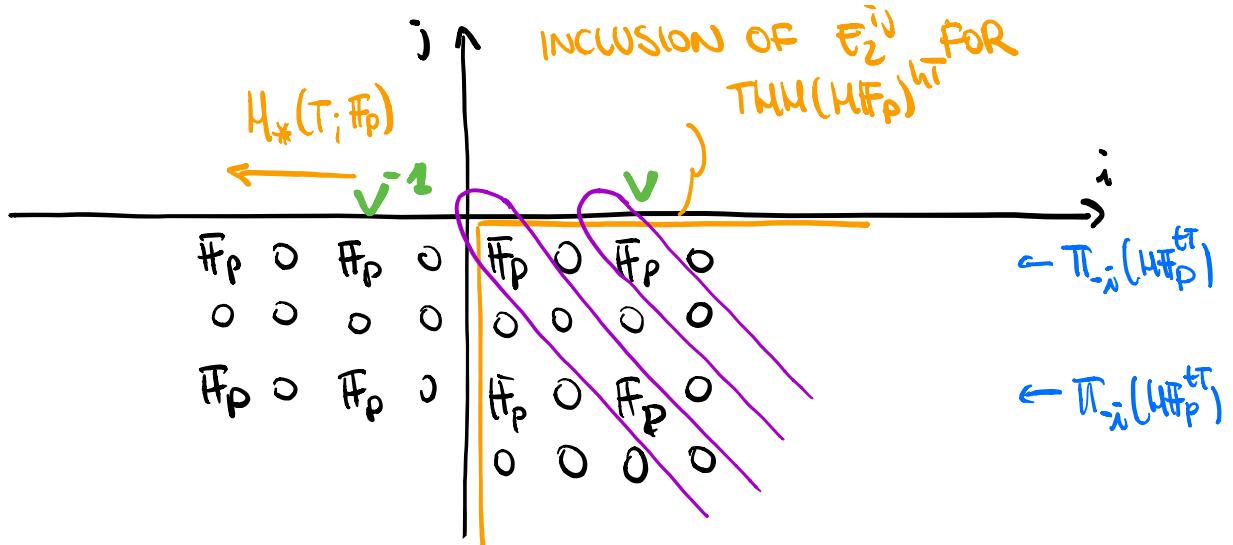
$\hat{h}^i(T, M)$   
 (TATE cohomology)

(LES of  $\Sigma HM_{ht} \rightarrow HM^{ht} \rightarrow HM^{tT}$  on homotopy grps)  
 for an Eilenberg MacLane spectrum

$$\rightarrow \pi_2(HM^{ht}) \rightarrow \pi_2(HM^{tT}) \xrightarrow{\cong} \pi_2(\Sigma HM_{ht})$$

$$\pi_0(HM^{ht}) \rightarrow \pi_0(HM^{tT}) \rightarrow \pi_0(\Sigma HM_{ht})$$

$$\pi_0(HM^{ht}) \xrightarrow{\cong} \pi_0(HM^{tT}) \rightarrow \pi_{-1}(\Sigma HM_{ht})$$



- The right bottom quotient is exactly the previous one, with the same multiplicative behaviour.
- For negative degrees we obtain exactly the same set of homotopy fixed points:  $\mathbb{Z}_p$
- Using  $v^{-1}$  we translate also in positive degrees

$$\pi_* \text{THH}(HF_p)^{tr} = \mathbb{Z}_p [\overset{\circ}{N}, N^{-1}]$$

### ③ MAPS

Note that the canonical map is obvious from the inclusion of the S.S.

$$\frac{\mathbb{Z}_p[\tilde{m}, N]}{(\tilde{m}N - p)} \xrightarrow{\text{can}} \mathbb{Z}_p[v, v^{-1}]$$

$$v \mapsto v$$

$$\tilde{m} \mapsto PV^{-1}$$

CANONICAL:  $M = 2K$  even is injective

$$\begin{array}{ccc} \underbrace{\pi_m T_{M\#}(M\#p)}_{\mathbb{Z}_p} & \xrightarrow{\text{ht}} & \overline{\text{Im}} T_{M\#}(M\#p)^{\text{ht}} \\ & & \underbrace{\mathbb{Z}_p} \end{array}$$

for  $m \leq 0$  ISOMORPHISM.

for  $m > 0$  has image  $p^K \mathbb{Z}_p$

- PROBENIUS:

$$\begin{array}{ccc} \mathbb{Z}_p[\tilde{m}, V] / \tilde{n}_{(\tilde{m}V - p)} & \xrightarrow{\varphi_p^{\text{ht}}} & \mathbb{Z}_p[V, V^{-1}] \\ V \mapsto & \mapsto \lambda V & \lambda \in \mathbb{Z}_p \\ \tilde{m} \mapsto & \mapsto \mu V^{-1} & \end{array}$$

Map is multiplicative and it's a map of  $\mathbb{Z}_p$ -algebras  
in particular

$$\tilde{m}V = p \longmapsto p = \lambda \mu$$

$$\begin{array}{ccccc} X^{\text{ht}} & \xrightarrow{\varphi_p^{\text{ht}}} & (X^{t\#p})^{\text{ht}} & \simeq X^{t\#} & \xrightarrow{\mu} M\#p^{\text{ht}} \\ \downarrow & & \downarrow \text{res} & & \downarrow \text{res} \\ X & \xrightarrow{\varphi_p} & X^{t\#p} & \xrightarrow{\mu} & M\#p^{t\#p} \end{array} \quad (*)$$

where  $\mu$  is the multiplication map:

$$\begin{array}{ccc}
 M\mathbb{F}_p & \xrightarrow{\text{So}} & \text{Thh}(M\mathbb{F}_p) = | M\mathbb{F}_p \subseteq M\mathbb{F}_p \wedge M\mathbb{F}_p \subseteq \dots | \\
 & \searrow \text{Id} & \downarrow \mu \quad \text{mult. } \mu \text{ at each level.} \\
 & & M\mathbb{F}_p = | M\mathbb{F}_p \subseteq M\mathbb{F}_p \subseteq M\mathbb{F}_p \dots |
 \end{array}$$

Look the diagram (\*) in  $\Pi_2$ :

$$\begin{array}{ccccc}
 \Pi_2 X^{hT} & \xrightarrow{\quad} & \Pi_2 X^{tT} & \xrightarrow{\quad} & \Pi_2 M\mathbb{F}_p^{tT} = \mathbb{F}_p \\
 \downarrow & \swarrow \text{AV} & \downarrow & \nearrow [\lambda] & \downarrow \\
 \Pi_2 X & \xrightarrow{\quad} & \Pi_2 X^{tC_p} & \xrightarrow{\quad} & \Pi_2 M\mathbb{F}_p^{tC_p} = \mathbb{F}_p \\
 \text{= 0} & \text{green curve} & & & \text{= 0} = [\alpha]
 \end{array}$$

- The bottom left corner is zero no anti-clockwise  $\nabla$  is sent to zero.
- On the top now  $\nabla$  maps to the reduction  $[\lambda]$  in  $\mathbb{F}_p$ .  
 $\Rightarrow \lambda$  is  $p$ -divisible ad  $\mu$  a unit

$$P = \lambda \mu \rightsquigarrow \frac{\lambda}{p} \cdot \mu = 1$$

CONCLUSION: opposite behavior of the canonical map. ISO for  $n \geq 0$  and image  $p^k \mathbb{Z}_p$  for  $n < 0$  even.

### • $TC(\mathbb{F}_p)$ :

L.E.S. in homotopy groups:

$$\pi_n THH(W\mathbb{F}_p)^{ht} \xrightarrow{\cong} \pi_n THH(W\mathbb{F}_p)^{tf}$$

$$\hookrightarrow \pi_0 TC(\mathbb{F}_p) \longrightarrow \pi_0 THH(W\mathbb{F}_p)^{ht} \xrightarrow{\circ} \pi_0 THH(W\mathbb{F}_p)^{tf}$$

$$\hookrightarrow \pi_{-n} TC(\mathbb{F}_p) \longrightarrow \pi_{-n} THH(W\mathbb{F}_p)^{ht} \xrightarrow{\cong} \pi_{-n} THH(W\mathbb{F}_p)^{tf}$$

...

- $\pi_n(\mathbb{F}_p^{ht} - \text{com})$  for  $n \neq 0$  is the difference of an ISO and a  $p$ -divisible map between  $\mathbb{Z}_p$  vs it's an ISO.
- $\pi_0(\mathbb{F}_p^{ht} - \text{com})$  at degree zero they are both the identity, 'cause rep of wing Spectra, both algebras on them  $\pi_0$  are reps of  $\mathbb{Z}_p$ -algebras. Hence is the zero map in degree 0.

$$TC_*(\mathbb{F}_p) = \mathbb{Z}_p[\varepsilon]/\varepsilon^2 \quad |\varepsilon| = -1$$