TC OF LOOP SPACES

<u>AIM</u>: Recover BHM's comp. of TC of suspension spectrum of loop space

Y connected based space, there is a pullback square

$$\begin{tikzcd} TC(\Sigma^\infty_+(\Omega Y)) \arrow[r] \arrow[d] & \Sigma(\Sigma^\infty_+ LY)_{hT} \arrow[d] \\ \Sigma^\infty_+ LY \arrow[r, "1-\tilde{\varphi}_p"] & \Sigma^\infty_+ LY \end{tikzcd}$$

after p-completion. (LY free loop space on Y)

"<u>Recall</u>": $\forall$ connected space Y, ΩY is an $\mathbb{E}_1$-gp in $\mathcal{S}$ (∞-category of spaces). Since

$\Sigma^\infty_+ : \mathcal{S} \longrightarrow Sp$ is symm. monoidal & preserves colimits,

$\mathbb{S}[\Omega Y] := \Sigma^\infty_+ \Omega Y$ is an $\mathbb{E}_1$-algebra in Sp

$\Rightarrow THH(\mathbb{S}[\Omega Y])$

By definition $THH(\mathbb{S}[\Omega Y])$ is the geom. realization of a cyclic spectrum

$$N(\Lambda^{op}) \to N(Ass^{\otimes}_{ACT}) \xrightarrow{\Omega Y^{\otimes}} S^{\otimes}_{ACT} \to Sp^{\otimes}_{ACT} \xrightarrow{\otimes} Sp \quad (*)$$

Since Σ^{∞}_{+} is nice, we can write $(*)$ as Σ^{∞}_{+} of the geom. realization of

$$B^{cyc}_{\bullet}\Omega Y : N(\Lambda^{op}) \to N(Ass^{\otimes}_{ACT}) \xrightarrow{\Omega Y^{\otimes}} S^{\otimes}_{ACT} \to S \quad (**)$$

For an E_1-monoid in S, let $B^{cyc}M$ the geom. realization of $(**)$ (replacing ΩY by M)

<u>Lemma</u> (NS, IV.3.1)

(i) The cyclic bar construction was with a canonical $\mathbb{T}$-map and $\mathbb{T}$-equiv. morphisms

$$\Psi_p : B^{cyc}M \longrightarrow (B^{cyc}M)^{hC_p} \quad \forall p \text{ prime}$$

(ii) There is a comm. square

$$\begin{array}{ccc} M & \longrightarrow & B^{cyc}M \\ \downarrow{\scriptstyle \Delta_p} & & \downarrow{\scriptstyle \Psi_p} \\ (M\times\ldots\times M)^{hC_p} & \longrightarrow & (B^{cyc}M)^{hC_p} \end{array}$$

(iii) Upon taking the Σ^∞_+, the map ψ_p refines the cyclotomic structure of $THH(\mathbb{S}[\Omega Y])$ i.e.

$$\begin{tikzcd}
\Sigma^\infty_+ B^{cyc} M \arrow[r, "\psi_p"] \arrow[d, "\varphi_p"] & \Sigma^\infty_+ (B^{cyc} M^{hC_p}) \arrow[d] \\
(\Sigma^\infty_+ B^{cyc} M)^{tC_p} & \Sigma^\infty_+ (B^{cyc} M)^{hC_p} \arrow[l, "can"]
\end{tikzcd}$$

"<u>proof</u>" Maps ψ_p are constructed in the same way as φ_p for THH, replacing Tate diagonal w/ $\Delta_p : M \longrightarrow (\underbrace{M \times \dots \times M}_{p})^{hC_p}$ $\square$

- We have also explicit description of $B^{cyc} M$ (Prop. IV.3.2). Assume that $M = \Omega Y$ for Y connected based space.

(i) There is a natural $\mathbb{T}$-equivariant equivalence $B^{cyc} M \simeq LY = Map(S^1, Y)$

(ii) Under this equivalence the $\mathbb{T} \cong \mathbb{T}/C_p$-equiv. map
$$\psi_p : B^{cyc} M \longrightarrow (B^{cyc} M)^{hC_p}$$

is identified with the map $LY \to LY^{hC_p}$ induced by the p-fold covering of S^1.

<u>COR</u> (NS IV.3.3)

There is a natural $\mathbb{T}$-equiv. equivalence
$THH(\mathbb{S}[\Omega Y]) \simeq \Sigma^\infty_+ LY$

Under this equivalence the Frobenius map
$\varphi_p : THH(\mathbb{S}[\Omega X]) \to THH(\mathbb{S}[\Omega Y])^{tC_p}$,

is given by the composite
$$\Sigma^\infty_+ LY \xrightarrow{f_p} (\Sigma^\infty_+ LY)^{hC_p} \longrightarrow (\Sigma^\infty_+ LY)^{tC_p}$$

- More generally we say that a Frobenius lift on a p-cyclotomic spectrum (X, φ_p) is a C_{p^∞}-equivariant factorization of φ_p as
$$X \xrightarrow{\tilde{\varphi}_p} X^{hC_p} \longrightarrow X^{tC_p}$$

"<u>Recall</u>": X p-cyclotomic spectrum bounded below, then $TC(X)^\wedge_p = TC(X^\wedge_p) = TC(X^\wedge_p ; p)$
In this case the C_{p^∞} action on $X^\wedge_p$ extends to a $\mathbb{T}$-action.

__PROP__ (NS IV.3.4)
For a bdd below p-cyclotomic, p-complete spectrum X w/ a Frobenius lift $\hat{\varphi}: X \to X^{hC_p}$ we have a pullback square of the form:

$$\begin{tikzcd}
TC(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma X_{hT} \arrow[d] \\
X \arrow[r, "\mathrm{Id} - \hat{\varphi}_p"] & X
\end{tikzcd}$$

__Pf__ We have a pullback diagram

$$\begin{tikzcd}
TC(X,p) \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & 0 \arrow[d] \\
X^{hT} \arrow[r, "\mathrm{Id} - \tilde{\varphi}_p^{hT}"] & X^{hT} \arrow[r] & (X^{tC_p})^{hT}
\end{tikzcd}$$

Now we take a pullback RHS-square and obtain:

$$\begin{tikzcd}
TC(X,p) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \Sigma X_{hT} \arrow[r] \arrow[d, "\mathrm{Tr}"] \arrow[dr, phantom, "\circlearrowleft"] & 0 \arrow[d] \\
X^{hT} \arrow[r, "\mathrm{Id} - \hat{\varphi}_p^{hT}"] & X^{hT} \arrow[r, "\mathrm{can}^{hT}"] & (X^{tC_p})^{hT}
\end{tikzcd}$$

$\therefore$ $TC(X,p) = TC(X)$

2. $X^{tT} \longrightarrow (X^{tC_p})^{hT}$ is a p-completion of X^{tT} plus if X is p-complete it is an $\simeq$.

$\Rightarrow$ hofiber of can^{hT} is ΣX_{hT}.

We are done after proving

<u>Lemma</u> (NS IV.3.5) (Same assumptions)

Pullback square:

$$\begin{tikzcd} X^{hT} \arrow[r, "\mathrm{Id}-\hat{\varphi}_p^{hT}"] \arrow[d] & X^{hT} \arrow[d] \\ X \arrow[r, "\mathrm{Id}-\hat{\varphi}_p"] & X \end{tikzcd}$$

<u>Proof</u>

Note that $T \cong T/C_p$-equiv. Map $\tilde{\varphi}_p : X \to X^{hC_p}$ is equivalently given by a natural transformation Ψ_p from $X \circ f_p : BT \to BT \to Sp$ to $X : BT \to Sp$ where f_p is induced by the degree p self map of T.

With this identification the map $\tilde{\varphi}_p^{hT} : X^{hT} \longrightarrow X^{hT}$ is given by the composite

$$X^{hT} \xrightarrow{f_p^*} (X \circ f_p)^{hT} \xrightarrow{\lim_{BT} \Psi_p} X^{hT}$$

Now since for every function $Y: BT \to Sp$ we have that

$$Y^{hT} = \lim_{BT} Y \qquad BT = \mathbb{C}P^\infty = \bigcup_{n \geq 1} \mathbb{C}P^n$$

we can write that

$$Y^{hT} = \lim_{\leftarrow} \lim_{\mathbb{C}P^n} Y$$

with notational fiber sequences

$$\Omega^{2n} Y \longrightarrow \lim_{\mathbb{C}P^n} Y \longrightarrow \lim_{\mathbb{C}P^{n-1}} Y$$

.... just idea of work ....

<u>RMK</u>

$$\begin{array}{ccc} \Sigma^\infty_+ LY & \xrightarrow{1-\tilde{\varphi}_p} & \Sigma^\infty_+ LY \\ \downarrow ev & & \downarrow ev \\ \Sigma^\infty_+ Y & \xrightarrow{1-\tilde{\varphi}_p = 0} & \Sigma^\infty_+ Y \end{array}$$

$$TC(\mathbb{S}[\Omega Y]) \simeq \Sigma^\infty_+ Y \oplus \mathrm{fib}(ev \circ tm).$$