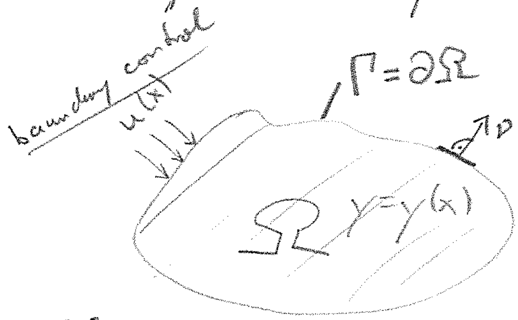
Optimal Control of Partial Differential Equations

1.1. Introduction

Here, we will consider optimal control of PDE problems.
There are different types of PDEs. We focus on linear elliptic PDEs.

In the next section, we introduce a few simple model problems.

1.2. Model problems



Optimal stationary boundary heating

Consider a body (e.g. a room) $\Omega \subset \mathbb{R}^3$ that is to be heated by a heat source u (constant in time) applied to its boundary $\Gamma = \partial\Omega$.

Aim: Choose u such that the stationary temperature distribution $y = y(x)$ in Ω is 'close to' a desired $y_\Omega = y_\Omega(x)$ (given)

~~Mathematical~~ Model:

"cost functional"

$$\min J(y,u) := \frac{1}{2}\int_\Omega |y(x) - y_\Omega(x)|^2\,dx + \frac{\lambda}{2}\int_\Gamma |u(x)|^2\,ds(x), \quad \lambda \geq 0$$

subject to

$$\begin{cases} -\Delta y = -\left(\dfrac{\partial^2 y}{\partial x_1^2} + \dfrac{\partial^2 y}{\partial x_2^2} + \dfrac{\partial^2 y}{\partial x_3^2}\right) = 0 & \text{in } \Omega \\ \dfrac{\partial y}{\partial \nu} = \alpha(u - y) & \text{on } \Gamma \end{cases}$$

"state equation"

and $\quad u_a(x) \leq u(x) \leq u_b(x)$ on Γ. "pointwise control constraints"

$\lambda \geq 0$: small parameter to include cost of heating and enforce regularity of controls;

$\nu = \nu(x)$: outward unit normal to Γ at $x \in \Gamma$;

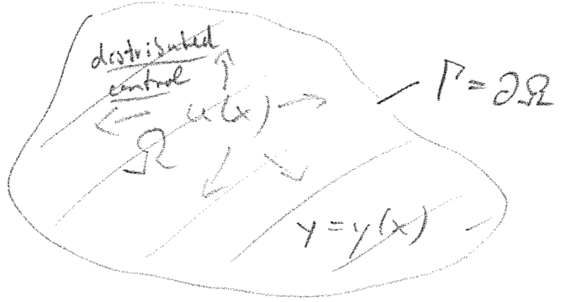
$\alpha > 0$: heat transmission coefficient (constant)
(More realistic: $a = a(x)$ and state eqn. $\operatorname{div}(a \nabla y) = 0$)

<u>Observation:</u> State equation is <u>linear</u> (with respect to y) elliptic PDE, cost functional is <u>quadratic</u>, control acts on the <u>boundary</u> of Ω.

$\rightarrow$ <u>linear-quadratic elliptic boundary control problem</u>

<u>Optimal heat source</u>

In a similar way, we can consider a heat source in the domain ('microwave') $\Omega \subset \mathbb{R}^3$.



<u>Aim:</u> Choose the distributed control $u = u(x)$ in Ω such that the stationary temperature distribution $y(x)$ is 'close to' a given, desired distribution $y_\Omega(x)$.

<u>Math. model:</u>

$$\min \; J(y,u) := \frac{1}{2} \int_\Omega |y(x) - y_\Omega(x)|^2 \, dx + \frac{\lambda}{2} \int_\Omega |u(x)|^2 \, dx$$

<u>"cost functional"</u>

subject to

$$\begin{cases} -\Delta y = \beta u & \text{in } \Omega \\ \quad\; y = 0 & \text{on } \Gamma \end{cases}$$

<u>"state eqn."</u> (assuming temperature is zero on boundary)

and $\quad u_a(x) \leq u(x) \leq u_b(x)$ <u>"pointwise control constraints"</u>

$\beta = \beta(x)$ given coefficient. (e.g. $\beta(x) = \chi_{\Omega_c}$ for some $\Omega_c \subset \Omega$)

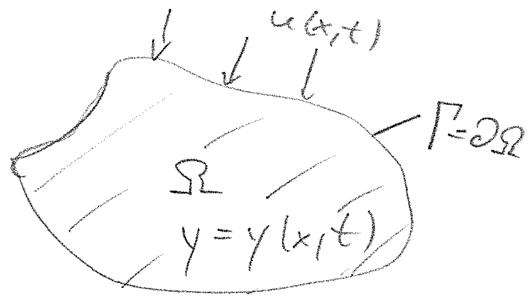
$\hookrightarrow$ characteristic function

$$\chi_{\Omega_c}(x) := \begin{cases} 1, & x \in \Omega_c \\ 0, & \text{else} \end{cases}$$

Observation: State equation is linear, elliptic PDE, cost functional is quadratic, control is distributed in Ω

$\to$ linear - quadratic elliptic control problem with distributed control.

Optimal non-stationary boundary control



$\Omega \subset \mathbb{R}^3$: potatoe to be roasted for some period of time $T>0$.

Temperature distribution $y = y(x,t)$.

Aim: Choose $u = u(x,t)$ such that the temperature distribution $y = y(x,t)$ is at a pleasant temperature y_Ω at final time T.

(Denote by $y_t = \frac{\partial y}{\partial t}$).

Math. model:

$$\min \; J(y,u) := \frac{1}{2} \int_\Omega |y(x,T) - y_\Omega(x)|^2 \, dx + \frac{\lambda}{2} \int_0^T \int_\Gamma |u(x,t)|^2 \, ds(x) dt$$

subject to

$$\begin{cases} y_t - \Delta y = 0 & \text{in } Q := \Omega \times (0,T) \\ \dfrac{\partial y}{\partial \nu} = \alpha (u - y) & \text{on } \Sigma := \Gamma \times (0,T) \\ y(x,0) = y_0(x) & \text{in } \Omega \end{cases}$$

and $\quad u_a(x,t) \le u(x,t) \le u_b(x,t)$ on $\Sigma = \Gamma \times (0,T)$.

Observation: State equation is a linear parabolic PDE, cost functional is quadratic, control acts on boundary

$\to$ linear - quadratic parabolic boundary control problem

Outlook

In this lecture we will focus on linear-quadratic elliptic optimal control problems with boundary control or distributed control.

We want to answer questions about

- existence of optimal controls
- necessary optimality conditions
- sufficient optimality conditions
- foundations of numerical methods

Along the way, we will present necessary mathematical tools from
- existence theory for linear elliptic PDE ('weak solutions');
- functional analysis (Sobolev spaces, differentiability in Banach spaces).

The material will follow quite closely the book of Tröltzsch, Chapters 1, 2 (and maybe 4), cf. reading list.

Initially, we will discuss some important basic concepts in finite dimensions.

1.3. Basic concepts for the finite-dimensional case

We will introduce some basic concepts of optimal control theory in finite dimensions avoiding the technical difficulties of PDE theory and functional analysis.

Finite-dimensional optimal control problems

Given:
- state space: $\mathbb{R}^n$ (vectors are column vectors)
- admissable set of controls (non-empty): $U_{ad} \subset \mathbb{R}^m$
- continuous cost functional $J : \mathbb{R}^n \times U_{ad} \to \mathbb{R}$
- matrix $A \in \mathbb{R}^{n\times n}$ and matrix $B \in \mathbb{R}^{n\times m}$

Problem: Find $y \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ which minimise $J(y,u)$ subject to $Ay = Bu$ and $u \in U_{ad}$. $\quad (*)$