Example: $f(u) = \frac{1}{2}|Su - y_d|^2 + \frac{\lambda}{2}|u|^2$

$$\Rightarrow \nabla f(u) = S^T(Su - y_d) + \lambda u,$$

$$f'(u)h = (\nabla f(u), h)_{\mathbb{R}^m} = (S^T(Su - y_d) + \lambda u, h)_{\mathbb{R}^m}$$

Variational inequality for an optimal control $\bar{u}$ reads:

$$(S^T(S\bar{u} - y_d) + \lambda\bar{u}, \tilde{u} - \bar{u})_{\mathbb{R}^m} \geq 0 \quad \text{for all } \tilde{u} \in U_{ad}.$$

Adjoint state

The necessary condition given above is a bit clumsy due to the appearance of $(A^{-1})^T$ and $S = A^{-1}B$. Also, in practical numerical realisations one strives to avoid computing the inverse A^{-1} if possible.

Therefore, we introduce the so-called adjoint state $\bar{p} \in \mathbb{R}^n$ by

$$\bar{p} = (A^{-1})^T \nabla_y J(\bar{y}, \bar{u})$$

The adjoint state $\bar{p}$ can be obtained by solving the linear system

$$A^T \bar{p} = \nabla_y J(\bar{y}, \bar{u}).$$

We call this the adjoint equation and $\bar{p}$ the adjoint state associated with $(\bar{y}, \bar{u})$.

Example: $J(y,u) = \frac{1}{2}|y - y_d|^2 + \frac{\lambda}{2}|u|^2$

$\Rightarrow$ adjoint equation: $A^T \bar{p} = \bar{y} - y_d$ since $\nabla_y J(y,u) = y - y_d$.

By introducing the adjoint state we can replace the variational inequality by the so-called <u>optimality system</u>:

$$\boxed{\begin{aligned} Ay &= Bu, \quad u \in U_{ad} \\ A^T p &= \nabla_y J(y,u), \\ \left(B^T p + \nabla_u J(y,u), v-u\right)_{\mathbb{R}^m} &\geq 0, \quad \text{for all } v \in U_{ad} \end{aligned}}$$

Every solution to the optimal control problem $(\bar{y}, \bar{u})$ must, together with the associated adjoint state $\bar{p}$, satisfy this system.

If there are <u>no constraints on u</u>, i.e. $U_{ad} = \mathbb{R}^m$, then $\tilde{u} - \bar{u}$ in the variational inequality can assume any value in $\mathbb{R}^m$, hence the variational inequality reduces to

$$B^T \bar{p} + \nabla_u J(\bar{y}, \bar{u}) = 0.$$

<u>Example</u>: $J(y,u) = \frac{1}{2}|Cy - y_d|^2 + \frac{\lambda}{2}|u|^2$ with $C \in \mathbb{R}^{n \times n}$

$$\Rightarrow \nabla_y J(y,u) = C^T(Cy - y_d), \quad \nabla_u J(y,u) = \lambda u$$

The optimality system reads

$$\boxed{\begin{aligned} Ay &= Bu, \quad u \in U_{ad} \\ A^T p &= C^T(Cy - y_d), \\ \left(B^T p + \lambda u, v-u\right)_{\mathbb{R}^m} &\geq 0, \quad \text{for all } v \in U_{ad}. \end{aligned}}$$

If $U_{ad} = \mathbb{R}^m$, then $B^T p + \lambda u = 0$. If $\lambda > 0$, we can solve this for $\bar{u}$ and obtain $\bar{u} = -\frac{1}{\lambda} B^T \bar{p}$.
Substitution in the optimality system yields:

$$\boxed{\begin{aligned} Ay &= -\frac{1}{\lambda} B B^T p, \\ A^T p &= C^T (Cy - y_d). \end{aligned}}$$

This linear system has to be satisfied by $(\bar{y}, \bar{p})$ and $\bar{u}$ can be recovered from $\bar{u} = -\frac{1}{\lambda} B^T \bar{p}$.

The optimality system can be conveniently formulated by using Lagrangian functions.

Lagrangians

Definition: The function $L : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$,

$$L(y,u,p) := J(y,u) - (Ay - Bu, p)_{\mathbb{R}^m}$$

is called the Lagrangian (function).

Remark: The adjoint state p plays the role of a Lagrangian multiplier corresponding to the state equation $Ay = Bu$.

The optimality system can now be written in the following compact form:

$$\boxed{\begin{aligned} &\nabla_p L(y,u,p) = 0, \; u \in U_{ad} \quad (\Leftrightarrow Ay = Bu, \; u \in U_{ad}) \\ &\nabla_y L(y,u,p) = 0, \quad (\Leftrightarrow A^T p = \nabla_y J(y,u)) \\ &(\nabla_u L(y,u,p), \tilde{u} - u)_{\mathbb{R}^m} \geq 0, \text{ for all } \tilde{u} \in U_{ad} \quad \left(\Leftrightarrow (B^T p + \nabla_u J(y,u), \tilde{u} - u)_{\mathbb{R}^m} \geq 0\right) \end{aligned}}$$

For $U_{ad} = \mathbb{R}^m$ it reads: $\boxed{\nabla_{(y,u,p)} L = 0}$.

Discussion of the variational inequality

Usually, we do not have $U_{ad} = \mathbb{R}^m$ but typically an admissable set defined by pointwise upper and lower bounds for the control, so-called box constraints:

$$U_{ad} = \{ u \in \mathbb{R}^m : \; u_a \le u \le u_b \}.$$

Here, $u_a, u_b \in \mathbb{R}^m$ are given vectors, inequalities are to be understood componentwise, i.e. $u_{a,i} \le u_i \le u_{b,i}$, $i = 1, \dots, m$.

Recalling the variational inequality

$$\left(B^T \bar{p} + \nabla_u J(\bar{y}, \bar{u}), \tilde{u} - \bar{u} \right)_{\mathbb{R}^m} \ge 0 \qquad \text{for all } \tilde{u} \in U_{ad}$$

$$\Longleftrightarrow \quad \left(B^T \bar{p} + \nabla_u J(\bar{y}, \bar{u}), \bar{u} \right)_{\mathbb{R}^m} \le \left(B^T \bar{p} + \nabla_u J(\bar{y}, \bar{u}), \tilde{u} \right)_{\mathbb{R}^m} \quad \text{for all } \tilde{u} \in U_{ad}$$

we understand that $\bar{u}$ solves the linear optimisation problem

$$\min_{u \in U_{ad}} \left(B^T \bar{p} + \nabla_u J(\bar{y}, \bar{u}), u \right)_{\mathbb{R}^m} = \min_{u \in U_{ad}} \sum_{i=1}^{m} \left(B^T \bar{p} + \nabla_u J(\bar{y}, \bar{u}) \right)_i u_i .$$

If U_{ad} is given as above, then from the fact that the u_i are independent of each other, it follows:

$$\left(B^T \bar{p} + \nabla_u J(\bar{y}, \bar{u}) \right)_i \bar{u}_i = \min_{u_{a,i} \le u_i \le u_{b,i}} \left(B^T \bar{p} + \nabla_u J(\bar{y}, \bar{u}) \right)_i u_i, \quad i = 1, \dots, m$$

Hence, <u>either</u>: $\left(B^T \bar{p} + \nabla_u J(\bar{y}, \bar{u}) \right)_i > 0 \implies \underline{\bar{u}_i = u_{a,i}}$

<u>or</u>: $\left(B^T \bar{p} + \nabla_u J(\bar{y}, \bar{u}) \right)_i < 0 \implies \underline{\bar{u}_i = u_{b,i}}$

<u>or</u>: $\left(B^T \bar{p} + \nabla_u J(\bar{y}, \bar{u}) \right)_i = 0 \implies$ no direct information. (but maybe information can be extracted from this equation)

Formulation as a Karush-Kuhn-Tucker system

So far, in the Lagrangian, we have only incorporated the state equation. We can introduce an extended Lagrangian, which also incorporates the box constraints.

We introduce the Lagrange multipliers

$$\mu_a := \left(B^T \bar{p} + \nabla_u J(\bar{y}, \bar{u}) \right)_+ ,$$
$$\text{and } \mu_b := \left(B^T \bar{p} + \nabla_u J(\bar{y}, \bar{u}) \right)_- ,$$

where $(\cdot)_+$ and $(\cdot)_-$ denote the positive part and negative part of the quantities, respectively, i.e. we have $\mu_{a,i} = (B^T \bar{p} + \nabla_u J(\bar{y}, \bar{u}))_i$ if the right hand side is positive, and zero otherwise. Likewise $\mu_{b,i} = \left| (B^T \bar{p} + \nabla_u J(\bar{y}, \bar{u}))_i \right|$ if the right-hand side is negative and zero otherwise.

Using these Lagrange multipliers for the inequality constraints, we can introduce the extended Lagrangian $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{2m} \to \mathbb{R}$,

$$\mathcal{L}(y, u, p, \mu_a, \mu_b) := J(y, u) - (Ay - Bu, p)_{\mathbb{R}^m} + (u_a - u, \mu_a)_{\mathbb{R}^m} + (u - u_b, \mu_b)_{\mathbb{R}^m}$$

Then we have the following theorem.

Theorem 1.4: Let A be invertible, U_{ad} given as above (box constraints), and $(\bar{u}, \bar{y})$ optimal control and state for $(*)$. Then there exist Lagrange multipliers $\bar{p} \in \mathbb{R}^n$ and $\mu_a, \mu_b \in \mathbb{R}^m$ such that the following Karush-Kuhn-Tucker conditions hold:

$$\boxed{\begin{aligned} &\nabla_y \mathcal{L}(\bar{y}, \bar{u}, \bar{p}, \mu_a, \mu_b) = 0, \\ &\nabla_u \mathcal{L}(\bar{y}, \bar{u}, \bar{p}, \mu_a, \mu_b) = 0, \\ &\mu_a \geq 0, \quad \mu_b \geq 0 \\ &(u_a - \bar{u}, \mu_a)_{\mathbb{R}^m} = (\bar{u} - u_b, \mu_b)_{\mathbb{R}^m} = 0. \end{aligned} \qquad \text{“KKT system”}}$$

$$\nabla_y \mathcal{L} = \nabla_y L = 0 \checkmark$$

<u>Proof:</u> $\mu_a \geq 0$ and $\mu_b \geq 0$ by definition of $(\cdot)_+$ and $(\cdot)_-$.

By definition $\mu_a - \mu_b = B^T \bar{p} + \nabla_u J(\bar{y}, \bar{u})$, hence

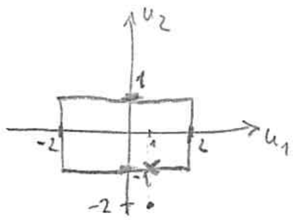
$$\nabla_u \mathcal{L}(\bar{y}, \bar{u}, \bar{p}, \mu_a, \mu_b) = \nabla_u J + B^T \bar{p} - \mu_a + \mu_b = 0.$$

The orthogonality relations hold since <u>either</u> ($u_{a,i} \leq \bar{u}_i$)

i) <u>$u_{a,i} < \bar{u}_i$:</u> (strict inequality) and therefore $(B^T \bar{p} + \nabla_u J(\bar{y}, \bar{u}))_i \leq 0$.
This implies, by definition of μ_a that $\mu_{a,i} = 0$, hence
$(u_{a,i} - \bar{u}_i)\mu_{a,i} = 0$; <u>or</u>

ii) <u>$u_{a,i} = \bar{u}_i$:</u> then $(B^T \bar{p} + \nabla_u J(\bar{y}, \bar{u}))_i > 0$ and $\mu_{a,i} > 0$,
$(u_{a_i} - \bar{u}_i)\mu_{a,i} = 0$ holds as well. $\square$

<u>Example:</u> $u_a = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$, $u_b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $J(y, u) = u_1^2 + u_2^2 + 2y$, $A = 1$, $B = (-1 \; 2)$



$u \in [-2, 2] \times [-1, 1]$, $y \in \mathbb{R}$, $p \in \mathbb{R}$ $\qquad Ay = Bu \Rightarrow y = Bu$

$\Rightarrow f(u) = u_1^2 + u_2^2 + 4u_2 - 2u_1 = (u_1 - 1)^2 + (u_2 + 2)^2 - 5.$

$A\bar{y} = B\bar{u} \Leftrightarrow \bar{y} = 2\bar{u}_2 - u_1$

$\nabla_y \mathcal{L}(\bar{y}, \bar{u}, \bar{p}, \mu_a, \mu_b) = 0 \Leftrightarrow A^T \bar{p} = \nabla_y J(\bar{y}, \bar{u}) \Leftrightarrow \bar{p} = 2$

$\nabla_u \mathcal{L}(\bar{y}, \bar{u}, \bar{p}, \mu_a, \mu_b) = 0 \Leftrightarrow \nabla_u J(\bar{y}, \bar{u}) + B^T \bar{p} = \mu_a - \mu_b$

$\Leftrightarrow 2\begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \mu_a - \mu_b$

Componentwise: i) Either: $2\bar{u}_1 - 2 > 0$ and $\bar{u}_1 = -2$ ↯
or: $2\bar{u}_1 - 2 < 0$ and $\bar{u}_1 = 2$ ↯
or: $2\bar{u}_1 - 2 = 0$ (hence $\boxed{\bar{u}_1 = 1}$). ✓

ii) Either: $2\bar{u}_2 + 4 > 0$ and $\boxed{\bar{u}_2 = -1}$ ✓
or: $2\bar{u}_2 + 4 < 0$ and $\bar{u}_2 = 1$ ↯
or: $2\bar{u}_2 + 4 = 0$ (and $\bar{u}_2 = -2$ ↯)

$\Rightarrow B^T \bar{p} + \nabla_u J(\bar{y}, \bar{u}) = 2\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 2 \end{pmatrix} =$

$\Rightarrow \mu_a = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and $\mu_b = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$