2. Linear-quadratic elliptic problems

In the first few sections we recall results and notions from functional analysis. We only present the results as they are necessary for what is to follow, we do not give proofs, but instead refer to the references in the Tröltzsch book.

2.1 Normed spaces

Definition: Let X be a linear space over $\mathbb{R}$. A mapping $\|\cdot\| : X \to \mathbb{R}$ is called a norm on X if for all $x, y \in X$ and $\lambda \in \mathbb{R}$ the following hold:

(i) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$

(ii) $\|x+y\| \leq \|x\| + \|y\|$ ("triangle inequality")

(iii) $\|\lambda x\| = |\lambda| \, \|x\|$ ("homogeneity")

If $\|\cdot\|$ is a norm on X, then $\{X, \|\cdot\|\}$ is called a normed space. If it is clear, which norm is used, one often shortens this to say 'X is a normed space'.

Examples: (i) $\mathbb{R}^n$ is a normed space when equipped with the Euclidean norm

$$|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}.$$

(ii) Space of continuous functions $x : [a,b] \to \mathbb{R}$ is a normed space, denoted by $C([a,b])$ with the maximum norm of $x(\cdot)$,

$$\|x\|_{C([a,b])} = \max_{t \in [a,b]} |x(t)|$$

(iii) Space of continuous real-valued functions, endowed with the L^2-norm,

$$\|x\|_{C_{L^2}([a,b])} = \left(\int_a^b |x(t)|^2 \, dt \right)^{1/2},$$

is a normed space and denoted by $C_{L^2}([a,b])$.

<u>Definition</u>: Let $\{X, \|\cdot\|\}$ be a normed space, and $(x_n)_{n=1}^{\infty} \subset X$ a sequence.

(i) (x_n) is <u>convergent</u> in X, if there exists $x \in X$ such that $\lim\limits_{n\to\infty} \|x_n - x\| = 0$.

(ii) We call this x the <u>limit</u> of the sequence, and write $\lim\limits_{n\to\infty} x_n = x$.

(iii) (x_n) is a <u>Cauchy sequence</u> if for any $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that for all $n, m > n_0(\varepsilon)$ it holds: $\|x_n - x_m\| < \varepsilon$.

Any convergent sequence is a Cauchy sequence, but the converse is in general false, see the following example.

<u>Example</u>: Consider the space $C_{L^2}([0,2])$ and the sequence $x_n(t) = \min\{1, t^n\}$ for $t \in [0,2]$, $n \in \mathbb{N}$.

Then $\|x_n - x_m\|^2_{C_{L^2}([0,2])}$

$$= \int_0^1 (t^n - t^m)^2 \, dt = \int_0^1 (t^{2n} - 2t^{n+m} + t^{2m}) \, dt$$

$$= \frac{1}{2n+1} - \frac{2}{n+m+1} + \frac{1}{2m+1} \leq \frac{1}{2m+1} \quad \text{for } m \leq n$$

$\Rightarrow$ (x_n) is Cauchy sequence.

However, its pointwise limit

$$x(t) = \lim_{n\to\infty} x_n(t) = \begin{cases} 0, & 0 \leq t < 1 \\ 1, & 1 \leq t \leq 2 \end{cases}$$

is not continuous on $[0,2]$ and hence not an element of $C_{L^2}([0,2])$.

Definition: A normed space $\{X, \|\cdot\|\}$ is complete if every Cauchy sequence in X converges, i.e. has a limit in X. A complete normed space is called a Banach space.

Examples: $\mathbb{R}^n$ and $C([a,b])$ are Banach spaces with respect to their natural norms, while $\{C_{L^2}([a,b]), \|\cdot\|_{C_{L^2}([a,b])}\}$ is not complete, and hence no Banach space.

In a Banach space an equivalent to the scalar product of two vectors in $\mathbb{R}^n$, which is fundamantal for the concept of orthogonality, does not necessarily exist.

Definition: Let H be a real linear space. A mapping $(\cdot,\cdot): H \times H \to \mathbb{R}$ is called scalar product on H if the following conditions are satisfied for all $u, v, u_1, u_2 \in H$ and $\lambda \in \mathbb{R}$:

(i) $(u,u) \geq 0$ and $(u,u) = 0 \Leftrightarrow u = 0$.

(ii) $(u,v) = (v,u)$ ("Symmetry")

(iii) $(u_1 + u_2, v) = (u_1, v) + (u_2, v)$

(iv) $(\lambda u, v) = \lambda (u,v)$

$\Big\}$ ("Linearity")

If $(\cdot,\cdot)$ is a scalar product on H, the $\{H, (\cdot,\cdot)\}$ is called a pre-Hilbert space.

Every pre-Hilbert space $\{H, (\cdot,\cdot)\}$ is a normed space with respect to its natural norm $\|u\| := \sqrt{(u,u)}$, and the Cauchy-Schwarz inequality holds: $|(u,v)| \leq \|u\| \, \|v\|$ for all $u, v \in H$.

A pre-Hilbert space is called a Hilbert space if it is complete with respect to the natural norm $\|u\| := \sqrt{(u,u)}$.

Examples: $\mathbb{R}^n$ is a pre-Hilbert space with respect to $(u,v) := u^T v$.

$C_{L^2}([a,b])$ is a pre-Hilbert space when equipped with $(u,v) = \int_a^b u(t)\, v(t)\, dt$.

$\mathbb{R}^n$ is a Hilbert space with respect to the standard scalar product, while $C_{L^2}([a,b])$ is not complete (see above) and hence not a Hilbert space.

2.2 L^p spaces and Sobolev spaces

In the following, $\Omega \subset \mathbb{R}^N$ denotes a non-empty, bounded and Lebesgue-measurable set with Lebesgue measure $|\Omega|$.

L^p spaces

For a measurable function $y: \Omega \to \mathbb{R}$ and $1 \le p < \infty$ we define

$$\|y\|_{L^p(\Omega)} := \left(\int_\Omega |y(x)|^p \, dx \right)^{1/p}$$

and $\|y\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x \in \Omega} |y(x)| = \inf\{C \ge 0 : |y(x)| \le C \text{ almost everywhere}^{**)} \text{ in } \Omega\}$

By esssup we dentote the "essential supremum" of a function, which excludes any suprema which change upon the removal of sets of Lebesgue measure zero, e.g. lower dimensional sets/single points.

Example: Consider $y: [0,1] \to \mathbb{R}$ with $y = \begin{cases} 1, & x = 0 \\ 0, & x \in (0,1] \end{cases}$

has maximum 1 on $[0,1]$, but $\operatorname*{ess\,sup}_{x \in [0,1]} y(x) = 0$.

For fixed $p \in [1,\infty]$ the (integrable) measurable functions*) $y: \Omega \to \mathbb{R}$ with $\|y\|_{L^p(\Omega)} < \infty$ constitute the Banach space $L^p(\Omega)$ with norm $\|\cdot\|_{L^p(\Omega)}$.

For $p = 2$, $L^2(\Omega)$ is a Hilbert space with scalar product

$$(u,v)_{L^2(\Omega)} = \int_\Omega u(t)\, v(t)\, dt.$$

*) more precisely: equivalence classes of functions

**) everywhere, except on a set of measure zero.

Hoelder inequality: For $y \in L^p(\Omega)$, $z \in L^{p'}(\Omega)$ with $\frac{1}{p} + \frac{1}{p'} = 1$ it holds:
$yz \in L^1(\Omega)$ and $\|yz\|_{L^1(\Omega)} \leq \|y\|_{L^p(\Omega)} \|z\|_{L^{p'}(\Omega)}$.

Sobolev space $H^1(\Omega)$

Let $\Omega \subset \mathbb{R}^N$ be bounded with Lipschitz boundary.
$y \in L^2(\Omega)$ belongs to the Sobolev space $H^1(\Omega)$ if and only if there exist n functions $z_1, \dots, z_n \in L^2(\Omega)$ with

$$\int_\Omega y(x) \frac{\partial \varphi(x)}{\partial x_i} \, dx = -\int_\Omega z_i(x) \varphi(x) \, dx \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

($C_0^\infty(\Omega)$ is the set of test functions which are smooth and have compact support in Ω, i.e. vanish on $\partial\Omega$.)

Then, we call $z_i =: \frac{\partial y}{\partial x_i}$ (and $\nabla y := \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$) the weak derivatives (and gradient, respectively) of y.

$H^1(\Omega)$ is a Banach space with norm

$$\|y\|_{H^1(\Omega)} = \left(\int_\Omega (y^2 + |\nabla y|^2 \, dx \right)^{1/2}.$$

$H^1(\Omega)$ becomes a Hilbert space with the scalar product

$$(u,v)_{H^1(\Omega)} = \int_\Omega u v \, dx + \int_\Omega \nabla u \cdot \nabla v \, dx.$$

Morally: "z_i is the weak i-th partial derivative of y, if it satisfies the integration-by-parts formula, as if y belonged to $C^1(\bar{\Omega})$. If a weak derivative exists, it is unique and hence we use the same symbols as for the (strong) derivative."