<u>Example</u>: Consider $y(x) = |x|$ in $\Omega = (-1,1)$.
It is easy to verify that the weak derivative is given by

$$z(x) = \left\{ \begin{matrix} -1, & x \in (-1,0) \\ +1, & x \in [0,1) \end{matrix} \right\} =: y'(x)$$

since for all $v \in C_0^\infty((-1,1))$ we have

$$\int_{-1}^{1} |x|\, v'(x)\, dx = \int_{-1}^{0} (-x)\, v'(x)\, dx + \int_{0}^{1} x\, v'(x)\, dx$$

$$= \left[-x\, v(x)\right]_{-1}^{0} - \int_{-1}^{0} (-1)\, v(x) + \left[x\, v(x)\right]_{0}^{1} - \int_{0}^{1} (+1)\, v(x)\, dx$$

$$= -\int_{-1}^{1} z(x)\, v(x)\, dx.$$

Note that the function value of y' at zero plays no role, since it does not change the integral values...

A subtle difficulty arises if we want to assign boundary values (on $\partial\Omega$) to functions in $H^1(\Omega)$. For example, in the example of the "optimal heat source" what does it mean that "$y = 0$ on $\partial\Omega$" for $y \in H^1(\Omega)$?
Since $\partial\Omega \subset \mathbb{R}^{N-1}$, we can change the values of any $y \in L^p(\Omega)$ arbitrarily on $\partial\Omega$ without affecting y as an element of $L^p(\Omega)$ (the norm defined by the integral does not change!).

Let us recall the notion of the <u>closure $\overline{E}$ of a set $E \subset X$</u> in a normed space $\{X, \|\cdot\|\}$,

$$\overline{E} = \{x \in X : x \text{ is the limit of some sequence } (x_n)_n \subset E\}.$$

We say that $E \subset X$ is <u>dense</u> in X if $\overline{E} = X$.
Using this notion, we can define another Sobolev space.

Definition: The closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$ is denoted by $H_0^1(\Omega)$.

Endowed with the norm $\|\cdot\|_{H^1(\Omega)}$, $H_0^1(\Omega)$ is a normed space, and, as a closed subspace of $H^1(\Omega)$, also a Banach space. By definition, $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$.

The elements of $H_0^1(\Omega)$ can be regarded as functions which vanish at the boundary. This is a consequence of the following result.

Theorem (Trace theorem): Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Then there exists a linear and continuous mapping $\tau: H^1(\Omega) \to L^2(\partial\Omega)$ such that for all $y \in H^1(\Omega) \cap C(\bar{\Omega})$ we have $(\tau y)(x) = y(x)$ for all $x \in \partial\Omega$, i.e. in the case of continuous functions in $H^1(\Omega)$, (τy) coincides with the restriction $y|_{\partial\Omega}$ of y on $\partial\Omega$. (τy) is called the trace of y on $\partial\Omega$, and the mapping τ is called the trace operator. It holds $\|y|_{\partial\Omega}\|_{L^2(\Omega)} \le c_\tau(\Omega)\|y\|_{H^1(\Omega)}\ \forall y \in H^1(\Omega)$, with some constant $c_\tau = c_\tau(\Omega)$.

In this sense, we can say that

$$H_0^1(\Omega) = \{ y \in H^1(\Omega) : \ y|_{\partial\Omega} = 0 \}.$$

An alternative norm on $H_0^1(\Omega)$ is defined by

$$\|y\|^2_{H_0^1(\Omega)} := \int_\Omega |\nabla y|^2 \, dx.$$

It is equivalent to the norm on $H^1(\Omega)$, i.e. there exist c_1, c_2 such that: $c_1 \|y\|_{H_0^1(\Omega)} \le \|y\|_{H^1(\Omega)} \le c_2 \|y\|_{H_0^1(\Omega)}\ \forall y \in H_0^1(\Omega)$.

2.3. Weak solutions to elliptic equations

Let us consider first for given $f \in L^2(\Omega)$

$$(*) \begin{cases} -\Delta y = f & \text{in } \Omega, \\ \quad y = 0 & \text{on } \partial\Omega. \end{cases}$$

Such right hand sides $f \in L^2(\Omega)$ can be very irregular. E.g. if $\Omega \subset \mathbb{R}^2$ is the unit square, and is subdivided in a "chessboard style" with f equal unity on "black", and zero on "white" squares. Obviously, $-\Delta y = f$ cannot have a classical solution $y \in C^2(\Omega) \cap C(\overline{\Omega})$ for such an f. Instead, we seek a so-called weak solution $y \in H_0^1(\Omega)$. Its definition is based on a weak (or variational) formulation of $(*)$.

Assume for the moment, that $y \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a classical solution of $(*)$ and Ω is a bounded Lipschitz domain. We multiply the PDE in $(*)$ by an arbitrary test function $\varphi \in C_0^\infty(\Omega)$ and integrate over Ω,

$$-\int_\Omega \Delta y \, \varphi \, dx = \int_\Omega f \varphi \, dx \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

Integrating by parts we obtain

$$\int_\Omega \nabla y \cdot \nabla \varphi \, dx - \underbrace{\int_{\partial\Omega} \partial_\nu y \, \varphi \, ds}_{=0} = \int_\Omega f \varphi \, dx, \quad \text{for all } \varphi \in C_0^\infty(\Omega),$$

where $\partial_\nu y$ denotes the normal derivative of y, i.e. the directional derivative of y in the direction of the outward unit normal ν to $\partial\Omega$. Recall $\partial_\nu y = \nabla y \cdot \nu$.
Since φ vanishes on $\partial\Omega$, the second term vanishes.

We obtain

$$(**) \quad \int_\Omega \nabla y \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx \qquad \text{for } \underline{\text{all}} \ \varphi \in C_0^\infty(\Omega).$$

Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, and for fixed y all terms depend continuously on φ, we conclude this equation holds for all $\varphi \in H_0^1(\Omega)$.

We call $y \in H_0^1(\Omega)$ a weak solution of $(*)$ if it satisfies the weak (or variational) formulation $(**)$ for all $\varphi \in H_0^1(\Omega)$.

<u>Remark</u>: 1) Observe, that if a weak solution y is sufficiently smooth, then it is a classical solution of $(*)$. Therefore, calling it a weak solution is justified.

2) The boundary condition in $(*)$ is encoded in the solution space $H_0^1(\Omega)$.

3) We require the solution of the second-order PDE in $(*)$ only to have (weak) first-order derivatives.

To be able to treat $(*)$ and more general problems by a unified approach, we define $V = H_0^1(\Omega)$ and

- bilinear form $a: V \times V \to \mathbb{R}$, $a[y, \varphi] := \int_\Omega \nabla y \cdot \nabla \varphi \, dx$,
- linear and continuous functional $F: V \to \mathbb{R}$, $F(\varphi) := \int_\Omega f \varphi \, dx = (f, \varphi)_{L^2(\Omega)}$

Then, $(**)$ attains the general form

$$\boxed{a[y, \varphi] = F(\varphi) \quad \text{for all } \varphi \in V.}$$

We denote by V^* the dual space of V, the space of linear and continuous functionals on V. Hence, $F \in V^*$.

The following result is of fundamental importance for the existence theory for linear elliptic equations.

Lemma 2.2 (Lax and Milgram): Let V be a real Hilbert space, and let $a: V \times V \to \mathbb{R}$ denote a bilinear form. Suppose, there exist positive constants α_0 and β_0 such that for all $y, \varphi \in V$:

(i) $|a[y,\varphi]| \leq \alpha_0 \|y\|_V \|\varphi\|_V$ ("boundedness")

(ii) $a[y,y] \geq \beta_0 \|y\|_V^2$ ("V-ellipticity")

Then, for every $F \in V^*$ the variational equality,

$$a[y,\varphi] = F(\varphi) \quad \text{for all } \varphi \in V,$$

admits a unique solution $y \in V$.
Moreover, it exists $c_a > 0$ (independent of F), such that

$$\|y\|_V \leq c_a \|F\|_{V^*},$$

where $\|F\|_{V^*} = \sup_{\|\varphi\|_V = 1} |F(\varphi)|$ is the norm on V^*.

We also require the following estimate.

Lemma 2.3 (Friedrichs inequality): Let Ω be a bounded Lipschitz domain. Then, there exists $c(\Omega) > 0$, such that

$$\int_\Omega |y|^2 dx \leq c(\Omega) \int_\Omega |\nabla y|^2 dx \quad \text{for all } y \in H_0^1(\Omega).$$

N.B. This holds only for $y \in H_0^1(\Omega)$, not for general $y \in H^1(\Omega)$.

Theorem 2.4: If Ω is a bounded Lipschitz domain, then for every $f \in L^2(\Omega)$, problem (*) has a unique weak solution $y \in H_0^1(\Omega)$. Moreover, it exists $c_p > 0$ which does not depend on f, such that

$$\|y\|_{H^1(\Omega)} \le c_p \, \|f\|_{L^2(\Omega)}.$$

Proof: We apply the Lax-Milgram lemma with $V = H_0^1(\Omega)$. We verify that the conditions (i) and (ii) are met.

ad (i):

$$|a[y,\varphi]| = \left| \int_\Omega \nabla y \cdot \nabla \varphi \, dx \right|$$

$|(u,v)| \le \|u\| \|v\| \ \forall u,v \in H$ — C-S-ineq.

$$\le \left(\int_\Omega |\nabla y|^2 \, dx \right)^{1/2} \left(\int_\Omega |\nabla \varphi|^2 \, dx \right)^{1/2}$$

$$\le \left(\int_\Omega (|y|^2 + |\nabla y|^2) \, dx \right)^{1/2} \left(\int_\Omega (|\varphi|^2 + |\nabla \varphi|^2) \, dx \right)^{1/2}$$

$$\le \|y\|_{H^1(\Omega)} \|\varphi\|_{H^1(\Omega)}.$$

ad (ii):

$$a[y,y] = \int_\Omega |\nabla y|^2 \, dx = \frac{1}{2} \int_\Omega |\nabla y|^2 \, dx + \frac{1}{2} \int_\Omega |\nabla y|^2 \, dx$$

Friedrichs ineq.

$$\ge \frac{1}{2} \int_\Omega |\nabla y|^2 \, dx + \frac{1}{2c(\Omega)} \int_\Omega |y|^2 \, dx$$

$$\ge \frac{1}{2} \min\left\{1, \frac{1}{c(\Omega)}\right\} \|y\|^2_{H^1(\Omega)}.$$

Since $|F(\varphi)| = |(f,\varphi)_{L^2(\Omega)}| \overset{\text{C.-S. ineq.}}{\le} \|f\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)} \|\varphi\|_{H^1(\Omega)}$,

we have $\|F\|_{V^*} = \sup_{\|\varphi\|_V = 1} |F(\varphi)| \le \|f\|_{L^2(\Omega)}$.