$\Rightarrow$ Lemma 2.2. yields existence and uniqueness.
Inserting the estimate for F, we obtain

$$\|y\|_{H^1(\Omega)} \leq C_a \|F\|_{V^*} \leq C_a \|f\|_{L^2(\Omega)},$$

which completes the proof. $\boxtimes$

In a similar way we can prove existence and uniqueness to elliptic boundary value problems of more general form. For example, consider:

$$(***)\begin{cases} -\Delta y + c_0 y = f & \text{in } \Omega \\ \partial_\nu y + \alpha y = g & \text{on } \Gamma = \partial\Omega \end{cases}$$

The functions $f \in L^2(\Omega)$, $g \in L^2(\Gamma)$, $c_0 \in L^\infty(\Omega)$, $\alpha \in L^\infty(\Gamma)$ are given. The boundary condition in $(***)$ is usually called a Robin-type boundary condition.

To obtain the weak form we multiply by an arbitrary $\varphi \in C^1(\bar{\Omega})$, integrate over Ω and integrate by parts:

$$-\int_\Gamma \varphi\, \partial_\nu y \, ds + \int_\Omega \nabla y \cdot \nabla\varphi \, dx + \int_\Omega c_0 y \varphi \, dx = \int_\Omega f \varphi \, dx \quad \text{for all } \varphi \in C^1(\bar{\Omega}).$$

Substituting the boundary condition from $(***)$ yields:

$$\underbrace{\int_\Omega \nabla y \cdot \nabla \varphi \, dx + \int_\Omega c_0 y \varphi \, dx + \int_\Gamma \alpha y \varphi \, ds}_{=: a[y,\varphi]} = \underbrace{\int_\Omega f\varphi \, dx + \int_\Gamma g \varphi \, ds}_{=: F(\varphi)} \quad \text{for all } \varphi \in C^1(\bar{\Omega}).$$

Noting that $C^1(\bar{\Omega})$ for Lipschitz domains is dense in $H^1(\Omega)$, we arrive at the following definition:

<u>Definition</u>: A function $y \in H^1(\Omega)$ is called a weak solution to (***) if the variational inequality
$$a[y,\varphi] = F(\varphi)$$
holds for all $\varphi \in H^1(\Omega)$.

The following theorem holds.

<u>Theorem 2.6</u>: Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz domain, and suppose $c_0 \in L^\infty(\Omega)$ and $\alpha \in L^\infty(\Gamma)$ are almost everywhere non-negative functions such that
$$\int_\Omega (c_0(x))^2 \, dx + \int_\Gamma (\alpha(x))^2 \, ds(x) > 0.$$
Then for every $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$, the boundary value problem (***) has a unique weak solution $y \in H^1(\Omega)$. Moreover,
$$\|y\|_{H^1(\Omega)} \le c_R \left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma)} \right)$$
with some constant $c_R > 0$ (independent of f, g).

<u>Proof</u>: See Thm. 2.6 in the Tröltzsch book.
The proof employs the Lemma of Lax-Milgram with $V = H^1(\Omega)$ and makes use of a generalised form of the Friedrichs inequality. $\boxtimes$

2.4. Continuous linear operators, functionals and weak convergence

Let $\{U, \|\cdot\|_U\}$ and $\{V, \|\cdot\|_V\}$ be normed spaces over $\mathbb{R}$.

Definition: • A mapping $A: U \to V$ is linear or a linear operator if $A(u+v) = Au + Av$ and $A(\lambda v) = \lambda Av$ for all $u, v \in U$ and $\lambda \in \mathbb{R}$.

• A linear mapping $f: U \to \mathbb{R}$ is called a linear functional.

• $A: U \to V$ is continuous on U if for any sequence $(u_n)_n \subset U$ with $\lim_{n\to\infty} \|u_n - u\| = 0$ it holds $\lim_{n\to\infty} \|Au_n - Au\|_V = 0$.

• A linear operator $A: U \to V$ is bounded if there is a constant $c(A) > 0$: $\|Au\|_V \leq c(A) \|u\|_U$ for all $u \in U$.

Theorem 2.8: A linear operator is bounded if and only if it is continuous.

Example: Let $U = V = C([0,1])$ and consider the integral operator A, defined by

$$(Au)(t) = \int_0^1 e^{t-s} u(s)\, ds, \quad t \in [0,1].$$

Obviously, A is a linear mapping from U into itself. To prove that A is continuous, we show it is bounded, and apply Thm 2.8:

$$|(Au)(t)| \leq e^t \int_0^1 e^{-s} |u(s)|\, ds \leq e^t (1 - e^{-1}) \max_{t\in[0,1]} |u(t)|$$

$$\leq (e-1) \|u\|_{C([0,1])}.$$

Hence, $\|Au\|_U = \max_{t\in[0,1]} |(Au)(t)| \leq (e-1) \|u\|_U$

$\Rightarrow$ A is bounded with $c(A) = e-1$.

$\Rightarrow$ A is continuous on U.

<u>Definition:</u> If $A: U \to V$ is a linear and continuous operator, we call

$$\|A\|_{\mathcal{L}(U,V)} = \sup_{\|u\|_U=1} \|Au\|_V = \sup_{u \in U \setminus \{0\}} \frac{\|Au\|_V}{\|u\|_U} < \infty$$

the <u>operator norm</u> of A, and denote by $\mathcal{L}(U,V)$ the normed space of all linear and continuous mappings from U to V. The space $\mathcal{L}(U,V)$ is complete (and hence a Banach space) if V is complete.

<u>Example:</u> The space of all continuous linear functionals $f: U \to \mathbb{R}$ on $\{U, \|\cdot\|_U\}$ is denoted by U^*. Obviously, $U^* = \mathcal{L}(U, \mathbb{R})$. The associated norm is

$$\|f\|_{U^*} = \sup_{\|u\|_U=1} |f(u)|.$$

Since $\mathbb{R}$ is complete, the dual space U^* is always a Banach space.

In a Hilbert space H, the elements of the dual space can be characterised by elements of H. This is the purpose of the following fundamental theorem.

<u>Theorem 2.9 (Riesz representation theorem):</u>

Let $\{H, (\cdot,\cdot)_H\}$ be a real Hilbert space. Then for any $F \in H^*$ there exists a unique $f \in H$ such that $\|F\|_{H^*} = \|f\|_H$ and $F(v) = (f, v)_H$ for all $v \in H$.

<u>Example:</u> The right hand side in (**) on p.23 is $F(v) = \int_\Omega f v \, dx \in (L^2(\Omega))^*$, it is uniquely characterised by $f \in L^2(\Omega)$.

Let U be a real Banach space with dual space U^*.
Fix arbitrary $u \in U$, let f vary over U^* and consider the mapping $F_u : U^* \to \mathbb{R}$ induced by u, $f \mapsto f(u)$.

F_u is linear and also continuous, since

$$|F_u(f)| = |f(u)| \leq \|u\|_U \, \|f\|_{U^*}$$

Hence, F_u belongs to the dual space $(U^*)^* =: U^{**}$ of U^*, called bidual space of U.

Since $u \mapsto F_u$ turns out to be injective[*)], we can identify u with F_u, in this sense we can interpret $u \in U$ as an element of U^{**}, and it holds $U \subset U^{**}$.
If this mapping is additionally surjective, i.e. if $U = U^{**}$, U is called a reflexive space.

Example: i) From Riesz represantation theorem (Thm 2.9) it follows that every Hilbert space, e.g. also $L^2(\Omega)$, is reflexive.

ii) $L^p(\Omega)$, $1 < p < \infty$, is also reflexive (see Tröltzsch boch, p. 43).

N.B. $L^\infty(\Omega)$ and $L^1(\Omega)$ are not refexive.

Definition: Let U be a real Banach space. A sequence $(u_n)_n \subset U$ converges weakly to $u \in U$ if

$$\lim_{n \to \infty} F(u_n) = F(u) \quad \text{for all } F \in U^*.$$

We denote weak convergence by '$\rightharpoonup$', i.e. we write $u_n \rightharpoonup u$ as $n \to \infty$.

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*) this mapping is called the 'canonical embedding'

The limit u is uniquely determined and is called the weak limit of the sequence.

Examples (cf. Tröltzsch book, pp. 44) show that little information about the pointwise behaviour can be extracted from a weak limit of a sequence. Therefore, this notion usually plays not a major role for numerical approximations, it is however of fundamental importance for proving existence results.

Remarks: (i) If $(u_n)_n$ converges strongly (i.e. with respect to the norm in U), then it also converges weakly to u:

$$u_n \to u \quad \Rightarrow \quad u_n \rightharpoonup u \quad \text{as } n\to\infty.$$

(ii) If U is finite-dimensional (e.g. $U=\mathbb{R}^n$), weak and strong convergence are equivalent.

(iii) If U is infinite-dimensional, there exists a sequence $(u_n)_n$ with $u_n \rightharpoonup 0$ $(n\to\infty)$, but at the same time $\|u_n\|_U = 1$ for all $n\in\mathbb{N}$. (cf. Tröltzsch book, pp. 44).

(iii) If $(u_n)_n$ converges weakly in U, then $(\|u_n\|_U)_n \subset \mathbb{R}$ is bounded (i.e. $(u_n)_n$ is bounded in U).

Theorem 2.10 (Alaoglu's theorem): If U is reflexive, any bounded set is weakly sequentially relatively compact, i.e. any bounded sequence $(u_n)_n \subset M \subseteq U$ contains a weakly convergent subsequence.

<u>Definition</u>: A subset M of a real Banach space is <u>weakly sequentially closed</u>, if it holds:
$(u_n)_n \subset M$ and $u_n \rightharpoonup u \ (n \to \infty) \implies u \in M$.

<u>Theorem 2.11.</u>: Every convex and closed subset of a Banach space is weakly sequentially closed.

<u>Theorem 2.12.</u>: Every continuous and convex functional $F: U \to \mathbb{R}$ on a Banach space U is weakly <u>lower semicontinuous</u>, i.e. for any sequence $(u_n)_n \subset U$:

$$u_n \rightharpoonup u \ (n \to \infty) \implies \liminf_{n \to \infty} F(u_n) \geq F(u).$$

The last two theorems show the importance of convexity for the treatment of optimisation problems in function spaces.

<u>Remark</u>: (i) Theorem 2.12 is important, because in general continuous functionals are not necissarily weakly sequentially continuous, e.g. $f(u) = \|u\|_U$ in $U = L^2(0, 2\pi)$ is not weakly sequentially continuous, consider the example on pp 44 in the Tröltzsch book, where a sequence $(u_n)_n$ is defined such that

$$u_n \rightharpoonup 0 \ (n \to \infty), \text{ but } \lim_{n \to \infty} f(u) = \lim_{n \to \infty} \|u_n\|_U = 1 \neq 0 = f(0).$$

(ii) The functional $f(u) = \|u\|_U$ is <u>continuous</u> on any Banach space U, since for all $\varepsilon > 0$ it exists $\delta = \varepsilon$ such that: $\|u - v\|_U < \delta \implies \big| \|u\|_U - \|v\|_U \big| \leq \|u - v\|_U < \delta = \varepsilon$ using the inverse triangle inequality, and it is <u>convex</u>, since for all $\lambda \in [0,1]$:

$$\|\lambda u + (1-\lambda) v\|_U \leq \lambda \|u\|_U + (1-\lambda) \|v\|_U,$$

using properties of the norm.