2.5. Existence of optimal controls

Our aim is to show existence of a solution to linear-quadratic elliptic optimal control problems, i.e. a pair of optimal control and associated optimal state.

We make the following general assumptions.

Assumption 2.13: Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with boundary Γ, and $\lambda \geq 0$, $y_\Omega \in L^2(\Omega)$, $y_\Gamma \in L^2(\Gamma)$, $\beta \in L^\infty(\Omega)$, $\alpha \in L^\infty(\Gamma)$ with $\alpha(x) \geq 0$ for almost every $x \in \Gamma$, as well as $u_a, u_b, v_a, v_b \in L^2(E)$ having the property $u_a(x) \leq u_b(x)$ and $v_a(x) \leq v_b(x)$ for almost every $x \in E$.

In these assumptions y_Ω, y_Γ are desired states ('targets'), α, β are coefficient functions, and u_a, u_b, v_a, v_b define the set of admissible controls acting on Ω and Γ, respectively.

We assume also we are given

- a state space Y of states y
- an admissible set $U_{ad} \subset U$ of admissible controls u
- a cost functional $J : Y \times U_{ad} \to \mathbb{R}$

As a first example, we consider the problem of the 'optimal stationary heat source' (cf. Chapter 1).

$$(*)\quad \begin{cases} \min J(y,u) = \frac{1}{2}\|y - y_\Omega\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|u\|^2_{L^2(\Omega)} \\ \text{subject to} \quad (**) \begin{cases} -\Delta y = \beta u & \text{in } \Omega \\ y = 0 & \text{on } \Gamma = \partial\Omega \end{cases} \\ \text{and} \quad u_a(x) \leq u(x) \leq u_b(x) \text{ for almost every } x \in \Omega. \end{cases}$$

From (*) a natural space for the controls appears to be $L^2(\Omega)$. Hence, we define

$$U_{ad} = \{ u \in L^2(\Omega) : u_a(x) \le u(x) \le u_b(x) \text{ for almost every } x \in \Omega \}.$$

Thanks to our existance results, for every $u \in U_{ad}$ there exists a unique $y \in H_0^1(\Omega)$, which is a solution to (**), called the state associated with u.
Hence, the state space is $Y = H_0^1(\Omega)$. To express the dependence of y on u, we sometimes write $y = y(u)$.
(the context should ensure that this is not confused with $y = y(x)$ for $x \in \bar{\Omega}$).

Definition: We call a control $\bar{u} \in U_{ad}$ optimal and $\bar{y} = \bar{y}(\bar{u})$ the associated optimal state if

$$J(\bar{y}, \bar{u}) \le J(y(u), u) \quad \text{for all } u \in U_{ad}.$$

To answer the question if they exist, we rewrite the problem (*) using the reduced functional.

Definition: We call $G : L^2 \to H_0^1(\Omega)$, $u \mapsto y(u)$, defined by Theorem 2.4 the control-to-state operator.

G is a linear mapping and, since $y(u)$ is bounded by the data u (see the estimate in Theorem 2.4), it is also continuous.

Since $\|y\|_{L^2(\Omega)} \le \|y\|_{H^1(\Omega)}$, $H^1(\Omega)$ and its subspace $H_0^1(\Omega)$ can be linearly and continuously embedded in $L^2(\Omega)$, i.e. there exists $E_Y : H^1(\Omega) \to L^2(\Omega)$ (the 'embedding operator') which assigns to each function in $H^1(\Omega)$ the same function in $L^2(\Omega)$. E_Y is linear and continuous.

We can now introduce $S = E_Y G$, which will always represent the part of y that appears in the cost functional. Here, $S : L^2(\Omega) \longrightarrow L^2(\Omega)$, $u \mapsto y(u)$.

Using this operator, problem $(*)$ can be rewritten using the reduced functional as a minimisation problem in the Hilbert space $L^2(\Omega)$, the so-called reduced problem:

$$\min_{u \in U_{ad}} f(u) := \frac{1}{2} \| Su - y_\Omega \|^2_{L^2(\Omega)} + \frac{\lambda}{2} \|u\|^2_{L^2(\Omega)}.$$

We can now prove the following theorem.

Theorem 2.14: Let $\{U, \|\cdot\|_U\}$ and $\{H, \|\cdot\|_H\}$ denote real Hilbert spaces, and let a non-empty, bounded, closed, and convex set $U_{ad} \subset U$, as well as some $y_d \in H$ and constant $\lambda \geq 0$ be given. Moreover, let $S : U \longrightarrow H$ be a continuous linear operator. Then, the problem

$$\min_{u \in U_{ad}} f(u) := \frac{1}{2} \| Su - y_d \|^2_H + \frac{\lambda}{2} \|u\|^2_U \qquad (***)$$

admits an optimal solution $\bar{u}$. If $\lambda > 0$ or S is injective, then the solution is unique.

Proof: Since $f(u) \geq 0$, there exists the infimum $j := \inf_{u \in U_{ad}} f(u)$, and there is a sequence $(u_n)_n \subset U_{ad}$ such that $f(u_n) \longrightarrow j$ as $n \to \infty$.
U_{ad} is bounded, and hence by Alaoglu's theorem (Theorem 2.10) there exists a subsequence

$(u_{n_k})_k \subset U_{ad}$ which converges weakly, i.e.

$$u_{n_k} \rightharpoonup \bar{u} \in \mathcal{U} \quad (k \to \infty)$$

for some weak limit $\bar{u}$. Since $\mathcal{U}_{ad}$ is convex and closed, it is also weakly sequentially closed, hence $\bar{u} \in \mathcal{U}_{ad}$.

Since S is continuous, f is continuous. Since f is also convex, by Theorem 2.12. it is weakly lower semicontinuous, hence

$$f(\bar{u}) \leq \liminf_{k \to \infty} f(u_{n_k}) = j.$$

Since $\bar{u} \in \mathcal{U}_{ad}$, it must hold $f(\bar{u}) = j$, and hence $\bar{u}$ is an optimal control.

The uniqueness follows from the strict convexity of f, which can be shown to hold if either $\lambda > 0$ or S is injective. ⊠

<u>Remark</u>: i) The proof only makes use of the fact that f is continuous and convex, hence the existence result holds for any functional $f: U \to \mathbb{R}$ with these properties.

ii) By Theorem 2.11, the assertion also holds if we replace the Hilbert space U by a reflexive Banach space.

Using Theorem 2.14, we can state the existence result for problem (*).

Theorem 2.15: Suppose assumptions 2.13 are fulfilled. Then, problem (*) has at least one optimal control $\bar{u}$. If, in addition, $\lambda > 0$ or $\beta \neq 0$ almost everywhere in Ω, then it is unique.

Proof: The claim follows from Theorem 2.14 with $U = H = L^2(\Omega)$, $y_d = y_\Omega$ and $S = E_Y G$.
The set $U_{ad} = \{u \in L^2(\Omega) : u_a \le u \le u_b \text{ a.e. in } \Omega\}$ is bounded, closed and convex. Hence, (*) admits at least one solution $\bar{u}$.
It is unique if $\lambda > 0$. If $\lambda = 0$, $\beta \neq 0$ almost everywhere in Ω implies that S is injective (Indeed, if $Su = 0$, then $y = 0$, and inserting this into (**) yields $\beta u = 0$ and thus $u = 0$ almost everywhere in Ω.), hence uniqueness also holds in this case. $\boxtimes$

What happens, if one (or both) of the constraints in U_{ad} is absent? Then, U_{ad} is no longer bounded, and hence not weakly sequentially relatively compact. However, we still have existence and uniqueness if $\lambda > 0$, as the following theorem shows.

Theorem 2.16: Suppose that U_{ad} is non-empty, closed and convex. If $\lambda > 0$, problem (***) has a unique optimal solution.

Proof: Since U_{ad} is non-empty, there exists some $u_0 \in U_{ad}$. Now note that if $\|u\|_U^2 > \frac{2}{\lambda} f(u_0)$, then

$$f(u) = \frac{1}{2}\|Su - y_d\|_H^2 + \frac{\lambda}{2}\|u\|_U^2 \ge \frac{\lambda}{2}\|u\|_U^2 > f(u_0).$$

Therefore, it is sufficient to search for an optimum in the closed, convex and bounded set $U_{ad} \cap \{u \in U : \|u\|_U^2 \le \frac{2}{\lambda} f(u_0)\}$.
Then proceed as in the proof of Theorem 2.14. $\boxtimes$

We can now also consider another example from Chapter 1, the '<u>optimal stationary boundary heating</u>':

$$(****)\begin{cases} \min\ J(y,u) := \frac{1}{2}\|y-y_\Omega\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|u\|^2_{L^2(\Gamma)} \\ \text{subject to} \quad (*****)\begin{cases} -\Delta y = 0 & \text{in } \Omega \\ \partial_\nu y = \alpha(u-y) & \text{on } \Gamma \end{cases} \\ \text{and} \quad u_a(x) \leq u(x) \leq u_b(x) \quad \text{for almost every } x \in \Gamma. \end{cases}$$

Assuming that $\int_\Gamma (\alpha(x))^2\,ds(x) > 0$, by Theorem 2.6, for any $u \in L^2(\Gamma)$, problem $(*****)$ has a unique weak solution $y = y(u) \in H^1(\Omega)$. The control space is chosen as $L^2(\Gamma)$. The set of admissible controls is

$$U_{ad} = \{u \in L^2(\Gamma) : u_a(x) \leq u(x) \leq u_b(x) \text{ a.e. on } \Gamma\}.$$

The operator $G : L^2(\Gamma) \to H^1(\Omega)$, $u \mapsto y(u)$ is continuous. We interpret G as a continuous linear operator from $L^2(\Gamma)$ into $L^2(\Omega)$, i.e. we set $S = E_Y G$ and $S : L^2(\Gamma) \to L^2(\Omega)$. We have the following result.

<u>Theorem 2.18</u>: Suppose that Assumptions 2.13 and $\int_\Gamma (\alpha(x))^2\,ds(x) > 0$ are satisfied.

Then $(****)$ has an optimal control, which is unique if $\lambda > 0$.

<u>Proof</u>: The result is a consequence of Theorem 2.14. $\boxtimes$

By virtue of Theorem 2.16, it carries over to the case of unbounded admissible sets U_{ad}.

At the end of this section, let us discuss a more general problem:

$$\begin{cases} \min J(y,u,v) := \frac{1}{2}\|y-y_\Omega\|^2_{L^2(\Omega)} + \frac{1}{2}\|y-y_\Gamma\|^2_{L^2(\Gamma)} + \frac{\lambda_v}{2}\|v\|^2_{L^2(\Omega)} + \frac{\lambda_u}{2}\|u\|^2_{L^2(\Gamma)} \\ \text{subject to} \quad \begin{cases} -\Delta y + c_0 y = \beta_\Omega v & \text{in } \Omega \\ \partial_\nu y + \alpha y = \beta_\Gamma u & \text{on } \Gamma \end{cases} \\ \text{and} \quad v_a(x) \le v(x) \le v_b(x) \quad \text{a.e. in } \Omega, \\ \text{and} \quad u_a(x) \le u(x) \le u_b(x) \quad \text{a.e. on } \Gamma. \end{cases}$$

Suppose, Assumptions 2.13 hold and assumptions of Theorem 2.6. are satisfied.

We begin the discussion by recalling that the appropriate state space is $Y = H^1(\Omega)$. Using the above assumptions,
$G : L^2(\Gamma) \times L^2(\Omega) \longrightarrow H^1(\Omega)$, $(u,v) \longmapsto y$, is a linear and continuous operator.

Define $S = E_Y G$, $S : L^2(\Gamma) \times L^2(\Omega) \to L^2(\Omega)$.

Define the <u>boundary observation operator</u> $S_\Gamma = \tau \circ G$ (τ: trace operator), $S_\Gamma : L^2(\Gamma) \times L^2(\Omega) \longrightarrow L^2(\Gamma)$, $(u,v) \longmapsto y|_\Gamma = \tau y$. S_Γ is also a linear and continuous operator (by the trace theorem, τ is linear and continuous).

The sets $V_{ad} = \{v \in L^2(\Omega) : v_a(x) \le v(x) \le v_b(x) \text{ a.e. in } \Omega\}$
and $U_{ad} = \{u \in L^2(\Gamma) : u_a(x) \le u(x) \le u_b(x) \text{ a.e. on } \Gamma\}$
are non-empty, bounded, convex and closed. After elimination of y in the cost functional, the reduced functional is again convex and continuous with respect to (v,u), therefore Theorem 2.14 applies, and it exists an optimal pair $(\bar v, \bar u) \in V_{ad} \times U_{ad}$, which is unique if $\lambda_v > 0$ and $\lambda_v > 0$.

By Theorem 2.16, for unbounded U_{ad}, existence follows for $\lambda_u > 0$ and $\lambda_v > 0$.