STOKES PRECONDITIONING FOR THE INVERSE POWER METHOD

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Computational stability analysis is the most powerful method for the precise determination of transition points and instability mechanisms in fluid flows. Yet the field of hydrodynamic stability has not undergone the advances seen in other areas of computational fluid dynamics. We present a new method for solving linear stability problems, requiring minimal computational time and storage.

In previous work [1,2], we showed how a code for time-stepping the Navier-Stokes equations, written schematically as $dU/dt = L + N$ could be adapted to find steady flows, i.e. roots of the equation $0 = L + N$, via Newton's method. Here, L stands for the viscous damping and pressure projection and N is the nonlinear advection term. The key idea is that the difference between two Forward-Euler/Backwards-Euler (FEBE) timesteps

$$
\begin{aligned}
U(t + \Delta t) - U(t) &= [(I - \Delta t L)^{-1}(I + \Delta t N) - I]U(t) \\
&= (I - \Delta t L)^{-1}\Delta t(N + L)U(t)
\end{aligned}
$$

yields an operator whose roots are identical to those of $(N + L)$, but whose Jacobian matrix is far better conditioned.

We now apply this idea to solve the related linear stability problem.

$$
Au \equiv (N_U + L)u = \lambda u
$$

where U is a steady flow and N_U is, e.g., $N_U u = -U \cdot \nabla u - u \cdot \nabla U$ if $N(U) = -U \cdot \nabla U$. Complete diagonalization of the large ($10^4 \times 10^4$ to $10^5 \times 10^5$) matrices A arising from realistic two or three dimensional flows is prohibitively expensive and wasteful, when only the eigenvalues producing instability are important. (*Leading* eigenvalues are those of largest real part.)

An alternative is to use the power method on $\exp(A\Delta t)$, since the leading eigenvalues of A are the dominant (largest magnitude) eigenvalues of $\exp(A\Delta t)$. Moreover, any code for time-stepping $du/dt = Au$ already acts repeatedly with an approximation to $\exp(A\Delta t)$, effectively carrying out the power method on the exponential. The fundamental drawback of this method is that the approximation to $\exp(A\Delta t)$ is only valid for small Δt. In this case, $\exp(A\Delta t)$ is near the identity, so each step accomplishes little to separate the eigenvectors. This method has nevertheless been successfully used to calculate leading eigenvalues near bifurcations in a variety of hydrodynamic configurations including spherical Couette flow [2], flow down an inclined plate [3], and channel flow [4, 5].

The *inverse power method* is the method of choice for finding eigenvalues closest to zero. The difficulty, as in steady-state solving, lies in the inversion of the large operator A, which may be

neither banded nor well-conditioned. However, A's condition number may be greatly improved by the same *Stokes preconditioner* we have used for steady-state solving.

We write:

$$Au_{n+1} = u_n \qquad\qquad (N_U + L)u_{n+1} = u_n \qquad\qquad (1)$$

$$CAu_{n+1} = Cu_n \qquad (I - \Delta tL)^{-1}\Delta t(N_U + L)u_{n+1} = (I - \Delta tL)^{-1}\Delta tu_n \qquad (2)$$

$$\tilde{A}u = b \qquad \left[(I - \Delta tL)^{-1}(I + \Delta tN_U) - I\right]u_{n+1} = (I - \Delta tL)^{-1}\Delta tu_n \qquad (3)$$

where the equations on the left define abbreviations for those on the right. The inversion of A in (1) has been replaced by that of $\tilde{A} \equiv CA$ in (2)-(3). This has two advantages. The first is that CA is the difference between two timesteps and hence already available in a FEBE timestepping code. The second is that for Δt *large*, $C = (I - \Delta tL)^{-1}\Delta t \approx -L^{-1}$, partially inverting (i.e. preconditioning) the badly conditioned operator $A = N_U + L$. In (2)-(3) Δt may be taken arbitrarily large, since it has become an algebraic parameter no longer playing the role of a timestep.

Conjugate gradient iteration [6] converges rapidly to the solution of the well conditioned system $\tilde{A}u = b$. Two inputs are required from the user:
–the right-hand-side b, which here is produced by taking a backwards Euler step on u_n.
–a subroutine which acts with the matrix $\tilde{A}$ on a vector, which here merely takes the difference between two widely spaced FEBE timesteps.

Complex conjugate eigenvalues signaling imminent Hopf bifurcations may have large imaginary parts and hence be far from zero. These can be detected by applying the inverse power method to $(A + A^T)/2$, a self-adjoint operator whose eigenvalues are the real parts of the eigenvalues of A. Complex shifts may also be applied to move imaginary eigenvalues to the origin.

One of the attractive features of the method is that the inversion speeds up as u_n approaches an eigenvector. The reason for this is that conjugate gradient methods construct the solution to $\tilde{A}u = b$ from the vectors $b, \tilde{A}b, \tilde{A}^2b, \ldots$. When b contains components of only K eigenvectors of $\tilde{A}$, u can be constructed from K powers of $\tilde{A}$ on b.

We apply the method to spherical Couette flow, the flow between differentially rotating concentric spheres, and to Rayleigh–Bénard convection to demonstrate its speed for realistic hydrodynamic problems. DB thanks the Nuffield Foundation for supporting this research.

References

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