EXPONENTIAL EQUIDISTRIBUTION OF STANDARD PAIRS FOR PIECEWISE EXPANDING MAPS OF METRIC SPACES

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Abstract. For a large class of piecewise expanding maps of a metric space we show the equidistribution of standard pairs at an exponential rate. As a corollary such systems have a unique absolutely continuous invariant probability measure with respect to which they are exponentially mixing. We allow for non-compact spaces and do not assume or use the existence of a Markov structure. Furthermore, we provide explicit estimates on the exponential rate of equidistribution and various constants involved.

1. Introduction

In this paper we set up a framework for proving statistical properties of piecewise expanding maps of metric spaces. In our main theorem we prove the exponential equidistribution of standard pairs with explicit estimates on the constants. As a corollary the system has a unique absolutely continuous invariant probability measure (ACIP) with respect to which it is exponentially mixing. Our framework contains non-Markov dynamical systems defined on a countably infinite partition of a non-compact or non-connected metric space. Furthermore, the systems under consideration are not required to satisfy the "big image property".

We make reasonable sufficient assumptions on the dynamical system, which in principle may be checked, possibly on a computer, and lead to explicit estimates. Such explicit estimates help one understand which parameters determine the rate of mixing of a system. They also make it possible to study perturbations of the system. Almost all of our assumptions are somehow necessary; otherwise, counterexamples can be constructed.

The study of piecewise expanding maps has a long history, starting with the existence of ACIPs for maps of the unit interval. One of the first major results in this direction was obtained by Lasota and Yorke [13] who set up a functional analytic framework and showed that piecewise $\mathscr{C}^2$ expanding maps of the unit interval (with a finite partition of monotonicity) admit finitely many ACIPs. Later, using a similar point of view, several authors proved the existence of ACIPs

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for multi-dimensional piecewise expanding maps under various extra assumptions [12, 11, 18, 7]. The functional analytic point of view has proven to be quite fruitful. Strong results on the statistical properties can be obtained once one constructs proper Banach spaces on which to study the spectrum of the transfer operator associated to the dynamical system. However, this approach does not lead to explicit estimates for example on the constants involved in the decay of correlations. These constants depend on the intrinsic properties of the dynamical system, hence to find them explicitly requires much better understanding of the obstructions to fast mixing.

In connection to the result of Lasota-Yorke, we remark that $\mathscr{C}^1$ regularity is not sufficient [10], and that in general some assumptions are needed related to the amount of expansion versus the growth rate of partition elements [20, 6]. In our setting, we provide a notion of complexity growth which is weaker than the usual assumption [8, 2], but under which the system still has good statistical properties. In some cases high regularity of the map makes up for the need for extra assumptions on the complexity growth of partition elements [19, 21, 5].

For maps defined on a countably infinite partition, Rychlik [17] obtained some results for maps of the interval using the functional analytic approach and in the non-compact setting Bugiel [4], Lenci [14] have studied Markov maps of $\mathbb{R}^n$.

Results on the rate of mixing for multi-dimentsional maps, which also provide explicit estimates on the constants are rare. Saussol [18] obtains such explicit estimates via the approach of Liverani [15] using Birkhoff cones; however, in his setting the space is a compact subset of $\mathbb{R}^d$ and he makes assumptions involving the ACIP of the system in order to obtain exponential mixing. We do not make any assumptions on the ACIP of the system to prove exponential mixing.

The essential ingredients of this article are standard families (introduced and developed by D. Dolgopyat and N. Chernov) and the method of coupling (introduced to dynamical systems by L.-S. Young). Both ingredients have been used in various setting by various authors [22, 8, 3, 23, 9, 16, 1].

In Section 2 we describe the assumptions on our dynamical system and state the main theorem (Theorem 1) modulo the definition of standard pairs. In Section 3 we define standard pairs and families. In Section 4 we show their invariance under the dynamics. Finally, in Section 5 we describe the coupling of standard families and prove our main theorem.

2. Setting

Let $(X, \mathbf{d})$ be a metric space, $\mathscr{B}$ the Borel sigma-algebra and $\mathbf{m}$ a σ-finite measure on the measurable space $(X, \mathscr{B})$. We consider a *non-singular piecewise invertible map* T on X with respect to the countable (mod 0)–partition $\mathcal{P} = \{O_h\}_{h\in\mathcal{H}}$ of open subsets of X. This means that $T : X \to X$ is surjective (mod 0) and that the restrictions $T : O_h \to T(O_h)$ are non-singular homeomorphisms of O_h onto $T(O_h)$. We also use h to denote an inverse branch of T and use $\mathcal{H}$ to denote the set of inverse branches of T. Accordingly we denote the set of inverse branches of T^n by $\mathcal{H}^n$, for $n \in \mathbb{N}$. We write Jh for the Radon-Nikodym derivative $d(\mathbf{m} \circ h)/d\mathbf{m}$.

The transfer operator $\mathscr{L} : \mathbf{L}^1(X, \mathbf{m}) \circlearrowleft$ is defined as the dual of the Koopman operator $U : \mathbf{L}^\infty(X, \mathbf{m}) \circlearrowleft$, $Ug = g \circ T$. By a change of variables, it follows that

$$\mathscr{L} f(x) = \sum_{h \in \mathcal{H}} f \circ h(x) \cdot Jh(x) \cdot \mathbb{1}_{T(O_h)}(x), \text{ for } \mathbf{m}\text{-a.e. } x \in X. \tag{2.1}$$

Note that $\mathscr{L}^n f(x) = \sum_{h \in \mathcal{H}^n} f \circ h(x) Jh(x) \mathbb{1}_{T^n(O_h)}(x)$, for every $n \in \mathbb{N}$.

We make the following assumptions on our dynamical system.

(1) Uniform expansion: For every $h \in \mathcal{H}$ and $\varepsilon > 0$, denote

$$\Lambda_h(\varepsilon) = \sup_{\{x, y \in T(O_h) : \mathbf{d}(x,y) < \varepsilon\}} \frac{\mathbf{d}(h(x), h(y))}{\mathbf{d}(x, y)}.$$

There exist $\varepsilon_0 > 0$ and $\Lambda \in (0, 1)$ such that for every $h \in \mathcal{H}$, $\Lambda_h(\varepsilon_0) \leq \Lambda < 1$. Set $\Lambda_h := \Lambda_h(\varepsilon_0)$. Note that for $h \in \mathcal{H}^n$, we can define Λ_h using T^n and it is easy to verify that for all $h \in \mathcal{H}^n$, $\Lambda_h(\varepsilon_0) \leq \Lambda^n < 1$.

(2) Bounded distortion: There exist $\alpha \in (0, 1)$ and $\tilde{D} \geq 0$ such that $\forall h \in \mathcal{H}$, $\forall x, y \in T(O_h)$ satisfying $\mathbf{d}(x, y) < \varepsilon_0$, holds

$$Jh(x) \leq e^{\tilde{D}\mathbf{d}(x,y)^\alpha} Jh(y). \tag{2.2}$$

Let $D = \tilde{D}/(1 - \Lambda^\alpha)$. As a consequence of uniform expansion, (2.2) holds for $h \in \mathcal{H}^n$ uniformly for all $n \in \mathbb{N}$ with D instead of $\tilde{D}$.

Definition 1. For a set $A \in X$, let $\partial_\varepsilon A = \{x \in A : \mathbf{d}(x, \partial A) < \varepsilon\}$, where $\partial A = \overline{A} \cap \overline{X \setminus A}$ is the topological boundary of A as a subset of X. It is easy to verify that the topological boundary of a set is preserved under a homeomorphism.

Fix $a_0 = D/(1 - \Lambda^\alpha)$.

(3) Dynamical Complexity: There exist constants $n_0 \in \mathbb{N}$ and $\sigma \in (0, e^{-a_0 \varepsilon_0^\alpha}(\Lambda^{-n_0} - 1))$ such that for every open set I, $\operatorname{diam} I \leq \varepsilon_0$, for every $\varepsilon < \varepsilon_0$,

$$\sum_{\{h \in \mathcal{H}^{n_0}, \mathbf{m}(I \cap O_h) > 0, \operatorname{diam} T^{n_0}(I \cap O_h) \leq \varepsilon_0\}} \frac{\mathbf{m}(h(\partial_\varepsilon T^{n_0} O_h) \cap (I \setminus \partial_{\Lambda^{n_0}\varepsilon} I))}{\mathbf{m}(\partial_{\Lambda^{n_0}\varepsilon} I)} \leq \sigma. \tag{2.3}$$

Moreover, there exists a constant $\bar{C} < \infty$ such that for every integer $1 \leq r < n_0$, for every $\varepsilon < \varepsilon_0$,

$$\sum_{\{h \in \mathcal{H}^r, \mathbf{m}(I \cap O_h) > 0, \operatorname{diam} T^r(I \cap O_h) \leq \varepsilon_0\}} \frac{\mathbf{m}(h(\partial_\varepsilon T^r O_h) \cap (I \setminus \partial_{\Lambda^r \varepsilon} I))}{\mathbf{m}(\partial_{\Lambda^r \varepsilon} I)} \leq \bar{C}. \tag{2.4}$$

Remark 1. This condition may seem difficult to verify at first because it requires (2.3) to be checked for every small open set I. However in many situations of interest it can be verified. For example if $X \subset \mathbb{R}^d$, $\mathbf{m} =$ Lebesgue measure and the boundaries of O_h are unions of sufficiently smooth $(d-1)$-dimensional manifolds (see also [8]). Condition (2.3) is in some sense a generalization of condition (PE5) of [18].

Remark 2. Often one can check (2.3) for $n_0 = 1$ in which case there is no need to check (2.4).

(4) Divisibility of large sets: There exists $C_{\varepsilon_0} > 0$ such that for every open set V with $\operatorname{diam} V \geq \varepsilon_0$ and any set $V_* \subset V$ of $\operatorname{diam} V_* < \varepsilon_0/3$ there exists a (mod 0)–partition $\{V_\ell\}_{\ell\in\mathcal{U}}$ of V into open sets such that $\operatorname{diam} V_\ell < \varepsilon_0$, $V_* \subset V_\ell$ for some $\ell \in \mathcal{U}$, and

$$\sup_{\ell\in\mathcal{U}} \frac{\mathbf{m}(\partial_\varepsilon V_\ell \setminus \partial_\varepsilon V)}{\mathbf{m}(V_\ell)} \leq C_{\varepsilon_0}\varepsilon, \text{ for every } \varepsilon < \varepsilon_0. \tag{2.5}$$

Let $\zeta_1 = e^{a_0\varepsilon_0^\alpha} C_{\varepsilon_0}$ and let $\zeta_2, \zeta_3, \zeta_4, M, B_0$ be as in Proposition 1. Let $\delta_0 = 1/(3B_0)$.

Remark 3. The Growth Lemma (Lemma 3) as well as its corollary (Corollary 2) may be of interest even if one is not interested in coupling, so it is worth pointing out that conditions **(1)-(3)** and a simplified version of **(4)** where $V_* = \varnothing$ suffice to establish Lemma 3 and Corollary 2.

Definition 2. A set $I \subset X$ is said to be δ_0-*regular* if I is open and $\mathbf{m}(I \setminus \partial_{\delta_0} I) > 0$.

Remark 4. We remark that in certain situations a δ_0-regular set must contain a ball $\mathcal{B}$ of a definite size. For example, if $(X, \mathbf{d})$ is a metric space in which open balls of radius $< \delta_0$ are connected, then if I is δ_0-regular, then every open ball of radius δ_0 centered at a point of $I \setminus \partial_{\delta_0} I$ is contained in I. Indeed, if $\mathcal{B}(x, \delta_0)$ were a ball centered at $x \in I \setminus \partial_{\delta_0} I$ so that $\mathcal{B}(x, \delta_0) \cap (X \setminus I) \neq \varnothing$, then $\mathcal{B}(x, \delta_0) \cap I$ and $\mathcal{B}(x, \delta_0) \cap (X \setminus \overline{I})$ would be non-empty open sets whose union is $\mathcal{B}(x, \delta_0)$, which contradicts the ball being connected.

More generally, if I is δ_0-regular and for every $x \in I$, $\mathbf{d}(x, \partial I) \leq \mathbf{d}(x, X \setminus I)$, then I contains a ball of radius δ_0. Indeed, since I is δ_0-regular, we can choose $y \in I$ so that $\mathbf{d}(y, \partial I) \geq \delta_0$. Then $\mathbf{d}(y, X \setminus I) \geq \delta_0$ hence the open ball of radius δ_0 centered at y is contained in I.

Definition 3. A set $\Omega \subset X$ is a *good overlap set* if it is open, $\mathbf{m}(\Omega) > 0$, $\mathbf{m}(\partial\Omega) = 0$, $\operatorname{diam}\Omega < \varepsilon_0/3$, and for every open set V containing Ω and every $\varepsilon < \varepsilon_0$,

$$\mathbf{m}(\partial_\varepsilon \Omega \setminus \partial_\varepsilon V) + \mathbf{m}(\partial_\varepsilon (V \setminus \bar{\Omega}) \setminus \partial_\varepsilon V) \leq C_X \mathbf{m}(\partial_\varepsilon V),$$

where C_X is a universal constant independent of Ω and V.

Let us denote $\delta = \delta_0$. Recall that M is specified in Proposition 1 and if $n_0 = 1$, then we can take $M = 1$.

(5) Positively linked: There exist $N_\delta \in \mathbb{N}$ with $N_\delta \geq M$, $\Delta_\delta > 0$, $\Gamma_{N_\delta} > 0$ and a collection $\mathcal{Q}_{N_\delta}$ whose elements are subsets of elements of $\mathcal{P}^{N_\delta}$, such that the following conditions hold.

- δ–density: Every δ–regular set $I \subset X$ contains an element of $\mathcal{Q}_{N_\delta}$.
- Overlapping images: For every $Q, \tilde{Q} \in \mathcal{Q}_{N_\delta}$ there exists N with $M \leq N \leq N_\delta$ such that $T^N Q \cap T^N \tilde{Q}$ contains a good overlap set Ω with $\mathbf{m}(\Omega) \geq \Delta_\delta > 0$. Note that N is a function from $\mathcal{Q}_\delta \times \mathcal{Q}_\delta$ into $\{M, M+1, \dots, N_\delta\}$.
- Positive weight: For every $Q \in \mathcal{Q}_\delta$, $N \in \mathcal{R}(N(Q, \cdot)) :=$ range of the function $N(Q, \cdot)$, and $h \in \mathcal{H}^N$ with $Q \subset O_h$, holds

$$\inf_{T^N(Q)} Jh \geq \Gamma_{N_\delta}. \tag{2.6}$$

Let $\gamma = (1/2)\varepsilon_0^{-2} e^{-2a_0\varepsilon_0^\alpha} \Delta^2 \Gamma_{N_\delta}$ and $\gamma_1 = (2/3)\gamma$ as in Lemma 6 and Lemma 7. Let n_1 be a positive integer such that $2a_0\Lambda^{\alpha n_1} + D < a_0$ and let $n_2 = k_0 n_0$, where k_0 is such that $3\vartheta^{k_0} + \zeta_2/B_0 < 1$ (Recall that n_0 was given by condition **(3)**).

Set

$$\bar{n} = N_\delta + \max\{n_1, n_2\};\ C_{\gamma_1} = (1-\gamma_1)^{-1};\ \gamma_2 = (1-\gamma_1)^{1/\bar{n}}.$$

Proper standard pairs (associated to parameters $a_0, \varepsilon_0, B_0 > 0$) will be defined in the next section; nevertheless, let us state our main goal.

Theorem 1. *Let $(X, \mathbf{d}, \mathbf{m})$ be a metric measure space and $T : X \circlearrowleft$ a piecewise expanding map satisfying hypotheses **(1)–(5)** involving parameters a_0, ε_0, B_0. Then there exist $C > 0$, $\gamma_2 \in (0,1)$ such that for every two proper standard pairs (I, ρ) and $(\tilde{I}, \tilde{\rho})$ with associated parameters a_0, ε_0, B_0,*

$$\|\mathscr{L}^m \rho - \mathscr{L}^m \tilde{\rho}\|_{\mathbf{L}^1} \leq C\gamma_2^m,\ \textit{for every } m \in \mathbb{N}. \tag{2.7}$$

The constants C and γ_2 are explicitly defined above with $C = 2C_{\gamma_1}$.

As a consequence of Theorem 1 there exists a unique absolutely continuous invariant measure with respect to which T is exponentially mixing.

Corollary 1. *Let $(X, \mathbf{d}, \mathbf{m})$ be a metric measure space and $T : X \circlearrowleft$ a piecewise expanding map satisfying hypotheses **(1)–(5)**. There exists a unique probability density $\wp \in \mathbf{L}^1(X, \mathbf{d}, \mathbf{m})$ such that $\mathscr{L}\wp = \wp$. Moreover, there exist $C > 0$, $\gamma_2 \in (0,1)$ such that for every proper probability standard family $\mathcal{G}$ with associated parameters a_0, ε_0, B_0,*

$$\|\mathscr{L}^m \rho_{\mathcal{G}} - \wp\|_{\mathbf{L}^1} \leq C\gamma_2^m,\ \textit{for every } m \in \mathbb{N}.$$

The constants C and γ_2 are explicitly defined above with $C = 2C_{\gamma_1}$.

Remark 5. If the original standard pairs (or standard families) are proper but have larger associated parameters, one can show using the growth lemma that they turn into proper standard families with the correct parameters a_0, ε_0, B_0 at an exponential rate.

3. Standard pairs and iterations

For $\alpha \in (0,1)$, and a function $\rho : I \to \mathbb{R}^+$ define

$$H(\rho) = \sup_{x,y\in I} \frac{|\ln \rho(x) - \ln \rho(y)|}{\mathbf{d}(x,y)^\alpha}. \tag{3.1}$$

All integrals where the measure is not indicated are with respect to the underlying measure $\mathbf{m}$.

Definition 4 (Standard pair)**.** A *standard pair* with associated parameters $a_0 > 0, \varepsilon_0 > 0$ is a pair (I, ρ) consisting of an open (mod 0) set $I \subset X$ and a function $\rho : I \to \mathbb{R}^+$ such that $\operatorname{diam} I < \varepsilon_0 \leq 1$, $\int_I \rho = 1$ and

$$H(\rho) \leq a_0. \tag{3.2}$$

Remark 6. We do *not* assume that I is connected.

Definition 5 (Standard family)**.** A *standard family* $\mathcal{G}$ is a set of standard pairs $\{(I_j, \rho_j)\}_{j\in\mathcal{J}}$ and an associated measure $w_{\mathcal{G}}$ on a countable set $\mathcal{J}$. The *total weight* of a standard family is denoted $|\mathcal{G}| := \sum_{j\in\mathcal{J}} w_j$. We say that $\mathcal{G}$ is a *proper* standard family if in addition there exists a constant $B_0 > 0$ such that,

$$|\partial_\varepsilon \mathcal{G}| := \sum_{j\in\mathcal{J}} w_{\mathcal{G}}(j) \int_{\partial_\varepsilon I_j} \rho_j \leq B_0 |\mathcal{G}| \varepsilon, \text{ for all } \varepsilon < \varepsilon_0, \tag{3.3}$$

where $\partial_\varepsilon I_j$ denotes the set of all points in I_j that have distance less than ε to the boundary of I_j. If $w_{\mathcal{G}}$ is a probability measure on $\mathcal{J}$, then $\mathcal{G}$ is called a *standard probability family.* Note that every standard family induces an absolutely continuous measure on X with the density $\rho_{\mathcal{G}} := \sum_{j\in\mathcal{J}} w_j \rho_j \mathbb{1}_{I_j}$. We say two standard families $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are *equivalent* if $\rho_{\mathcal{G}} = \rho_{\tilde{\mathcal{G}}}$.

The following simple lemma provides a useful consequence of log-Hölder regularity (3.2).

Lemma 1. *If (I, ρ) satisfies $H(\rho) \leq a_0$ and $diam(I) \leq \varepsilon_0$, then for every $J, J' \subset I$ with $\mathbf{m}(J)\mathbf{m}(J') \neq 0$,*

$$\inf_I \rho \asymp_{a_0} \mathcal{A}_J \rho \asymp_{a_0} \mathcal{A}_{J'} \rho \asymp_{a_0} \sup_I \rho, \tag{3.4}$$

where $\mathcal{A}_J \rho = \mathbf{m}(J)^{-1} \int_J \rho$ is the average of ρ on J and $C_1 \asymp_a C_2$ means $e^{-a_0 \varepsilon_0^\alpha} C_1 \leq C_2 \leq e^{a_0 \varepsilon_0^\alpha} C_1$.

Proof. The lemma follows from the fact that if (I, ρ) satisfies $H(\rho) \leq a_0$, then for every $x, y \in I$,

$$e^{-a_0 \mathbf{d}(x,y)^\alpha} \rho(y) \leq \rho(x) \leq e^{a_0 \mathbf{d}(x,y)^\alpha} \rho(y).$$

□

Given a standard family $\mathcal{G}$ with parameters a_0, ε_0, we define its n-th iterate as follows.

Definition 6 (Iteration)**.** Let $\mathcal{G}$ be a standard family with index set $\mathcal{J}$ and weight $w_{\mathcal{G}}$. For $(j,h) \in \mathcal{J} \times \mathcal{H}^n$ such that $\operatorname{diam} T^n(I_j \cap O_h) \geq \varepsilon_0$, let $\mathcal{U}_{(j,h)}$ be the index set of any[1] (mod 0)–partition $\{U_\ell\}_{\ell \in \mathcal{U}_{(j,h)}}$ of $T^n(I_j \cap O_h)$ into open sets such that

$$\operatorname{diam} U_\ell < \varepsilon_0. \tag{3.5}$$

and such that, setting $V = T^n(I_j \cap O_h)$ and $V_\ell = T^n(I_j \cap O_h) \cap U_\ell$,

$$\sup_{\ell \in \mathcal{U}} \frac{\mathbf{m}(\partial_\varepsilon V_\ell \setminus \partial_\varepsilon V)}{\mathbf{m}(V_\ell)} \leq C_{\varepsilon_0} \varepsilon, \text{ for every } \varepsilon \leq \varepsilon_0. \tag{3.6}$$

For $(j,h) \in \mathcal{J} \times \mathcal{H}^n$ such that $\operatorname{diam} T^n(I_j \cap O_h) < \varepsilon_0$ set $\mathcal{U}_{(j,h)} = \varnothing$. Define

$$\mathcal{J}_n := \{(j,h,\ell) | (j,h) \in \mathcal{J} \times \mathcal{H}^n, \ell \in \mathcal{U}_{(j,h)}, \mathbf{m}(I_j \cap O_h) > 0\}.^2 \tag{3.7}$$

For every $j_n := (j,h,\ell) \in \mathcal{J}_n$, define $I_{j_n} := T^n(I_j \cap O_h) \cap U_\ell$ and $\rho_{j_n} := \rho_j \circ h \cdot Jh \cdot z_{j_n}^{-1}$, where $z_{j_n} := \int_{I_{j_n}} \rho_j \circ h Jh$. Define $\mathcal{G}_n := \{(I_{j_n}, \rho_{j_n})\}_{j_n \in \mathcal{J}_n}$ and associate to it the measure given by

$$w_{\mathcal{G}_n}(j_n) = z_{j_n} w_{\mathcal{G}}(j). \tag{3.8}$$

Remark 7. Comparing the definition of the transfer operator applied to a density with the definition of $\mathcal{G}_n$ and the measure associated to it, we see that

$$\mathscr{L}^n \rho_{\mathcal{G}} = \rho_{\mathcal{G}_n}. \tag{3.9}$$

This is the main connection between the evolution of densities under $\mathscr{L}^n$ and the evolution of standard families.

Remark 8. A simple change of variables shows that for every standard family $\mathcal{G}$ and every $n \in \mathbb{N}$, $|\mathcal{G}_n| = |\mathcal{G}|$. That is, the total weight does not change under iterations. We will make use of this fact throughout the article.

[1]There may be many choices for such "artificial chopping". One can make different choices at different iterations. Later we require that the artificial chopping does not chop the *good overlap set*. That such a choice exists is contained in our assumption **(4)** on divisibility of large sets.

[2]When $\mathcal{U}_{(j,h)} = \varnothing$, by (j,h,ℓ) we mean (j,h).

4. Invariance of standard families

The first thing to show is the invariance of standard families under iterations of $\mathscr{L}_g^n$. Recall that $a_0 = D/(1-\Lambda^\alpha)$.

Lemma 2. *Suppose (I,ρ) is a standard pair satisfying $H(\rho) \le a_0$ and (I_n,ρ_n) is an image of it under T^n for some $n \in \mathbb{N}$, as in Definition 6. Then* $\operatorname{diam}(I_n) < \varepsilon_0$, *$\int_{I_n} \rho_n = 1$ and*

$$H(\rho_n) \le a_0(\Lambda^{\alpha n} + {a_0}^{-1} D). \tag{4.1}$$

Proof. Using the definition of $H(\cdot)$, noting its properties under multiplication and composition, and using the expansion of the map, it follows that

$$H(\rho_{j_n}) \le H(Jh) + \Lambda^{\alpha n} H(\rho_j).$$

By (2.2) we have $H(Jh) \le D$, and by assumption $H(\rho_j) \le a_0$, finishing the proof of (4.1). $\square$

Lemma 3 (Growth Lemma)**.** *Suppose $\varepsilon_0 > 0$, $n_0 \in \mathbb{N}$ and σ are as in our assumptions. Suppose $\mathcal{G}$ is a standard family with parameters a_0, ε_0. Then for every $\varepsilon \le \varepsilon_0$ we have*

$$|\partial_\varepsilon \mathcal{G}_{n_0}| \le (1 + e^{a_0 \varepsilon_0^\alpha} \sigma)|\partial_{\Lambda^{n_0}\varepsilon} \mathcal{G}| + \zeta_1 |\mathcal{G}| \varepsilon, \tag{4.2}$$

where $\zeta_1 = e^{a_0 \varepsilon_0^\alpha} C_{\varepsilon_0}$.

Proof. Suppose $\varepsilon < \varepsilon_0$. We write n for n_0. We have, by definition, $|\partial_\varepsilon \mathcal{G}_n| = \sum_{j_n} w_{\mathcal{G}_n}(j_n) \int_{\partial_\varepsilon I_{j_n}} \rho_{j_n}$. We split the sum into two parts according to whether $\mathcal{U}_{(j,h)} = \varnothing$ or $\mathcal{U}_{(j,h)} \ne \varnothing$.

Suppose $\mathcal{U}_{(j,h)} = \varnothing$. By a change of variables,

$$w_{j_n} \int_{\partial_\varepsilon I_{j_n}} \rho_{j_n} = w_j \int_{h(\partial_\varepsilon I_{j_n})} \rho_j.$$

For every $h \in \mathcal{H}^n$ we can write

$$h(\partial_\varepsilon I_{j_n}) = (h(\partial_\varepsilon I_{j_n}) \setminus \partial_{\Lambda^n \varepsilon} I_j) \cup (\partial_{\Lambda^n \varepsilon} I_j \cap O_h). \tag{4.3}$$

The integral over $\partial_{\Lambda^n\varepsilon} I_j \cap O_h$ is easily estimated by $|\partial_{\Lambda^n\varepsilon}\mathcal{G}|$. To estimate the integral of ρ_j over $h(\partial_\varepsilon I_{j_n}) \setminus \partial_{\Lambda^n\varepsilon} I_j$ we compare it to $\int_{\partial_{\Lambda^n\varepsilon} I_j} \rho_j$. As a result, we need to estimate the following

$$\sum_h \frac{\mathbf{m}(h(\partial_\varepsilon T(I_j \cap O_h)) \setminus (\partial_{\Lambda^n\varepsilon} I_j \cap O_h))}{\mathbf{m}(\partial_{\Lambda^n\varepsilon} I_j)} \le \sum_h \frac{\mathbf{m}(h(\partial_\varepsilon T O_h) \cap (I_j \setminus \partial_{\Lambda^n\varepsilon} I_j))}{\mathbf{m}(\partial_{\Lambda^n\varepsilon} I_j)}. \tag{4.4}$$

By (2.3), the right-hand side of (4.4) is bounded by $\leq \sigma$. Therefore,

$$\sum_j w_j \sum_h \int_{h(\partial_\varepsilon I_{jn})\setminus \partial_{\Lambda\varepsilon} I_j} \rho_j \leq e^{a_0 \varepsilon_0^\alpha} \sigma |\partial_{\Lambda^n \varepsilon} \mathcal{G}|.$$

Now suppose that $\mathcal{U}_{(j,h)} \neq \varnothing$. By Definition 6, $w_{jn} \int_{\partial_\varepsilon I_{jn}} \rho_{jn}$ is bounded by $\leq \sum_j w_j \sum_{h,\ell} \int_{\partial_\varepsilon I_{jn}} \rho_j \circ hJh$. Let us split the integral over two sets by writing

$$\partial_\varepsilon I_{jn} = (\partial_\varepsilon I_{jn} \setminus \partial_\varepsilon T^n(I_j \cap O_h)) \cup (\partial_\varepsilon I_{jn} \cap \partial_\varepsilon T^n(I_j \cap O_h) \cap U_\ell). \tag{4.5}$$

Since $H(\rho_j \circ hJh) \leq a_0$ and $\operatorname{diam} I_{jn} \leq \varepsilon_0$ we get by Lemma 1 that the sum over the first term of (4.5) is bounded by

$$\leq \sum_j w_j \sum_{h,\ell} e^{a_0 \varepsilon_0^\alpha} \frac{\mathbf{m}(\partial_\varepsilon I_{jn}) \setminus \partial_\varepsilon T^n(I_j \cap O_h)}{\mathbf{m}(I_{jn})} \int_{I_{jn}} \rho_j \circ hJh.$$

Using (3.6), this is bounded by $\leq e^{a_0 \varepsilon_0^\alpha} C_{\varepsilon_0} \varepsilon \sum_j w_j \sum_h \int_{T^n(I_j \cap O_h)} \rho_j \circ hJh$, which is equal to $e^{a_0 \varepsilon_0^\alpha} C_{\varepsilon_0} |\mathcal{G}_n| \varepsilon$. Recall that $|\mathcal{G}_n| = |\mathcal{G}|$. The sum over the second term of (4.5) is equal to $\sum_j w_j \sum_h \int_{\partial_{T^n(I_j \cap O_h)}} \rho_j$ and was already included in the estimate of the paragraph above, so we do not need to add it again. □

Recall that n_0 is such that $\Lambda^{n_0}(1 + e^{a_0 \varepsilon_0^\alpha} \sigma) < 1$. Iterating Lemma 3 leads to the following. The proof is standard so we omit it.

Corollary 2. *Set $\vartheta_1 := \Lambda^{n_0}(1 + e^{a_0 \varepsilon_0^\alpha} \sigma)$, $\zeta_2 = \zeta_1(1 - \vartheta_1)^{-1}$, $\zeta_3 = (1 + \bar{C})$ and $\zeta_4 = (1 + \zeta_2(1 + \bar{C}))$. For every $k \in \mathbb{N}$ and $\varepsilon \leq \varepsilon_0$,*

$$|\partial_\varepsilon \mathcal{G}_{kn_0}| \leq (1 + e^{a_0 \varepsilon_0^\alpha} \sigma)^k |\partial_{\Lambda^{kn_0}\varepsilon} \mathcal{G}| + \zeta_2 |\mathcal{G}| \varepsilon. \tag{4.6}$$

Moreover, for every $m \in \mathbb{N}$ that does not divide n_0 and for every $\varepsilon \leq \varepsilon_0$,

$$|\partial_\varepsilon \mathcal{G}_m| \leq \zeta_3 (1 + e^{a_0 \varepsilon_0^\alpha} \sigma)^{m/n_0} |\partial_{\Lambda^m \varepsilon} \mathcal{G}| + \zeta_4 |\mathcal{G}| \varepsilon. \tag{4.7}$$

Proposition 1. *Suppose $\mathcal{G}$ is a standard family with parameters a_0, ε_0 and for some $\tilde{B} > 0$, $|\partial_\varepsilon \mathcal{G}| < \tilde{B} |\mathcal{G}| \varepsilon$ for $\varepsilon < \varepsilon_0$. Then for every $m \in \mathbb{N}$ with $m/n_0 \in \mathbb{N}$ and every $\varepsilon < \varepsilon_0$,*

$$|\partial_\varepsilon \mathcal{G}_m| \leq B_0 |\mathcal{G}| \varepsilon (\tilde{B} \vartheta_1^{m/n_0} / B_0 + \zeta_2 / B_0). \tag{4.8}$$

For every $m \in \mathbb{N}$ and $\varepsilon < \varepsilon_0$,

$$|\partial_\varepsilon \mathcal{G}_m| \leq B_0 |\mathcal{G}| \varepsilon (\tilde{B} \zeta_3 \vartheta_2^m / B_0 + \zeta_4 / B_0). \tag{4.9}$$

Hence if we fix $B_0 = 2\zeta_4$ and M by $\tilde{B} \zeta_3 \vartheta_2^M / B_0 = 1/2$, then for every $m \geq M$ $|\partial_\varepsilon \mathcal{G}_m| \leq B_0 |\mathcal{G}| \varepsilon$, for every $\varepsilon \leq \varepsilon_0$.

5. Coupling

In the previous section we justified the viewpoint of iterating standard families. Now we proceed to explain an inductive procedure to "couple" a small amount of mass of two proper standard families after a fixed number of iterations. During the coupling procedure standard families are modified in a controlled way. The following two lemma's are related to such modifications.

Lemma 4 (Splitting into constant and remainder)**.** *Consider two singleton standard families $\mathcal{G}_1 = \{(I_1, \rho_1)\}$ and $\mathcal{G}_2 = \{(I_2, \rho_2)\}$ with associated weights $w_1, w_2 > 0$. Suppose $w_2 \le w_1$ and let $c = (1/2)\varepsilon_0^{-1} e^{-a_0 \varepsilon_0^\alpha}$. Define*

$$\begin{aligned} \bar{\rho}_1 &= \frac{w_2}{w_1} c / \int_{I_1} \frac{w_2}{w_1} c = 1/\mathbf{m}(I_1),\ \mathring{\rho}_1 = (\rho_1 - \frac{w_2}{w_1} c) / \int_{I_1} (\rho_1 - \frac{w_2}{w_1} c), \\ \bar{\rho}_2 &= c / \int_{I_2} c = 1/\mathbf{m}(I_2),\ \mathring{\rho}_2 = (\rho_2 - c) / \int_{I_2} (\rho_2 - c). \end{aligned} \tag{5.1}$$

Set $\bar{w}_1 = w_1 \int_{I_1} (w_2/w_1) c = c w_2 \mathbf{m}(I_1)$, $\mathring{w}_1 = w_1 \int_{I_1} (\rho_1 - (w_2/w_1) c)$ and $\bar{w}_2 = w_2 \int_{I_2} c = c w_2 \mathbf{m}(I_2)$, $\mathring{w}_2 = w_2 \int_{I_2} (\rho_2 - c)$.

Then $H(\bar{\rho}_{1,2}) \le a_0$, $H(\mathring{\rho}_{1,2}) \le 2a$ and the families[3] *$\{(I_1, \bar{\rho}_1), (I_1, \mathring{\rho}_1)\}$, $\{(I_2, \bar{\rho}_2), (I_2, \mathring{\rho}_2)\}$ with their associated weights $\{\bar{w}_1, \mathring{w}_1\}$, $\{\bar{w}_2, \mathring{w}_2\}$ are equivalent to $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively.*

Proof. The functions $\bar{\rho}_{1,2}$ are constants, so they clearly satisfy $H(\bar{\rho}_{1,2}) \le a_0$ on their domains. Let us show that $\mathring{\rho}_2$ satisfies $H(\mathring{\rho}_{1,2}) \le 2a_0$. Indeed, since c is chosen such that $\inf \rho \ge 2c$, we have

$$\frac{\rho_2(x) - c}{\rho_2(y) - c} \le 1 + \frac{|\rho_2(x) - \rho_2(y)|}{\rho_2(y) - c} \le 1 + 2 \frac{|\rho_2(x) - \rho_2(y)|}{\rho_2(y)}.$$

Using $H(\rho_2) \le a_0$, the right hand side is further bounded by $1 + 2|e^{a_0 |x-y|^\alpha} - 1| \le e^{2|x-y|^\alpha}$. As for $\mathring{\rho}_1$, it also satisfies $H(\mathring{\rho}_1) \le 2a_0$ for the same reason since $w_2/w_1 \le 1$.

To check the equivalence of $\mathcal{G}_1$ and the family $\{(I_1, \bar{\rho}_1), (I_1, \mathring{\rho}_1)\}$ with associated weights $\{\bar{w}_1, \mathring{w}_1\}$, we check that $w_1 \rho_1 = \bar{w}_1 \bar{\rho}_1 + \mathring{w}_1 \mathring{\rho}_1$. Indeed, by construction, $\bar{w}_1 \bar{\rho}_1 = c w_2$ and $\mathring{w}_1 \mathring{\rho}_1 = w_1(\rho_1 - (w_2/w_1) c) = w_1 \rho_1 - w_2 c$. Hence the sum is $w_1 \rho_1$. The equivalence of $\mathcal{G}_2$ to the corresponding family $\{(I_2, \bar{\rho}_2), (I_2, \mathring{\rho}_2)\}$ with associated weights $\{\bar{w}_2, \mathring{w}_0\}$ is also easy to check. $\square$

Lemma 5 (Chopping out the overlap)**.** *Consider two singleton standard families $\mathcal{G} = \{(I, c)\}$ and $\tilde{\mathcal{G}} = \{(\tilde{I}, \tilde{c})\}$ with associated weights $w, \tilde{w}$. Suppose that $I \cap \tilde{I}$ contains a good overlap set Ω as defined in Definition 3. Denote $A_0 = \tilde{A}_0 = \Omega$,*

[3]These families are not standard families, but will become standard after several iterations. We refer to them as *pre*-standard families.

$A_1 = I \setminus \bar{\Omega}$ and $\tilde{A}_1 = \tilde{I} \setminus \bar{\Omega}$. Note that the latter two sets can be empty if $I \cap \tilde{I}$ is a good overlap set.

There exists a standard family equivalent to $\mathcal{G}$ obtained by replacing $\{(I, c)\}$ by $\{(A_j, 1/\mathbf{m}(A_j))\}_{j=0}^{1}$ and associated weights $\{cw\mathbf{m}(A_j)\}_{j=0}^{1}$.[4] Similarly there exists a standard family equivalent to $\tilde{\mathcal{G}}$ obtained by replacing $\{(\tilde{I}, \tilde{c})\}$ by $\{(\tilde{A}_j, 1/\mathbf{m}(\tilde{A}_j))\}_{j=0}^{1}$ and associated weights $\{\tilde{c}\tilde{w}\mathbf{m}(\tilde{A}_j)\}_{j=0}^{1}$. Note that if $cw = \tilde{c}\tilde{w}$, then $cw\mathbf{m}(A_0) = \tilde{c}\tilde{w}\mathbf{m}(\tilde{A}_0)$.

Proof. To show the existence of a standard family equivalent to $\mathcal{G}$ obtained by replacing $\{(I, c)\}$ by $\{(A_j, 1/\mathbf{m}(A_j))\}_{j=0}^{1}$ and associated weights $\{cw\mathbf{m}(A_j)\}_{j=0}^{1}$, we only need to show that each element of $\{(A_j, 1/\mathbf{m}(A_j))\}_{j=0}^{1}$ is a standard pair.[5] By Definition 3, Ω is open and $\operatorname{diam}\Omega < \varepsilon_0$ hence $(\Omega, 1/\mathbf{m}(\Omega))$ with associated weight $cw\mathbf{m}(\Omega)$ is a standard pair. The set $I \setminus \bar{\Omega}$ is also open and since $\operatorname{diam} I < \varepsilon_0$, $\operatorname{diam}(I \setminus \bar{\Omega}) < \varepsilon_0$. Therefore, $(I \setminus \bar{\Omega}, 1/\mathbf{m}(I \setminus \bar{\Omega}))$ is also a standard pair. Similar statements hold about $\tilde{\mathcal{G}}$. Note that $\mathbf{m}(\bar{\Omega}) = \mathbf{m}(\Omega)$ and similarly for $I \setminus \bar{\Omega}$ because $\mathbf{m}(\partial\Omega) = 0$. □

We are ready now to *couple* a small amount of weight of two standard pairs.

Lemma 6. *Suppose $\mathcal{G} = \{(I, \rho)\}$ and $\tilde{\mathcal{G}} = \{(\tilde{I}, \tilde{\rho})\}$ are singleton standard families with δ-regular domains $I, \tilde{I}$. Let N_δ, Δ_δ and Γ_{N_δ} be as in the assumptions. There exist pre-standard families $\mathcal{G}^*_{N_\delta}$, $\tilde{\mathcal{G}}^*_{N_\delta}$ such that*

$$\begin{aligned} &\rho_{\mathcal{G}^*_{N_\delta}} - \rho_{\tilde{\mathcal{G}}^*_{N_\delta}} = \rho_{\mathcal{G}_{N_\delta}} - \rho_{\tilde{\mathcal{G}}_{N_\delta}}; \text{ and,} \\ &|\mathcal{G}^*_{N_\delta}| \le |\mathcal{G}| - \min\{|\mathcal{G}|, |\tilde{\mathcal{G}}|\}\gamma, \\ &|\tilde{\mathcal{G}}^*_{N_\delta}| \le |\tilde{\mathcal{G}}| - \min\{|\mathcal{G}|, |\tilde{\mathcal{G}}|\}\gamma, \end{aligned} \tag{5.2}$$

where $\gamma = (1/2)\varepsilon_0^{-2} e^{-2a_0\varepsilon_0^\alpha} \Delta^2 \Gamma_{N_\delta}$.

Proof. Since the sets $I, \tilde{I}$ are regular sets, by the positively-linked assumption (namely δ-density), they each contain an element of $\mathcal{Q}_{N_\delta}$, namely $Q, \tilde{Q}$. Moreover, there exists N with $M \le N \le N_\delta$ such that the standard families $\mathcal{G}_N$ and $\tilde{\mathcal{G}}_N$ contain standard pairs (I_1, ρ_1) and (I_2, ρ_2) with associated weights w_1, w_2 whose intersection contains a good overlap set Ω, with $\mathbf{m}(\Omega) \ge \Delta_\delta > 0$. Here we have also used the fact that the artificial chopping of Definition 6 is done avoiding the overlap Ω.

Let us write $\Delta = \Delta_\delta$. We assume without loss of generality that $w_2 = \min\{w_1, w_2\}$. Apply Lemma 4 to replace the standard pairs (I_1, ρ_1), (I_2, ρ_2) by $\{(I_1, \bar{\rho}_1), (I_1, \mathring{\rho}_1)\}$, $\{(I_2, \bar{\rho}_2), (I_2, \mathring{\rho}_2)\}$ with associated weights $\{\bar{w}_1, \mathring{w}_1\}$, $\{\bar{w}_2, \mathring{w}_2\}$.

[4] with the convention that if A_1 is empty, then we do not include it in the collection.

[5] The statement about equivalence is a consequence of $cw\mathbb{1}_I = cw\mathbb{1}_{A_0} + cw\mathbb{1}_{A_1}$.

Now consider just $(I_1, \bar\rho_1)$ and $(I_2, \bar\rho_2)$. These are constant standard pairs and by definition (see Lemma 4) they satisfy $\bar w_1 \bar\rho_1 = \bar w_2 \bar\rho_2$. Now apply Lemma 5 to further replace these standard pairs by $\{(A_j, 1/\mathbf{m}(A_j))\}_{j=0}^1$, $\{(\tilde A_j, 1/\mathbf{m}(\tilde A_j))\}_{j=0}^1$ with associated weights $\{\bar\rho_1 \bar w_1 \mathbf{m}(A_j)\}_{j=0}^1$, $\{\bar\rho_2 \bar w_2 \mathbf{m}(\tilde A_j)\}_{j=0}^1$. Note that $A_0 = \tilde A_0$ and $\bar\rho_1 \bar w_1 = \bar\rho_2 \bar w_2 = c w_2$, so the elements corresponding to $j = 0$ are exactly the same in both families.

At this point we have replaced $\mathcal{G}_N$ by the family

$$(\mathcal{G}_N \setminus \{(I_1, \rho_1)\}) \cup \{(I_1, \mathring\rho_1)\} \cup \{(A_j, 1/\mathbf{m}(A_j))\}_{j=0}^1,$$

where $w_1\rho_1 = \mathring w_1 \mathring\rho_1 + \sum_{j=0}^1 (\bar\rho_1 \bar w_1 \mathbf{m}(A_j)) 1/\mathbf{m}(A_j) \mathbb{1}_{A_j}$.

To complete the modification of our standard families and obtain $\mathcal{G}_N^*, \tilde{\mathcal{G}}_N^*$, we remove the common element $(A_0, 1/\mathbf{m}(A_0))$ from both collections.

The weight of the removed element is $\bar\rho_1 \bar w_1 \mathbf{m}(A_0) = c \min\{w_1, w_2\}\mathbf{m}(A_0)$, which by definition of c (from Lemma 4) and $\mathbf{m}(A_0) = \mathbf{m}(\Omega) \geq \Delta$, is bounded by $\geq (1/2)\varepsilon_0^{-1} e^{-a_0 \varepsilon_0^\alpha} \Delta w_2$. Recall that w_2 is the weight of (I_2, ρ_2), which is a standard pair in $\tilde{\mathcal{G}}_N$. Hence for some $h_2 \in \mathcal{H}^N$ we have

$$\begin{aligned} w_2 = w_{\tilde{\mathcal{G}}} \int_{I_2} \tilde\rho \circ h_2 J h_2 \geq w_{\tilde{\mathcal{G}}} \int_{\Omega} \tilde\rho \circ h_2 J h_2 \geq w_{\mathcal{G}} \mathbf{m}(\Omega) \inf_{T^N(\tilde Q)} J h_2 \inf_{\tilde I} \tilde\rho \\ \geq w_{\tilde{\mathcal{G}}} \Delta \Gamma_N e^{-a_0 \varepsilon_0^\alpha} \mathbf{m}(\tilde I)^{-1} \int_{\tilde I} \tilde\rho. \end{aligned} \tag{5.3}$$

Since $\int_{\tilde I} \tilde\rho = 1$, we have $w_2 \geq \Delta \Gamma_N e^{-a_0\varepsilon_0^\alpha} \varepsilon_0^{-1} w_{\tilde{\mathcal{G}}}$. Therefore, the weight of the removed element $(A_0, 1/\mathbf{m}(A_0))$ is bounded by $\geq (1/2)\varepsilon_0^{-2} e^{-2a_0\varepsilon_0^\alpha} \Delta^2 \Gamma_{N_\delta} \min\{|\mathcal{G}|, |\tilde{\mathcal{G}}|\}$. Setting $\gamma = (1/2)\varepsilon_0^{-2} e^{-2a_0\varepsilon_0^\alpha} \Delta^2 \Gamma_{N_\delta}$,

$$|\mathcal{G}_{N_\delta}^*| \leq |\mathcal{G}| - \min\{|\mathcal{G}|, |\tilde{\mathcal{G}}|\}\gamma.$$

One also gets a similar estimate for $|\tilde{\mathcal{G}}_{N_\delta}^*|$. □

Now we remove the restriction that $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are singleton standard families.

Lemma 7. *Suppose $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are standard families and each satisfy $|\partial_\varepsilon \mathcal{G}| \leq B_0 |\mathcal{G}|\varepsilon$ for $\varepsilon < \varepsilon_0$. There exist pre-standard families $\mathcal{G}_N^*$, $\tilde{\mathcal{G}}_N^*$ such that*

$$\begin{aligned} &\rho_{\mathcal{G}_N^*} - \rho_{\tilde{\mathcal{G}}_N^*} = \rho_{\mathcal{G}_N} - \rho_{\tilde{\mathcal{G}}_N}; \text{ and,} \\ &|\mathcal{G}_N^*| \leq |\mathcal{G}| - \min\{|\mathcal{G}|, |\tilde{\mathcal{G}}|\}\gamma_1, \\ &|\tilde{\mathcal{G}}_N^*| \leq |\tilde{\mathcal{G}}| - \min\{|\mathcal{G}|, |\tilde{\mathcal{G}}|\}\gamma_1, \end{aligned} \tag{5.4}$$

where $\gamma_1 = (2/3)\gamma$.

Proof. Recall that we chose $\delta = \delta_0 = 1/(3B_0)$ so that $|\partial_\delta \mathcal{G}| \le B_0|\mathcal{G}|\delta < (1/3)|\mathcal{G}|$. Let $\mathcal{G}_L \subset \mathcal{G}$ be the collection of standard pairs $(I,\rho) \in \mathcal{G}$ such that $\int_{I\setminus\partial_\delta I} \rho > 0$. Note that for such standard pairs, I is necessarily a δ-regular set. We have $|\mathcal{G}_L| \ge 2/3|\mathcal{G}|$. Let $\mathcal{G}_S = \mathcal{G} \setminus \mathcal{G}_L$.

Let $(\mathcal{G}_L)_N$ be the image of $\mathcal{G}_L$. Note that $(\mathcal{G}_L)_N = \cup_{(I,\rho)\in\mathcal{G}}(\mathcal{G}_\rho)_N$, where $\mathcal{G}_\rho$ is a singleton standard family containing only (I,ρ). Thinking of $\mathcal{G}_L$ (and similarly $\tilde{\mathcal{G}}_L$) as a union of singleton standard families we can apply Lemma 6. However, an intermediate technical step is necessary to properly justify the application of Lemma 6. In the following paragraph we describe this intermediate step.

Suppose (I,ρ) is an element of $\mathcal{G}_L$ and it has associated weight v. We replace this element by countably many elements which are the same except that their weights are given by $\{v\tilde{v}/|\tilde{\mathcal{G}}_L|\}_{\tilde{v}\in\tilde{\mathcal{G}}_L}$. Here we have slightly abused notation and labeled standard pairs by their weights. Similarly, we replace every element in $\tilde{\mathcal{G}}$ of weight $\tilde{v}$ by elements of weight $\{v\tilde{v}/|\mathcal{G}_L|\}_{v\in\mathcal{G}_L}$. For every $v\tilde{v}/|\tilde{\mathcal{G}}_L| \in \mathcal{G}_L$, there exists a *matching* element $v\tilde{v}/|\mathcal{G}_L| \in \tilde{\mathcal{G}}_L$. We apply Lemma 6 to these two elements. As a result, the weight $v\tilde{v}/|\tilde{\mathcal{G}}_L|$ is reduced by $\min\{v\tilde{v}/|\tilde{\mathcal{G}}_L|, v\tilde{v}/|\mathcal{G}_L|\}\gamma$. Do this for all elements $v\tilde{v}/|\tilde{\mathcal{G}}_L| \in \mathcal{G}_L$. Then the total weight $|\mathcal{G}_L|$ is reduced by

$$\sum_v \sum_{\tilde{v}} \min\{v\tilde{v}/|\tilde{\mathcal{G}}_L|, v\tilde{v}/|\mathcal{G}_L|\}\gamma = \sum_v \sum_{\tilde{v}} v\tilde{v}\min\{1/|\tilde{\mathcal{G}}_L|, 1/|\mathcal{G}_L|\}\gamma.$$

Observe that this is just $\min\{|\mathcal{G}_L|, |\tilde{\mathcal{G}}_L|\}\gamma$. Note that we have just described a matching of weights and nothing else. This intermediate step does not affect any other characteristics of our standard families.

With the above considerations, we obtain $|(\mathcal{G}_L)^*_N| \le |\mathcal{G}_L| - \min\{|\mathcal{G}_L|, |\tilde{\mathcal{G}}_L|\}\gamma$. Similarly, $|(\tilde{\mathcal{G}}_L)^*_N| \le |\tilde{\mathcal{G}}_L| - \min\{|\mathcal{G}_L|, |\tilde{\mathcal{G}}_L|\}\gamma$. Since the standard pairs in $\mathcal{G}_S$ are not modified, we have $|(\mathcal{G}_S)^*_N| = |\mathcal{G}_S|$. Since $|\mathcal{G}^*_N| = |(\mathcal{G}_S)^*_N| + |(\mathcal{G}_L)^*_N|$, we have

$$\begin{aligned} |\mathcal{G}^*_N| &\le |\mathcal{G}_S| + |\mathcal{G}_L| - \min\{|\mathcal{G}_L|, |\tilde{\mathcal{G}}_L|\}\gamma \\ &\le |\mathcal{G}| - (2/3)\min\{|\mathcal{G}|, |\tilde{\mathcal{G}}|\}\gamma. \end{aligned}$$

Similar estimate is obtained for $|\tilde{\mathcal{G}}^*_N|$.

□

Remark 9 (Recovery of regularity). The families $\mathcal{G}^*_N$ and $\tilde{\mathcal{G}}^*_N$ are pre-standard families because by Lemma 4, an element (I_N, ρ_N) in one of these families only satisfies $H(\rho_N) \le 2a_0$. Let

$$n_1 = \lceil (\alpha \ln(\Lambda))^{-1} \ln(1/2 - D/(2a_0)) \rceil.$$

Then applying Lemma 2 we get $H(\rho_{N+n_1}) \le 2a_0(\Lambda^{\alpha n_1} + (2a_0)^{-1}D) < a_0$. Therefore, $\mathcal{G}_{N+n_1}, \tilde{\mathcal{G}}_{N+n_1}$ are standard families.

Remark 10 (Recovery of boundary). We also have to worry about the boundary of the standard family after modification. Recall that during modification, we first split a standard pair into two, a constant one and the remainder, then we further split the constant one into at most two pieces. The latter modification also modifies the boundary. However, since the splitting is done on a *good overlap set*, by a crude estimate the splitting increases the total boundary of the family by a factor of three.

Hence $|\partial_\varepsilon \mathcal{G}^*_N| \le 3|\partial_\varepsilon \mathcal{G}_N|$ and since $N \ge M$, this is bounded by $\le 3B_0|\mathcal{G}|\varepsilon$ for every $\varepsilon < \varepsilon_0$. In order to recover from this, we iterate $\mathcal{G}^*_N$ in multiples of n_0 and use (4.8). Indeed,

$$|\partial_\varepsilon \mathcal{G}^*_{N+kn_0}| := |\partial_\varepsilon (\mathcal{G}^*_N)_{kn_0}| \le B_0|\mathcal{G}^*_N|\varepsilon(3\vartheta_1^k + \zeta_2/B_0) \le B_0|\mathcal{G}|\varepsilon(3\vartheta_1^k + \zeta_2/B_0).$$

To finish the estimate recall our choice of B_0 and note that we just need to choose $k = k_0 \in \mathbb{N}$ such that $3\vartheta_1^{k_0} + \zeta_2/B_0 < 1$. Let $n_2 = k_0 n_0$ and $\bar{n} = N_\delta + \max\{n_1, n_2\}$.

As a corollary of the above remarks we get the following *recovered* version of Lemma 7, which can be iterated.

Proposition 2. *Suppose $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are proper standard families. There exist proper standard families $\mathcal{G}^*_{\bar{n}}$, $\tilde{\mathcal{G}}^*_{\bar{n}}$ such that*

$$\begin{aligned} &\rho_{\mathcal{G}^*_{\bar{n}}} - \rho_{\tilde{\mathcal{G}}^*_{\bar{n}}} = \rho_{\mathcal{G}_{\bar{n}}} - \rho_{\tilde{\mathcal{G}}_{\bar{n}}};\ \textit{and},\\ &|\mathcal{G}^*_{\bar{n}}| \le |\mathcal{G}| - \min\{|\mathcal{G}|, |\tilde{\mathcal{G}}|\}\gamma_1,\\ &|\tilde{\mathcal{G}}^*_{\bar{n}}| \le |\tilde{\mathcal{G}}| - \min\{|\mathcal{G}|, |\tilde{\mathcal{G}}|\}\gamma_1. \end{aligned} \tag{5.5}$$

We are ready to prove our main theorem.

Proof of Theorem 1. Using the above Corollary repeatedly, and noting that the weight of the families remain equal before and after modification,

$$\begin{aligned} \left\|\mathscr{L}^{2\bar{n}}\rho - \mathscr{L}^{2\bar{n}}\tilde{\rho}\right\|_{\mathbf{L}^1} &\le |\mathcal{G}^{**}_{2\bar{n}}| + |\tilde{\mathcal{G}}^{**}_{2\bar{n}}| \le (1-\gamma_1)|\mathcal{G}^*_{\bar{n}}| + (1-\gamma_1)|\tilde{\mathcal{G}}^*_{\bar{n}}|\\ &\le (1-\gamma_1)^2|\mathcal{G}| + (1-\gamma_1)^2|\tilde{\mathcal{G}}|\\ &\le 2(1-\gamma_1)^2. \end{aligned}$$

For a general $m \in \mathbb{N}$, write $m = k\bar{n} + r$, where $0 \le r < \bar{n}$. Using $\|\mathscr{L}^r\rho\|_{\mathbf{L}^1} \le \|\rho\|_{\mathbf{L}^1}$, we obtain $\|\mathscr{L}^m\rho - \mathscr{L}^m\tilde{\rho}\|_{\mathbf{L}^1} \le \left\|\mathscr{L}^{k\bar{n}}\rho - \mathscr{L}^{k\bar{n}}\tilde{\rho}\right\|$. This is bounded by $\le 2(1-\gamma_1)^k \le 2(1-\gamma_1)^{((m/\bar{n})-1)} = 2(1-\gamma_1)^{-1}\left((1-\gamma_1)^{1/\bar{n}}\right)^m$ □

A simple consequence of Proposition 2 is that for every proper probability standard pair ρ, the sequence $\{\mathscr{L}^m\rho\}_m$ is a Cauchy sequence in $\mathbf{L}^1(X, \mathcal{B}, \mathbf{m})$ hence it has a limit $\wp \in \mathbf{L}^1$. Moreover, this limit does not depend on the choice of the starting standard pair ρ. Indeed, for $n > m$, $\|\mathscr{L}^m\rho - \mathscr{L}^n\rho\|_{\mathbf{L}^1} = \left\|\rho_{\mathcal{G}_m} - \rho_{\tilde{\mathcal{G}}_m}\right\|_{\mathbf{L}^1}$, where $\mathcal{G} = \{(I,\rho)\}$ and $\tilde{\mathcal{G}} = \mathcal{G}_{n-m}$, which are proper probability standard families. Applying Proposition 2 repeatedly (as in the proof of Theorem 1), shows that $\left\|\rho_{\mathcal{G}_m} - \rho_{\tilde{\mathcal{G}}_m}\right\|_{\mathbf{L}^1}$ can be made arbitrarily small if n and m are sufficiently large and the result follows.

As an end remark, let us show that certain regular functions can be represented as proper standard families. Therefore, two such functions converge exponentially to one another under iteration.

Definition 7. We say that $V \subset X$ is $(a_0, \varepsilon_0, B_0)$-*nice* if there exists a (mod 0)-partition $\{V_\ell\}$ of V into countably many open sets such that

- for every ℓ, $\operatorname{diam} V_\ell \le \varepsilon_0$;
- $\mathbf{m}(\partial_\varepsilon V_\ell) \le e^{-a_0\varepsilon_0^\alpha} B_0 \varepsilon \mathbf{m}(V_\ell)$ for every $\varepsilon \le \varepsilon_0$.

Suppose $f \in \mathscr{C}^\alpha(X,\mathbb{R})$ is a bounded, Hölder continuous function supported on a (a_0,ε_0,B_0)-*nice* set $V \subset X$ with $\mathbf{m}(V) < \infty$ and $a_0 > 0$. Choosing $c = |f|_\alpha/a_0 + \sup|f|$, we can write $f = f + c - c$, where $H(f+c) \le a_0$. It is easy to see that $f + c$ can be written as a proper standard family with parameters a_0, ε_0, B_0. Indeed, the normalized restrictions of $f + c$ to sets V_ℓ form a standard family $\mathcal{G}$ and $|\partial_\varepsilon \mathcal{G}| \le e^{a_0\varepsilon_0^\alpha} \sum_\ell w_\ell \frac{\mathbf{m}(\partial_\varepsilon V_\ell)}{\mathbf{m}(V_\ell)} \le B_0 \varepsilon |\mathcal{G}|$. Note that $\mathcal{G}$ is no longer a *probability* standard family, but it is a standard family and we can apply Proposition 2 to it.

Suppose $f, g \in \mathscr{C}^\alpha(X,\mathbb{R})$ are bounded, Hölder continuous functions supported on (a_0,ε_0,B_0)-*nice* sets V_f, V_g such that $\mathbf{m}(V_f) = \mathbf{m}(V_g) < \infty$ and $\int_{V_f} f = \int_{V_g} g$. Write $f = f + c - c$ and $g = g + c - c$, where $c = \max\{|f|_\alpha, |g|_\alpha\}/a_0 + \max\{\sup|f|, \sup|g|\}$. Suppose our dynamical system satisfies conditions **(1)-(5)** with parameters a_0, ε_0, B_0. Then, applying Proposition 2,

$$\begin{aligned} \|\mathscr{L}^m f - \mathscr{L}^m g\|_{\mathbf{L}^1} &= \|\mathscr{L}^m(f+c) - \mathscr{L}^m(g+c)\|_{\mathbf{L}^1} \\ &\le \left\|\rho_{\mathcal{G}_m} - \rho_{\tilde{\mathcal{G}}_m}\right\|_{\mathbf{L}^1} \le 2C_{\gamma_1}\gamma_2^m |\mathcal{G}|, \end{aligned}$$

where $|\mathcal{G}| = \int_{V_f}(f+c) \le \frac{2\mathbf{m}(V_f)}{a_0} \max\{\|f\|_{\mathscr{C}^\alpha}, \|g\|_{\mathscr{C}^\alpha}\}$.

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