In this section we construct the toric variety X_Σ corresponding to a fan Σ.

Definition 3.1.1. A ***toric variety*** is an irreducible variety X containing a torus $T_N \simeq (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of T_N on itself extends to an algebraic action of T_N on X. (By algebraic action, we mean an action $T_N \times X \to X$ given by a morphism.)

Definition 3.1.2. A ***fan*** Σ in $N_\mathbb{R}$ is a finite collection of cones $\sigma \subseteq N_\mathbb{R}$ such that:

(a) Every $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone.

(b) For all $\sigma \in \Sigma$, each face of σ is also in Σ.

(c) For all $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of each (hence also in Σ).

Furthermore, if Σ is a fan, then:

- The ***support*** of Σ is $|\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma \subseteq N_\mathbb{R}$.
- $\Sigma(r)$ is the set of r-dimensional cones of Σ.

Definition 2.8. *Let M be a lattice and let $N = \mathrm{Hom}(M, \mathbb{Z})$ be the dual lattice.*

*A **strongly convex rational polyhedral cone** $\sigma \subset N_\mathbb{R} = N \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ is*

- *a **cone**, that is, if $v \in \sigma$ and $\lambda \in \mathbb{R}$, $\lambda \geq 0$ then $\lambda v \in \sigma$;*
- ***polyhedral**, that is, σ is the intersection of finitely many half spaces;*
- ***rational**, that is, the half spaces are defined by equations with rational coefficients;*
- ***strongly convex**, that is, σ contains no linear spaces other than the origin.*

RECALL: $\sigma \subseteq N_\mathbb{R}$ A CONE. $\sigma^\vee = \{m \in M_\mathbb{R} \mid \langle m, v\rangle \geq 0\}$ IS THE DUAL CONE

WE DEFINE $U_\sigma = \mathrm{SPEC}(k[S_\sigma])$ WHERE:

$$S_\sigma = \sigma^\vee \cap M$$

(NOTICE THE DIPENDENCE FROM THE LATTICE)

RECALL 2.0: $M = \mathbb{Z}^2$, σ DEFINED BY $e_1, e_2 \in \mathbb{R}^2$ THEN:

$\sigma^\vee = \mathrm{SPAN}_{\mathbb{N}}(e_1, e_2) \subseteq M$ SO $U_\sigma = \mathrm{SPEC}(\mathbb{C}[x,y]) = \mathbb{A}^2$

DONE IN LECTURE 2

Lemma 2.10. *Let $\tau \subset \sigma \subset N_\mathbb{R}$ be a face of the cone σ. Then we may find $u \in S_\sigma$ such that* $A_\sigma = k[S_\sigma]$

(1) $\tau = \sigma \cap u^\perp$,

(2)

$$S_\tau = S_\sigma + \mathbb{Z}^+(-u),$$

(3) *A_τ is a localisation of A_σ, and*

(4) *U_τ is a principal open subset of U_σ.* $U_\tau = (U_\sigma)_{x^m}$

IMMEDIATE FROM 3

3

$k[S_\tau] = k[S_\sigma + \mathbb{Z}^+(-u)]$
$= k[S_\sigma][x^{-m}] =$
$= k[S_\sigma]_{\{m\}}$

Given a fan F, we get a collection of affine toric varieties, one for every cone of F. It remains to check how to glue these together to get a toric variety. Suppose we are given two cones σ and τ belonging to F. The intersection is a cone ρ which is also a cone belonging to F. Since ρ is a face of both σ and τ there are natural inclusions

(SEPARATION LEMMA: TO SEE THAT ARE LOCALIZATION IN χ^m AND χ^{-m})

$$(U_\sigma)_{\chi^m} = U_\rho \subset U_\sigma \qquad \text{and} \qquad U_\rho \subset U_\tau.$$

We glue U_σ to U_τ using the natural identification of the common open subset U_ρ. Compatibility of gluing follows automatically from the fact that the identification is natural and from the combinatorics of the fan. It is clear that the resulting scheme is of finite type over the groundfield. Separatedness follows from:

Lemma 2.16. *Let σ and τ be two cones whose intersection is the cone ρ.*

If ρ is a face of each then the diagonal map

$$U_\rho \longrightarrow U_\sigma \times U_\tau,$$

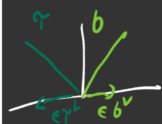
is a closed embedding.

Proof. This is equivalent to the statement that the natural map

$$A_\sigma \otimes A_\tau \longrightarrow A_\rho,$$

is surjective. For this, one just needs to check that

$$S_\rho = S_\sigma + S_\tau.$$



One inclusion is easy; the RHS is contained in the LHS. For the other inclusion, one needs a standard fact from convex geometry, which is called the *separation lemma: there is a vector $u \in S_\sigma \cap S_{-\tau}$ such that simultaneously

$$\rho = \sigma \cap u^\perp \qquad \text{and} \qquad \rho = \tau \cap u^\perp.$$

By the first equality $S_\rho = S_\sigma + \mathbb{Z}(-u)$. As $u \in S_{-\tau}$ we have $-u \in S_\tau$ and so the LHS is contained in the RHS. □

So we have shown that given a fan F we can construct a normal variety $X = X(F)$. It is not hard to see that the natural action of the torus corresponding to the zero cone extends to an action on the whole of X. Therefore $X(F)$ is indeed a toric variety.

(THE TORUS IS THE IDENTIFICATION OF THE TORUS IN U_σ THAT COMES FROM THE $\{0\}$-FACE)

X IS NORMAL SINCE THE U_σ ARE NORMAL AS SEEN IN A PREVIOUS LECTURE. (σ IS SATURATED)

SUMIHIRO *Let X be a normal separated toric variety with torus T_N. Then there exists a fan Σ in $N_\mathbb{R}$ such that $X \simeq X_\Sigma$.*

* **Lemma 1.2.13** (Separation Lemma). *Let σ_1, σ_2 be polyhedral cones in $N_\mathbb{R}$ that meet along a common face $\tau = \sigma_1 \cap \sigma_2$. Then*

$$\tau = H_m \cap \sigma_1 = H_m \cap \sigma_2$$

for any $m \in \mathrm{Relint}(\sigma_1^\vee \cap (-\sigma_2)^\vee)$.

Example 3.1.9. Consider the fan Σ in $N_{\mathbb{R}} = \mathbb{R}^2$ in Figure 2, where $N = \mathbb{Z}^2$ has standard basis e_1, e_2.

Here we show all points in the cones inside a rectangular viewing box (all figures of fans in the plane in this chapter will be drawn using the same convention.)

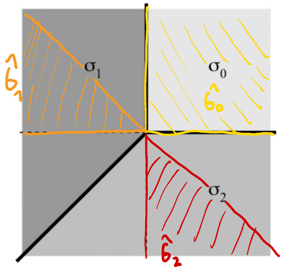


Figure 2. The fan Σ for $\mathbb{P}^2$

, we expect $X_\Sigma \simeq \mathbb{P}^2$, and we will show this in detail. The fan Σ has three 2-dimensional cones $\sigma_0 = \mathrm{Cone}(e_1, e_2)$, $\sigma_1 = \mathrm{Cone}(-e_1 - e_2, e_2)$, and $\sigma_2 = \mathrm{Cone}(e_1, -e_1 - e_2)$, together with the three rays $\tau_{ij} = \sigma_i \cap \sigma_j$ for $i \neq j$, and the origin. The toric variety X_Σ is covered by the affine opens

$$U_{\sigma_0} = \mathrm{Spec}(\mathbb{C}[S_{\sigma_0}]) \simeq \mathrm{Spec}(\mathbb{C}[x,y])$$
$$U_{\sigma_1} = \mathrm{Spec}(\mathbb{C}[S_{\sigma_1}]) \simeq \mathrm{Spec}(\mathbb{C}[x^{-1}, x^{-1}y])$$
$$U_{\sigma_2} = \mathrm{Spec}(\mathbb{C}[S_{\sigma_2}]) \simeq \mathrm{Spec}(\mathbb{C}[xy^{-1}, y^{-1}]).$$

Moreover, by Proposition 3.1.3, the gluing data on the coordinate rings is given by

$$g_{10}^* : \mathbb{C}[x,y]_x \simeq \mathbb{C}[x^{-1}, x^{-1}y]_{x^{-1}}$$
$$g_{20}^* : \mathbb{C}[x,y]_y \simeq \mathbb{C}[xy^{-1}, y^{-1}]_{y^{-1}}$$
$$g_{21}^* : \mathbb{C}[x^{-1}, x^{-1}y]_{x^{-1}y} \simeq \mathbb{C}[xy^{-1}, y^{-1}]_{xy^{-1}}.$$

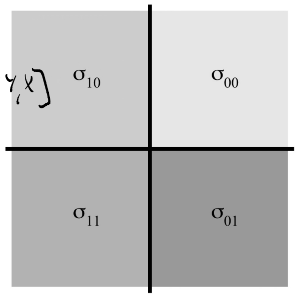
It is easy to see that if we use the usual homogeneous coordinates (x_0, x_1, x_2) on $\mathbb{P}^2$, then $x \mapsto \frac{x_1}{x_0}$ and $y \mapsto \frac{x_2}{x_0}$ identifies the standard affine open $U_i \subseteq \mathbb{P}^2$ with $U_{\sigma_i} \subseteq X_\Sigma$. Hence we have recovered $\mathbb{P}^2$ as the toric variety X_Σ. $\Diamond$

MAYBE BEFORE THE NEXT SLIDES:

When $n = m = 1$, we obtain the fan $\Sigma \subseteq \mathbb{R}^2 \simeq N_{\mathbb{R}}$ pictured in Figure 3 on the next page. Here, we can use an elementary gluing argument to show that this fan gives $\mathbb{P}^1 \times \mathbb{P}^1$. Label the 2-dimensional cones $\sigma_{ij} = \sigma_i \times \sigma'_j$ as above. Then

$$\mathrm{Spec}(\mathbb{C}[S_{\sigma_{00}}]) \simeq \mathrm{Spec}(\mathbb{C}[x,y])$$
$$\mathrm{Spec}(\mathbb{C}[S_{\sigma_{10}}]) \simeq \mathrm{Spec}(\mathbb{C}[x^{-1},y])$$
$$\mathrm{Spec}(\mathbb{C}[S_{\sigma_{11}}]) \simeq \mathbb{C}[x^{-1},y^{-1}]$$
$$\mathrm{Spec}(\mathbb{C}[S_{\sigma_{01}}]) \simeq \mathbb{C}[x,y^{-1}].$$

$\mathbb{C}[x,y,x^{-1}] \to \mathbb{C}[x^{-1},y,x]$



We see that if U_0 and U_1 are the standard affine open sets in $\mathbb{P}^1$, then $U_{\sigma_{ij}} \simeq U_i \times U_j$ and it is easy to check that the gluing makes $X_\Sigma \simeq \mathbb{P}^1 \times \mathbb{P}^1$. $\Diamond$

Proposition 3.1.14. *Suppose we have fans Σ_1 in $(N_1)_{\mathbb{R}}$ and Σ_2 in $(N_2)_{\mathbb{R}}$. Then*

$$\Sigma_1 \times \Sigma_2 = \{\sigma_1 \times \sigma_2 \mid \sigma_i \in \Sigma_i\}$$

is a fan in $(N_1)_{\mathbb{R}} \times (N_2)_{\mathbb{R}} = (N_1 \times N_2)_{\mathbb{R}}$ and

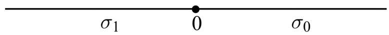
$$X_{\Sigma_1 \times \Sigma_2} \simeq X_{\Sigma_1} \times X_{\Sigma_2}.$$

DEPENDS ON THE EMBEDDINGS:

Example 3.1.11. We classify all 1-dimensional normal toric varieties as follows. We may assume $N = \mathbb{Z}$ and $N_{\mathbb{R}} = \mathbb{R}$. The only cones are the intervals $\sigma_0 = [0,\infty)$ and $\sigma_1 = (-\infty, 0]$ and the trivial cone $\tau = \{0\}$. It follows that there are only four possible fans, which gives the following list of toric varieties:

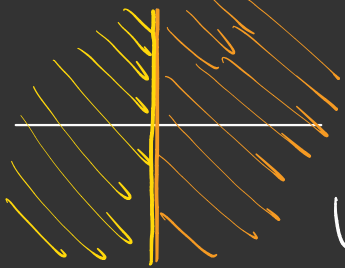
$$\{\tau\}, \text{ which gives } \mathbb{C}^*$$
$$\{\sigma_0,\tau\} \text{ and } \{\sigma_1,\tau\}, \text{ both of which give } \mathbb{C}$$
$$\{\sigma_0,\sigma_1,\tau\}, \text{ which gives } \mathbb{P}^1.$$

Here is a picture of the fan for $\mathbb{P}^1$:



$M = \mathbb{Z}^2$

3.1.7. In $N_{\mathbb{R}} = \mathbb{R}^2$, consider the fan Σ with cones $\{0\}$, $\mathrm{Cone}(e_1)$, and $\mathrm{Cone}(-e_1)$. Show that $X_\Sigma \simeq \mathbb{P}^1 \times \mathbb{C}^*$.



INTERSECTION

$U_{e_1} = \mathrm{Spec}\,(\mathbb{C}[x,y,y^{-1}])$ $U_{-e_1} = \mathrm{Spec}\,(\mathbb{C}[x^{-1},y,y^{-1}])$

$U_{\{0\}} = \mathrm{Spec}\,(\mathbb{C}[x,y,x^{-1},y^{-1}])$

$\mathrm{cone}(e_1) \cap \mathrm{cone}(-e_1) = \{0\}$

$U_{e_1} = \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[y]_y$ ....

$\underset{x}{\mathbb{P}^1} \times \underset{y}{\mathbb{C}^*}$

In this section, we will study the orbits for the action of T_N on the toric variety X_Σ. Our main result will show that there is a bijective correspondence between cones in Σ and T_N-orbits in X_Σ. The connection comes ultimately from looking at limit points of the one-parameter subgroups of T_N defined in §1.1.

Points and Semigroup Homomorphisms. It will be convenient to use the intrinsic description of the points of an affine toric variety U_σ recall [1]

$S_\sigma \ni m$
$P \mapsto (m \mapsto \chi^m(P))$
RECALL: $\{\chi^m\}_{m \in S_\sigma}$ IS A BASIS OF $\mathbb{C}[S_\sigma]$

1. • Points of U_σ are in bijective correspondence with semigroup homomorphisms $\gamma : S_\sigma \to \mathbb{C}$. WHERE $\mathbb{C} = (\mathbb{C}, \cdot)$ SEMI GROUP
2. • For each cone σ we have a point of U_σ defined by

$$m \in S_\sigma \longmapsto \begin{cases} 1 & m \in S_\sigma \cap \sigma^\perp = \sigma^\perp \cap M \\ 0 & \text{otherwise.} \end{cases}$$

   This is a semigroup homomorphism since $\sigma^\vee \cap \sigma^\perp$ is a face of $\sigma^\vee$. Thus, if $m, m' \in S_\sigma$ and $m + m' \in S_\sigma \cap \sigma^\perp$, then $m, m' \in S_\sigma \cap \sigma^\perp$. We denote this point by γ_σ and call it the *distinguished point* corresponding to σ. (IS A CLOSED POINT)
3. • The point γ_σ is fixed under the T_N-action if and only if $\dim \sigma = \dim N_\mathbb{R}$ (Corollary 1.3.3).
4. • If $\tau \preceq \sigma$ is a face, then $\gamma_\tau \in U_\sigma$. This follows since $\sigma^\perp \subseteq \tau^\perp$.

Limits of One-Parameter Subgroups. , the limit points of one-parameter subgroups are exactly the distinguished points for the cones in the fan We now show that this is true for all affine toric varieties.

Proposition 3.2.2. *Let $\sigma \subseteq N_\mathbb{R}$ be a strongly convex rational polyhedral cone and let $u \in N$. Then*

$$u \in \sigma \iff \lim_{t \to 0} \lambda^u(t) \text{ exists in } U_\sigma.$$

$\lambda^m(t)$ = ACTION OF $(1 : t^{m_1} : \dots : t^{m_n})$

Moreover, if $u \in \operatorname{Relint}(\sigma)$, then $\lim_{t\to 0} \lambda^u(t) = \gamma_\sigma$.

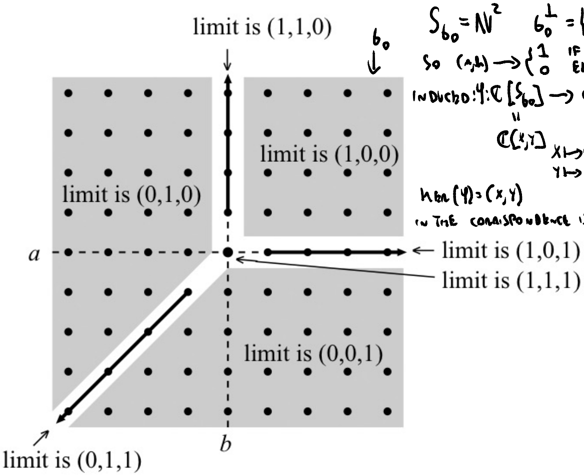
IT'S A COUNT, NOT PARTICULARLY ENLIGHTNING

Example 3.2.1. Consider $\mathbb{P}^2 \simeq X_\Sigma$ for the fan Σ from Figure 2 of §3.1. The torus $T_N = (\mathbb{C}^*)^2 \subseteq \mathbb{P}^2$ consists of points with homogeneous coordinates $(1,s,t)$, $s,t \neq 0$. For each $u = (a,b) \in N = \mathbb{Z}^2$, we have the corresponding curve in $\mathbb{P}^2$:

$$\lambda^u(t) = (1, t^a, t^b).$$

We are abusing notation slightly; strictly speaking, the one-parameter subgroup λ^u is a curve in $(\mathbb{C}^*)^2$, but we view it as a curve in $\mathbb{P}^2$ via the inclusion $(\mathbb{C}^*)^2 \subseteq \mathbb{P}^2$.

We start by analyzing the limit of $\lambda^u(t)$ as $t \to 0$. The limit point in $\mathbb{P}^2$ depends on $u = (a,b)$. It is easy to check that the pattern is as follows:



$S_{\sigma_0} = \mathbb{N}^2$ $\quad \sigma_0^\perp = \{0\}$

SO $(a,b) \longrightarrow \begin{cases} 1 & \text{IF } a=b=0 \\ 0 & \text{ELSE} \end{cases}$

INDUCED: $\varphi : \mathbb{C}[S_{\sigma_0}] \longrightarrow \mathbb{C}$

$\mathbb{C}[S_{\sigma_0}] = \mathbb{C}[x,y]$ $\quad x \mapsto 0$, $y \mapsto 0$

$\ker(\varphi) = (x,y)$

IN THE CORRISPONDENCE IS $(1,0,0)$

Figure 6. $\lim_{t\to 0} \lambda^u(t)$ for $u = (a,b) \in \mathbb{Z}^2$

For instance, suppose $a, b > 0$ in $\mathbb{Z}$. These points lie in the first quadrant. Here, it is obvious that $\lim_{t\to 0}(1,t^a,t^b) = (1,0,0)$. Next suppose that $a = b < 0$ in $\mathbb{Z}$, corresponding to points on the diagonal in the third quadrant. Note that

$$(1,t^a,t^b) = (1,t^a,t^a) \sim (t^{-a},1,1)$$

since we are using homogeneous coordinates in $\mathbb{P}^2$. Then $-a > 0$ implies that $\lim_{t\to 0}(t^{-a},1,1) = (0,1,1)$. You will check the remaining cases in Exercise 3.2.1.

The regions of N described in Figure 6 correspond to cones of the fan Σ. In each case, the set of u giving one of the limit points equals $N \cap \mathrm{Relint}(\sigma)$, where $\mathrm{Relint}(\sigma)$ is the *relative interior* of a cone $\sigma \in \Sigma$. In other words, we have recovered the structure of the fan Σ by considering these limits!

The Torus Orbits. Now we turn to the T_N-orbits in X_Σ. We saw above that each cone $\sigma \in \Sigma$ has a distinguished point $\gamma_\sigma \in U_\sigma \subseteq X_\Sigma$. This gives the torus orbit

$$O(\sigma) = T_N \cdot \gamma_\sigma \subseteq X_\Sigma.$$

In order to determine the structure of $O(\sigma)$, we need the following lemma, which you will prove in Exercise 3.2.4.

Lemma 3.2.4. *Let σ be a strongly convex rational polyhedral cone in $N_\mathbb{R}$. Let N_σ be the sublattice of N spanned by the points in $\sigma \cap N$, and let $N(\sigma) = N/N_\sigma$.*

ABELIAN GROUP

(a) *There is a perfect pairing*

$$\langle\ ,\ \rangle : \sigma^\perp \cap M \times N(\sigma) \to \mathbb{Z},$$

induced by the dual pairing $\langle\ ,\ \rangle : M \times N \to \mathbb{Z}$.

↪ $N = \mathrm{Hom}(M, \mathbb{Z})$

(b) *The pairing of part* (a) *induces a natural isomorphism*

$$\mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*) \simeq T_{N(\sigma)},$$

where $T_{N(\sigma)} = N(\sigma) \otimes_\mathbb{Z} \mathbb{C}^*$ is the torus associated to $N(\sigma)$. □

To study $O(\sigma) \subseteq U_\sigma$, we recall how $t \in T_N$ acts on semigroup homomorphisms. If $p \in U_\sigma$ is represented by $\gamma : \mathsf{S}_\sigma \to (\mathbb{C}, \cdot)$ then by Exercise 1.3.1, the point $t \cdot p$ is represented by the semigroup homomorphism

$$t \cdot \gamma : m \longmapsto \chi^m(t)\gamma(m). \tag{3.2.5}$$

Lemma 3.2.5. *Let σ be a strongly convex rational polyhedral cone in $N_\mathbb{R}$. Then*

$$\begin{aligned} O(\sigma) &= \{\gamma : \mathsf{S}_\sigma \to \mathbb{C} \mid \gamma(m) \neq 0 \Leftrightarrow m \in \sigma^\perp \cap M\} \\ &\simeq \mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*) \simeq T_{N(\sigma)}, \end{aligned}$$

RECALL THAT γ_σ IS FIXED IFF $\mathrm{Dim}(\sigma) = \mathrm{Dim}(N_\mathbb{R})$. AND IN THIS CASE $\sigma^\perp = \{0\}$

where $N(\sigma)$ is the lattice defined in Lemma 3.2.4.

Proof. The set $O' = \{\gamma : \mathsf{S}_\sigma \to \mathbb{C} \mid \gamma(m) \neq 0 \Leftrightarrow m \in \sigma^\perp \cap M\}$ contains γ_σ and is invariant under the action of T_N described in (3.2.5).

Next observe that $\sigma^\perp$ is the largest vector subspace of $M_\mathbb{R}$ contained in $\sigma^\vee$. Hence $\sigma^\perp \cap M$ is a *subgroup* of $\mathsf{S}_\sigma = \sigma^\vee \cap M$. If $\gamma \in O'$, then restricting γ to $m \in \mathsf{S}_\sigma \cap \sigma^\perp = \sigma^\perp \cap M$ yields a *group homomorphism* $\widehat{\gamma} : \sigma^\perp \cap M \to \mathbb{C}^*$ (Exercise 3.2.5). Conversely, if $\widehat{\gamma} : \sigma^\perp \cap M \to \mathbb{C}^*$ is a group homomorphism, we obtain a semigroup homomorphism $\gamma \in O'$ by defining

$$\gamma(m) = \begin{cases} \widehat{\gamma}(m) & \text{if } m \in \sigma^\perp \cap M \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $O' \simeq \mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*)$.

Now consider the exact sequence

$$0 \longrightarrow N_\sigma \longrightarrow N \longrightarrow N(\sigma) \longrightarrow 0. \tag{3.2.6}$$

Tensoring with $\mathbb{C}^*$ and using Lemma 3.2.4, we obtain a surjection

$$T_N = N \otimes_\mathbb{Z} \mathbb{C}^* \longrightarrow T_{N(\sigma)} = N(\sigma) \otimes_\mathbb{Z} \mathbb{C}^* \simeq \mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*).$$

The bijections

$$T_{N(\sigma)} \simeq \mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*) \simeq O'$$

are compatible with the T_N-action, so that T_N acts transtively on O'. Then $\gamma_\sigma \in O'$ implies that $O' = T_N \cdot \gamma_\sigma = O(\sigma)$, as desired. □

On $\mathbb{P}^2$

Now we relate this to the T_N-orbits in $\mathbb{P}^2$. By considering the description $\mathbb{P}^2 \simeq (\mathbb{C}^3 \setminus \{0\})/\mathbb{C}^*$, you will see in Exercise 3.2.1 that there are exactly seven T_N-orbits in $\mathbb{P}^2$:

$$\begin{aligned}
O_1 &= \{(x_0,x_1,x_2) \mid x_i \neq 0 \text{ for all } i\} \ni (1,1,1)\\
O_2 &= \{(x_0,x_1,x_2) \mid x_2 = 0, \text{ and } x_0,x_1 \neq 0\} \ni (1,1,0)\\
O_3 &= \{(x_0,x_1,x_2) \mid x_1 = 0, \text{ and } x_0,x_2 \neq 0\} \ni (1,0,1)\\
O_4 &= \{(x_0,x_1,x_2) \mid x_0 = 0, \text{ and } x_1,x_2 \neq 0\} \ni (0,1,1)\\
O_5 &= \{(x_0,x_1,x_2) \mid x_1 = x_2 = 0, \text{ and } x_0 \neq 0\} = \{(1,0,0)\}\\
O_6 &= \{(x_0,x_1,x_2) \mid x_0 = x_2 = 0, \text{ and } x_1 \neq 0\} = \{(0,1,0)\}\\
O_7 &= \{(x_0,x_1,x_2) \mid x_0 = x_1 = 0, \text{ and } x_2 \neq 0\} = \{(0,0,1)\}.
\end{aligned}$$

This list shows that each orbit contains a unique limit point. Hence we obtain a correspondence between cones σ and orbits O by

$$\sigma \text{ corresponds to } O \iff \lim_{t\to 0} \lambda^u(t) \in O \text{ for all } u \in \operatorname{Relint}(\sigma).$$

We will soon see that these observations generalize to all toric varieties X_Σ. $\Diamond$

FINAL BOMB !!

Theorem 3.2.6 (Orbit-Cone Correspondence). *Let X_Σ be the toric variety of the fan Σ in $N_\mathbb{R}$. Then:*

(a) *There is a bijective correspondence*

$$\begin{aligned}
\{\text{cones } \sigma \text{ in } \Sigma\} &\longleftrightarrow \{T_N\text{-orbits in } X_\Sigma\}\\
\sigma &\longleftrightarrow O(\sigma) \simeq \operatorname{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*).
\end{aligned}$$

(b) *Let $n = \dim N_\mathbb{R}$. For each cone $\sigma \in \Sigma$, $\dim O(\sigma) = n - \dim \sigma$.*

(c) *The affine open subset U_σ is the union of orbits*

$$U_\sigma = \bigcup_{\tau \preceq \sigma} O(\tau).$$

IDEA:

Proof of Theorem 3.2.6. Let O be a T_N-orbit in X_Σ. Since X_Σ is covered by the T_N-invariant affine open subsets $U_\sigma \subseteq X_\Sigma$ and $U_{\sigma_1} \cap U_{\sigma_2} = U_{\sigma_1 \cap \sigma_2}$, there is a unique minimal cone $\sigma \in \Sigma$ with $O \subseteq U_\sigma$. We claim that $O = O(\sigma)$. Note that part (a) will follow immediately once we prove this claim.

To prove the claim, let $\gamma \in O$ and consider those $m \in S_\sigma$ satisfying $\gamma(m) \neq 0$. In Exercise 3.2.6, you will show that these m's lie on a face of $\sigma^\vee$. But faces of $\sigma^\vee$ are all of the form $\sigma^\vee \cap \tau^\perp$ for some face $\tau \preceq \sigma$ by Proposition 1.2.10. In other words, there is a face $\tau \preceq \sigma$ such that

$$\{m \in S_\sigma \mid \gamma(m) \neq 0\} = \sigma^\vee \cap \tau^\perp \cap M.$$

This easily implies $\gamma \in U_\tau$ (Exercise 3.2.6), and then $\tau = \sigma$ by the minimality of σ. Hence $\{m \in S_\sigma \mid \gamma(m) \neq 0\} = \sigma^\perp \cap M$, and then $\gamma \in O(\sigma)$ by Lemma 3.2.5. This implies $O = O(\sigma)$ since two orbits are either equal or disjoint.

Part (b) follows from Lemma 3.2.5 and (3.2.6).

Next consider part (c). We know that U_σ is a union of orbits. If τ is a face of σ, then $O(\tau) \subseteq U_\tau \subseteq U_\sigma$ implies that $O(\tau)$ is an orbit contained in U_σ. Furthermore, the analysis of part (a) easily implies that any orbit contained in U_σ must equal $O(\tau)$ for some face $\tau \preceq \sigma$.