(CHOW GROUP AND LENGHT)

DEF.
LET X AN INTEGRALE SCHEME OF FINITE TYPE$/k$ OF $\dim X = n$.
THEN THE CLASS GROUP $Cl(X)$ IS:

$$Z^1(X) = \bigoplus_{\substack{V \subseteq X \\ \text{INT. SUBSCHEME} \\ \text{OF CODIM} = 1}} \mathbb{Z}[V]$$

QUOTIENT BY THE IMAGE OF:

$$k(X)^* \xrightarrow{\text{DIV}} Z^1(X)$$
$$\varphi \longmapsto \text{DIV}(\varphi) = \sum_{V \subseteq X} e_V(\varphi)$$

WHERE e_V: (IF X IS NORMAL e_V IS JUST THE EVALUATION IN $O_{X,V}$ A DVR) $\quad O_{X,\eta_V} = O_{X,V}$

$\bar{X} \to X \supseteq V \quad V_1, \dots, V_r$ IRRED. COMP. OF $\pi^{-1}(V)$ THEN e_{V_i} IS THE VALUATION
$e_{V_i}: k(\bar{X})^* = k(X) \to \mathbb{Z}$. WE DEFINE $e_V = \sum_i [k(V_i):k(V)]\, e_{V_i}(\varphi)$

NORMALIZATION

$\pi: V_i \to V$ IS A FINITE MAP AND $\dim(V) = \dim(V_i)$ THEN IS DOMINANT. SO $[k(V_i):k(V)]$ IS FINITE.

THE LOCAL RING OVER THE GENERIC POINT OF V

LEMMA: IF $\varphi \in O_{X,V}$, THEN $e_V(\varphi) = \ell_{O_{X,V}}(O_{X,V}/(\varphi))$

RMK: $O_{X,V}$ HAS DIMENSION 1 SINCE $V \subseteq X$ HAS CODIMENSION 1 AND SINCE WE QUOTIENT THE $\dim(O_{X,V}/(\varphi)) = 0$.

PROOF: IF X IS NORMAL AT V (I.E. AT THE GENERIC POINT OF V) THEN $O_{X,V}$ IS A DVR AND THE $\ell(O_{X,V}/(\varphi)) = \mathrm{VAL}_{O_{X,V}}(\varphi) = e_V(\varphi)$ $\Downarrow$ (REALLY EASY CASE)

IN GENERAL CALL $R = O_{X,V}$. SINCE THE STATEMENT IS LOCAL (AROUND THE GENERIC POINT OF V), THEN WE CAN ASSUME X TO BE AFFINE.

LOCAL STATEMENT

$$\begin{array}{ccc} \mathrm{SPEC}(\bar{R}) & \longrightarrow & \bar{X} \\ \downarrow & & \downarrow \\ \mathrm{SPEC}(R) & \longrightarrow & X \end{array}$$

HENCE $X = \mathrm{SPEC}(A)$, SO $V = V(p)$ FOR SOME PRIME $p \subseteq A$ THEN $R = O_{X,V} = O_{X,p} = A_p$
AND WE HAVE $\bar{R} = \bar{A}_p$ IS A SEMILOCAL RING OF DIM 1, $\bar{R}_{p_i} := O_{\bar{X},p_i}$.

IS A FINITE EXTENSION OF A_p THAT IS LOCAL
WITH ALL NON ZERO PRIME IDEAL ARE MAXIMAL

BASIC CASE

$$LHS. = \sum [k(V_i) : k(P)]\, e_{V_i}(\varphi) = \sum [k(P_i) : k(P)]\, \ell_{\bar R_{P_i}}\left(\bar R_{P_i}/(\varphi)\right) = \sum_i \ell_R\left(\bar R_{P_i}/(\varphi)\right) =$$

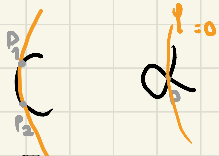
$$= \ell_R\left(\bigoplus_i \bar R_{P_i}/(\varphi)\right) = \ell_R\left(\bar R/(\varphi)\right)$$

IF $0 = M_0 \subseteq \dots \subseteq M_\ell = \bar R_{P_i}/(\varphi)$ THE MAX CHAIN THAT MISURE THE LENGHT AS $\bar R_{P_i}$-MODULE THEN $M_{n+1}/M_n \cong k(P_i)$. IN THE CASE OF R-MODULE THE QUOTIENT IS $k(P)$.

$\bar R$ IS NOETH OF DIM = 1 SO
$\bar R/(\varphi)$ " " " " = 0 SO
IS ARTINIAN WITH MAX IDEAL
$P_1, \dots, P_n$ THEN $\bar R/(\varphi) = \bigoplus \bar R_{P_i}/(\varphi)$

$$RHS. = \ell_{O_{X,V}}\left(O_{X,V}/(\varphi)\right)$$ SO WE NEED $\ell_R\left(R/(\varphi)\right) = \ell_{\bar R}\left(\bar R/(\varphi)\right)$

EXAMPLE:



$\bar X$ X

$R/(\varphi) = k[\varepsilon]/(\varepsilon^2)$

$\bar R/(\varphi) = k \times k$

AND NOT ISOMORPHIC BUT HAVE THE SAME LENGHT.

{FOR EXAMPLE $R = O_{X,(0,0)} = k[x,y]_{(x,y)}/(y^2 - x^2(x+1))$
$(\varphi) = (x+y)$

$\boxed{\text{TOOL}}$: IF $0 \to M' \to M \to M'' \to 0$ AN EXACT SEQUENCE OF R-MODULE AND $\ell_R(M')$, $\ell_R(M'')$ FINITE THEN $\ell_R(M)$ IS FINITE AND $\ell_R(M) = \ell_R(M') + \ell_R(M'')$.

WE LOOK AT THE SEQUENCE:

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & k \arrow[d] & \\
0 \arrow[r] & R \arrow[r, hook] \arrow[d, "\cdot\varphi"] & \bar R \arrow[r] \arrow[d, "\cdot\varphi"] & Q \arrow[r] \arrow[d, "\cdot\varphi"] & 0 \\
0 \arrow[r] & R \arrow[r, hook] \arrow[d, two heads] & \bar R \arrow[r] \arrow[d, two heads] & Q \arrow[r] \arrow[d] & 0 \\
 & R/(\varphi) \arrow[r] & \bar R/(\varphi) \arrow[r] & Q/(\varphi) \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

[Q IS A f.g. R-MODULE SINCE $\bar R$ IS ALSO f.g. AND HAS FINITE LENGHT SINCE $\bar R$ HAS IT.]

FOR THE SNAKE LEMMA WE HAVE THE EXACT:

$$0 \to k \to R/(\varphi) \to \bar R/(\varphi) \to Q/(\varphi) \to 0$$

SO SUMMING THE LENGHT WE HAVE:

$$\ell_R(k) - \ell_R\left(R/(\varphi)\right) + \ell_R\left(\bar R/(\varphi)\right) - \ell_R\left(Q/(\varphi)\right) = 0$$

TO CONCLUDE IS ENOUGH TO PROVE $\ell_R(k) = \ell_R\left(Q/(\varphi)\right)$ AND THIS FOLLOWS FROM IN THE SAME WAY.

SO WE HAVE $\ell_R\left(R/(\varphi)\right) = \ell_R\left(\bar R/(\varphi)\right)$

* THE NORMALIZATION USUALLY IS NOT FINITE SO THIS ARGUMENT FAIL, IT WORKS HERE ONLY BECAUSE WE HAVE FINITE ALGEBRAS OVER A FIELD SO FOR SCHEMES OF FINITE TYPE OVER SPEC(k)."

COROLLARY $\varphi, \psi \in O_{X,V}$ THEN $\ell_{O_{X,V}}\left(O_{X,V}/(\varphi\psi)\right) = \ell_{O_{X,V}}\left(O_{X,V}/(\varphi)\right) + \ell_{O_{X,V}}\left(O_{X,V}/(\psi)\right)$

Proof.

USING THE FORMULA AND THAT $\mathrm{VAL}_P(\varphi\psi) = \mathrm{VAL}_P(\varphi) + \mathrm{VAL}_P(\psi)$

CIT: "THIS IS USEFUL TO DEFINE THE CLASS GROUP ALSO WHEN THE SCHEME IS NOT NORMAL SO $O_{X,V}$

is not a DVR. Using $l_{\mathcal{O}_{X,V}}$ instead of val_p"

DEF: X scheme of finite type over $Spec(k)$.
We define the cycles of dimension d as $Z_d = \bigoplus_{\substack{V\subseteq X \\ \text{int. subscheme} \\ \text{of dim} = d}} \mathbb{Z}[V]$.

Moreover, $Z_*(X) = \bigoplus_{d\geq 0} Z_d(X)$.

Using this we called Chow group of X: (as cokernel)

$$\bigoplus_{\substack{W\subseteq X \text{ integral} \\ \dim(W)=d+1}} k(W)^* \xrightarrow{div} Z_d(X) \longrightarrow CH_d(X)$$

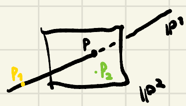
This is an equivalence and is called rational equivalence

i) If $\dim X = m \Rightarrow CH^m(X) = Z^m(X)$.

ii) If X is integral, dim m, then $CH_{m-1}(X) = Cl(X)$. since $\exists!\, W\subseteq X$ int. of $\dim = m$.

iii) If X is equidimensional of dim m, we define $CH^i(X) := CH_{m-i}(X)$.

Example: (why equidim.) $X =$



$P_1 \sim P$ $\quad P\sim P_2$ since $\mathbb{P}^1$ pass through P_1, P and $\exists$ a line passing through P, P_2. But $codim(P_1)=1$ $\quad codim(P_2)=2$

So codimension is **bad** for not equidimensional scheme.

Fact: X scheme $f.type/k$ and $Y\subseteq X$ closed subscheme. Then:

$$0 \to Z_m(Y) \hookrightarrow Z_m(X) \twoheadrightarrow Z_m(X\setminus Y) \to 0$$

$$V\subseteq X \longmapsto \begin{cases} V\cap (X\setminus Y) & \text{if } V\not\subseteq Y \\ 0 & \text{if } V\subseteq Y \end{cases}$$

is exact.

Proposition: The above induce the exact sequence:

$$CH_*(Y) \to CH_*(X) \twoheadrightarrow CH_*(X\setminus Y) \to 0$$

Rmk: In general $CH_*(Y) \to CH_*(X)$ is well defined since if $V\subseteq Y$ is a linear combination of divisors of function of subscheme of Y, using the same function with $W\subseteq Y\subseteq X \Rightarrow W\subseteq X$ then $V\subseteq X$ is also linear combination.

Example: ($CH_*(Y) \to CH_*(X)$ is not injective)

$X = \mathbb{P}^2$, $Y\subseteq \mathbb{P}^2$ a smooth curve of degree ≥ 3, $P,q\in Y$ $P\neq q$. Then $[P-q]\neq 0$ in $CH_0(Y) = Cl(Y)$ since $genus(Y)\geq 1$ so any two points are never equivalent.
But in $CH_0(\mathbb{P}^2)=0$ so $[P-q]=0$

Proof of proposition: Exercise

PROPER PUSH FORWARD

"CHOW GROUPS ARE A SORT OF HOMOLOGY THEORY FOR SCHEME"

(*) LET $f: X \to Y$ PROPER, WE CAN DEFINE $f_*: Z_k(X) \to Z_k(Y)$ AND PRESERVE DIMENSION.

IF $V \subseteq X$ A d-DIM. SUBVARIETY (CLOSED, INTEGRAL SUBSCHEME) THEN $W = f(V) \subseteq Y$ IS A SUBVARIETY.

CASE 1) $\dim W < \dim V$: WE DEFINE $f_*(V) = 0$ (KILL THE BLOW DOWN)

CASE 2) $\dim W = \dim V$: THEN $f|_V: V \to W$ IS PROPER AND SURJECTIVE BETWEEN SAME DIMENSION

USING THAT GENERICALLY THE DIMENSION OF THE FIBER IS $\dim(Y) - \dim(X) = 0$ WE HAVE

THAT THIS MAP IS GENERICALLY FINITE. (DIM IS UPPER SEMICONTINOUS)

SO IT'S WELL DEFINE A DEGREE THAT IS $[k(V) : k(W)]$.

WE DEFINE $f_* V = [k(V) : k(W)]\, W$.

STATEMENT

LEMMA: IF $f: V \to W$ A PROPER MORPHISM OF VARIETY THAT IS SURJECTIVE AND GENERICALLY FINITE. SO $k(V)/k(W)$ IS A FINITE EXTENSION AND $N_{k(V)/k(W)}: k(V)^* \to k(W)^*$

THEN IF $\varphi \in k(V)^*$, $f_*(\operatorname{Div} \varphi) = \operatorname{Div}(N_{k(V)/k(W)}(\varphi))$.

FULTON FOR THE PROOF 👉 1.4 THEOREM

Definition 1.4. If X is a complete scheme, i.e., X is proper over $S = \operatorname{Spec}(K)$, K the ground field, and $\alpha = \sum_P n_P[P]$ is a zero-cycle on X, the *degree* of α, denoted $\deg(\alpha)$, or $\int_X \alpha$, is defined by

$$\deg(\alpha) = \int_X \alpha = \sum_P n_P [R(P) : K].$$

Equivalently, $\deg(\alpha) = p_*(\alpha)$, where p is the structure morphism from X to S, and $A_0 S = \mathbb{Z}[S]$ is identified with $\mathbb{Z}$. By the theorem, rationally equivalent cycles have the same degree. We extend the degree homomorphism to all of $A_* X$,

$$\int_X : A_* X \to \mathbb{Z}$$

by defining $\int_X \alpha = 0$ if $\alpha \in A_k X$, $k > 0$. For any morphism $f: X \to Y$ of complete schemes, and any $\alpha \in A_* X$,

$$\int_X \alpha = \int_Y f_*(\alpha),$$

a special case of functoriality. We often write $\int$ in place of $\int_X$.

WE ASSUME FOR A MOMENT: $CH_d(\mathbb{A}^m) = \begin{cases} \mathbb{Z} & d = m \\ 0 & d \neq m \end{cases}$

PROPOSITION: $L^d = \mathbb{P}^d \subseteq \mathbb{P}^m$ LINEAR SUBSPACE OF DIMENSION d. THEN $CH_d(\mathbb{P}^m)$ IS GENERATED BY $[L_d]$

AND $\int_{\mathbb{P}^m}: CH_0(\mathbb{P}^m) \to \mathbb{Z}$ IS AN ISOMORPHISM.

PROOF.

INDUCTION ON m.

$m = d$) $L_d = \mathbb{P}^m$ AND IS CLEAR $CH_m(\mathbb{P}^m) = \mathbb{Z} = \langle [\mathbb{P}^m] \rangle$.

$m > d$) WE HAVE THE SES. $0 \to \mathbb{P}^{m-1} \to \mathbb{P}^m \to \mathbb{A}^m \to 0$ PASSING TO CHOW GROUPS:

$$\langle [L_d] \rangle = CH_d(\mathbb{P}^{m-1}) \twoheadrightarrow CH_d(\mathbb{P}^m) \to CH_d(\mathbb{A}^m) = 0$$

↓

FLAT PULL BACK

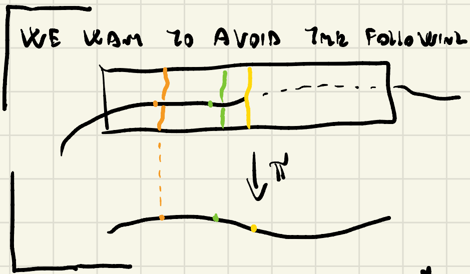
$[L_d] \mapsto [L_d]$

$f : X \to Y$ FLAT OF RELATIVE DIMENSION r

B.2.5. A morphism $f\colon X \to Y$ is *flat* if for $U \subset Y$, $U' \subset X$ affine open sets with $f(U') \subset U$, the induced map $f^*\colon A(U) \to A(U')$ makes $A(U')$ a flat $A(U)$-module. Equivalently, for all subvarieties V of X, with $W = \overline{f(V)}$, $\mathscr{O}_{V,X}$ is a flat $\mathscr{O}_{W,Y}$-module.

A morphism $f\colon X \to Y$ has *relative dimension n* if for all subvarieties V of Y, and all irreducible components V' of $f^{-1}(V)$, $\dim V' = \dim V + n$. If f is flat, Y is irreducible, and X has pure dimension equal to $\dim Y + n$, then f has relative dimension n, and all base extensions $X \times_Y Y' \to Y'$ have relative dimension n (cf. [H] III.9.6, [EGA] IV.14.2).

WE WANT TO AVOID THE FOLLOWING FENOMENA:



THIS IS A FLAT MAP WHOSE GENERIC FIBER IS A CURVE AND A POINT

NOW WE WANT TO DEFINE $f^* : Z_*(Y) \to Z_*(X)$. IF $V \subseteq Y$ OF DIM $= d$ THEN $f^{-1}(V)$ IS EQUIDIMENSIONAL OF DIMENSION $d+r$.

DEF.:

IN GENERAL, IF X A SCHEME AND $Z \subseteq X$ A SUBSCHEME OF PURE DIMENSION d THEN WE CAN DEFINE THE FUNDAMENTAL CLASS $[Z] \in Z_d(X)$. IF $V_1, \dots, V_r$ ARE IRREDUCIBLE COMPONENT OF Z THEN WE DEFINE $[Z] = \sum_i \ell(\mathcal{O}_{Z,V_i})[V_i]$

(ON A CURVE IF YOU HAVE SOMETHING LIKE $V(f) = Z$ IN X THEN $[Z] = \sum_{P \in Z} \mathrm{VAL}_P(f)$ $[P] = \sum_{P \in Z} \ell(\mathcal{O}_{X,P}/(f))[P] = \sum_{P \in Z} \ell(\mathcal{O}_{Z,P})[P]$)

COMPONENT

GENERALIZE THIS

DEF. $f^* V = [f^{-1}(V)] \in Z_{d+r}(X)$ THE FUNDAMENTAL CLASS

EXERCISE:

I) f^* IS FUNCTORIAL

II) f^* PRESERVE RATIONAL EQUIVALENCE $\rightsquigarrow f^* : CH_*(Y) \to CH_*(X)$

FULTON FOR HINTS