Projective Geometric Invariant Theory

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Introduction

We extend the theory of affine GIT developed in the previous talks to construct GIT quotients for reductive group actions on projective schemes. The idea is that we would like construct our GIT quotient by gluing affine GIT quotients. In order to do this we would like to cover our scheme X by affine open subsets which are invariant under the group action and glue the affine GIT quotients of these affine open subsets of X. However, it may not be possible to cover all of X by such compatible open invariant affine subsets.

For a projective scheme X with an action of a reductive group G, there is not a canonical way to produce an open subset of X which is covered by open invariant affine subsets. Instead, this will depend on a choice of an equivariant projective embedding $X \hookrightarrow \mathbb{P}^n$, where G acts on $\mathbb{P}^n$ by a linear representation $G \to \mathrm{GL}_{n+1}$. Firstly we will define GIT quotient assu,ing to have one of those equivariant embedding and the we will traslate the theory to more genereal setting via the linearisation of the G-action on X. Again our GIT quotient will be a good quotient (for the semi-stable points) and geometric quotient if restricted to the stable ones.

1 Construction of the projective GIT quotient.

Definition 1.1. *Let X be a projective scheme with an action of an affine algebraic group G. A **linear G-equivariant projective embedding of** X is a group homomorphism $G \to \mathrm{GL}_{n+1}$ and a G-equivariant projective embedding $X \hookrightarrow \mathbb{P}^n$. We will often simply say that the G-action on $X \hookrightarrow \mathbb{P}^n$ is **linear** to mean that we have a linear G-equivariant projective embedding of X as above.*

Suppose we have a linear action of a reductive group G on a projective scheme $X \subset \mathbb{P}^n$. Then the action of G on $\mathbb{P}^n$ lifts to an action of G on the affine cone $\mathbb{A}^{n+1}$ over $\mathbb{P}^n$. Since the projective embedding $X \subset \mathbb{P}^n$ is G-equivariant, there is an induced action of G on the affine cone $\tilde{X} \subset \mathbb{A}^{n+1}$ over $X \subset \mathbb{P}^n$. More precisely, we have

$$\mathcal{O}\left(\mathbb{A}^{n+1}\right)=k\left[x_0,\ldots,x_n\right]=\bigoplus_{r\geq 0} k\left[x_0,\ldots,x_n\right]_r=\bigoplus_{r\geq 0} H^0\left(\mathbb{P}^n,\mathcal{O}_X(r)\right)$$

and if $X\subset\mathbb{P}^n$ is the closed subscheme associated to a homogeneous ideal $I(X)\subset k\left[x_0,\ldots,x_n\right]$, then $\tilde{X}=\operatorname{Spec} R(X)$ where $R(X)=k\left[x_0,\ldots,x_n\right]/I(X)$.

The k-algebras $\mathcal{O}\left(\mathbb{A}^{n+1}\right)$ and $R(X)$ are graded by homogeneous degree and, as the G-action on $\mathbb{A}^{n+1}$ is linear it preserves the graded pieces, so that the invariant subalgebra

$$\mathcal{O}\left(\mathbb{A}^{n+1}\right)^G=\bigoplus_{r\geq 0} k\left[x_0,\ldots,x_n\right]_r^G$$

is a graded algebra and, similarly, $R(X)^G=\oplus_{r\geq 0}R(X)_r^G$. By Nagata's theorem, $R(X)^G$ is finitely generated, as G is reductive. The inclusion of finitely generated graded k-algebras $R(X)^G\hookrightarrow R(X)$ determines a rational morphism of projective schemes

$$X\dashrightarrow\operatorname{Proj} R(X)^G$$

whose indeterminacy locus is the closed subscheme of X defined by the homogeneous ideal $R(X)_+^G:=\oplus_{r>0}R(X)_r^G$.

Definition 1.2. *For a linear action of a reductive group G on a projective scheme $X\subset\mathbb{P}^n$, we define the* ***nullcone*** *N to be the closed subscheme of X defined by the homogeneous ideal $R(X)_+^G$ in $R(X)$ (strictly speaking the nullcone is really the affine cone $\tilde{N}$ over N). We define the* ***semistable set*** *$X^{ss}=X-N$ to be the open subset of X given by the complement to the nullcone. More precisely, $x\in X$* ***is semistable*** *if there exists a G-invariant homogeneous function $f\in R(X)_r^G$ for $r>0$ such that $f(x)\neq 0$. By construction, the semistable set is the open subset which is the domain of definition of the rational map*

$$X\dashrightarrow\operatorname{Proj} R(X)^G$$

We call the morphisms $X^{ss}\to X/\!\!/G:=\operatorname{Proj} R(X)^G$ the GIT quotient of this action.

Theorem 1.3. *For a linear action of a reductive group G on a projective scheme $X\subset\mathbb{P}^n$, the GIT quotient $\varphi: X^{ss}\to X/\!\!/G$ is a good quotient of the G-action on the open subset X^{ss} of semistable points in X. Furthermore, $X/\!\!/G$ is a projective scheme.*

Proof. Let $\varphi: X^{ss}\to Y:=X/\!\!/G$ denote the projective GIT quotient. In order to prove that is a good quotient we want to use the fact that this propriety is local on the target!
For $f\in R_+^G$, the open affine subsets $Y_f\subset Y$ form a basis of the open sets on Y. Since $f\in R(X)_+^G\subset R(X)_+$, we can also consider the open affine subset $X_f\subset X$ and, by construction of φ, we have that $\varphi^{-1}\left(Y_f\right)=X_f$. Let $\tilde{X}_f$ (respectively $\tilde{Y}_f$) denote the affine cone over X_f (respectively Y_f). Then

$$\mathcal{O}\left(Y_f\right) \cong \mathcal{O}\left(\tilde{Y}_f\right)_0 \cong \left(\left(R(X)^G\right)_f\right)_0 \cong \left(\left(R(X)_f\right)_0\right)^G \cong \left(\mathcal{O}\left(\tilde{X}_f\right)_0\right)^G \cong \mathcal{O}\left(X_f\right)^G$$

and so the corresponding morphism of affine schemes $\varphi_f : X_f \to Y_f \cong \operatorname{Spec} \mathcal{O}\left(X_f\right)^G$ is an affine GIT quotient, and so also a good quotient. The morphism $\varphi : X^{ss} \to Y$ is obtained by gluing the good quotients $\varphi_f : X_f \to Y_f$. $\square$

1.1 Stable points

We recall that as $\varphi : X^{ss} \to X/\!\!/G$ is a good quotient, for two semistable points x_1, x_2 in X^{ss}, we have

$$\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \cap X^{ss} \neq \emptyset \Longleftrightarrow \varphi\left(x_1\right) = \varphi\left(x_2\right).$$

Furthermore, the preimage of each point in $X/\!\!/G$ contains a unique closed orbit. The presence of non-closed orbits in the semistable locus will prevent the good quotient $\varphi : X^{ss} \to X/\!\!/G$ from being a geometric quotient. So using the same idea used for the affine case we define the so called stable points:

Definition 1.4. *Consider a linear action of a reductive group G on a closed subscheme $X \subset \mathbb{P}^n$. Then a point $x \in X$ can be:*

- ***stable*** *if $\dim G_x = 0$ and there is a G-invariant homogeneous polynomial $f \in R(X)_+^G$ such that $x \in X_f$ and the action of G on X_f is closed.*
- ***unstable*** *if it is not semistable.*

Notice that the set of not stable points is contained in the set of unstable points but isn't equal.

Proposition 1.5. *The stable and semistable sets X^s and X^{ss} are open in X.*

Proof. X^{ss} is the complement of the nullcone that is closed. X^{ss} is defined by $\dim G_x$ inside $X_c := \cup X_f$ where the union is taken over $f \in R(X)_+^G$ such that the action of G on X_f is closed, then is an open in an open. $\square$

Theorem 1.6. *For a linear action of a reductive group G on a closed subscheme $X \subset \mathbb{P}^n$, let $\varphi : X^{ss} \to Y := X/\!\!/G$ denote the GIT quotient. Then there is an open subscheme $Y^s \subset Y$ such that $\varphi^{-1}\left(Y^s\right) = X^s$ and that the GIT quotient restricts to a geometric quotient $\varphi : X^s \to Y^s$.*

Proof. Let Y_c be the union of Y_f for $f \in R(X)_+^G$ such that the G-action on X_f is closed and let X_c be the union of X_f over the same index set so that $X_c = \varphi^{-1}\left(Y_c\right)$. Then $\varphi : X_c \to Y_c$ is constructed by gluing $\varphi_f : X_f \to Y_f$ for $f \in R(X)_+^G$ such that the G-action on X_f is closed. Each φ_f is a good quotient and as the action on X_f is closed, φ_f is also a geometric quotient. Hence $\varphi : X_c \to Y_c$ is a geometric quotient, since it is a local propriety. Then one can show that this restrict to $\varphi : X^s \to Y^s$ that remain a geometric quotient. $\square$

Notice that if in the definition of stability we don't assume $\dim G_x = 0$ we have that the set of stable points is X_c that still open and restricted to this the quotient is geometric, so why we ask for the extra condition?
Well, in the original definition by Mumford this condition was omitted, but now is commonly used since it is better suited to moduli problems.

Example 1.7. *Consider the linear action of $G = \mathbb{G}_m$ on $X = \mathbb{P}^n$ by*

$$t \cdot [x_0 : x_1 : \cdots : x_n] = \left[t^{-1}x_0 : tx_1 : \cdots : tx_n\right]$$

In this case $R(X) = k[x_0, \ldots, x_n]$ which is graded into homogeneous pieces by degree. It is easy to see that the functions $x_0x_1, \ldots, x_0x_n$ are all G-invariant. In fact, we claim that these functions generate the ring of invariants. To prove the claim, suppose we have $f \in R(X)$; then

$$f = \sum_{\underline{m}=(m_0,\ldots,m_n)} a(\underline{m}) x_0^{m_0} x_1^{m_1} \ldots x_n^{m_n}$$

and, for $t \in \mathbb{G}_m$,

$$t \cdot f = \sum_{\underline{m}=(m_0,\ldots,m_n)} a(\underline{m}) t^{m_0 - m_1 - \cdots - m_n} x_0^{m_0} x_1^{m_1} \ldots x_n^{m_n}$$

Then f is G-invariant if and only if $a(\underline{m}) = 0$ for all $\underline{m} = (m_0, \ldots, m_n)$ such that $m_0 \neq \sum_{i=1}^n m_i$. If m satisfies $m_0 = \sum_{i=1}^n m_i$, then

$$x_0^{m_0} x_1^{m_1} \ldots x_n^{m_n} = (x_0x_1)^{m_1} \ldots (x_0x_n)^{m_n}$$

that is, if f is G-invariant, then $f \in k[x_0x_1, \ldots x_0x_n]$. Hence

$$R(X)^G = k[x_0x_1, \ldots, x_0x_n] \cong k[y_0, \ldots, y_{n-1}]$$

and after taking the projective spectrum we obtain the projective variety $X /\!\!/ G = \mathbb{P}^{n-1}$. The explicit choice of generators for $R(X)^G$ allows us to write down the rational morphism

$$\varphi : X = \mathbb{P}^n \dashrightarrow X /\!\!/ G = \mathbb{P}^{n-1}$$
$$[x_0 : x_1 : \cdots : x_n] \mapsto [x_0x_1 : \cdots : x_0x_n]$$

and its clear from this description that the nullcone

$$N = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : x_0 = 0 \ \text{or} \ (x_1, \cdots, x_n) = 0\}$$

is the projective variety defined by the homogeneous ideal $I = (x_0x_1, \cdots, x_0x_n)$. In particular,

$$X^{ss} = \bigcup_{i=1}^n X_{x_0x_i} = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : x_0 \neq 0 \ \text{and} \ (x_1, \ldots, x_n) \neq 0\} \cong \mathbb{A}^n - \{0\}$$

where we are identifying the affine chart on which $x_0 \neq 0$ in $\mathbb{P}^n$ with $\mathbb{A}^n$. Therefore

$$\varphi : X^{ss} = \mathbb{A}^n - \{0\} \longrightarrow X /\!\!/ G = \mathbb{P}^{n-1}$$

is a good quotient of the action on X^{ss}. As the preimage of each point in $X/\!\!/G$ is a single orbit, this is also a geometric quotient. Moreover, every semistable point is stable as all orbits are closed in $\mathbb{A}^n - \{0\}$ and have zero dimensional stabilisers.

1.2 A description of the k-points of the GIT quotient.

In general it can be difficult to determine which points are semistable and stable as it is necessary to have a description of the graded k-algebra of invariant functions. For this reason we introduce the following notion:

Definition 1.8. *Let G be a reductive group acting linearly on $X \subset \mathbb{P}^n$. A k-point $x \in X(k)$ is said to be* ***polystable*** *if it is semistable and its orbit is closed in X^{ss}. We say two semistable k-points are* **S-*equivalent*** *if their orbit closures meet in X^{ss}. We write this equivalence relation on $X^{ss}(k)$ as $\sim_{\text{S-equiv.}}$ and let $X^{ss}(k)/\sim_{\text{S-equiv.}}$ denote the S-equivalence classes of semistable k-points.*

How they relate with stable points? The following lemma tell us that all the stable points are polystable:

Lemma 1.9. *Let G be a reductive group acting linearly on $X \subset \mathbb{P}^n$. A k-point $x \in X(k)$ is stable if and only if x is semistable and its orbit $G \cdot x$ is closed in X^{ss} and its stabiliser G_x is zero dimensional.*

Idea: Suppose x is stable and $x' \in \overline{G \cdot x} \cap X^{\mathrm{ss}}$; then $\varphi(x') = \varphi(x)$ and so $x' \in \varphi^{-1}(\varphi(x)) \subset \varphi^{-1}(Y^s) = X^s$. As G acts on X^s with zero-dimensional stabiliser, this action must be closed as the boundary of an orbit is a union of orbits of strictly lower dimension. Therefore, $x' \in G \cdot x$ and so the orbit $G \cdot x$ is closed in X^{ss}.

Lemma 1.10. *Let G be a reductive group acting linearly on $X \subset \mathbb{P}^n$ and let $x \in X(k)$ be a semistable k-point; then its orbit closure $\overline{G \cdot x}$ contains a unique polystable orbit. Moreover, if x is semistable but not stable, then this unique polystable orbit is also not stable.*

Proof. The first statement follows from Corollary 3.32: φ is constant on orbit closures and the preimage of a k-point under φ contains a orbit which is closed in X^{ss}; this is the polystable orbit. For the second statement we note that if a semistable orbit $G \cdot x$ is not closed, then the unique closed orbit in $\overline{G \cdot x}$ has dimension strictly less than $G \cdot x$ by Proposition 3.15 and so cannot be stable. □

Corollary 1.11. *Let G be a reductive group acting linearly on $X \subset \mathbb{P}^n$. For two semistable points $x, x' \in X^{ss}$, we have $\varphi(x) = \varphi(x')$ if and only if x and x' are S-equivalent. Moreover, there is a bijection of sets*

$$X /\!\!/ G(k) \cong X^{ps}(k)/G(k) \cong X^{ss}(k)/\sim_{S\text{-}equiv.}$$

where $X^{ps}(k)$ is the set of polystable k-points.

2 Linearisations

2.1 Remarks on ample line bundle

An abstract projective scheme X does not come with a prespecified embedding in a projective space. However, an ample line bundle L on X (or more precisely some power of L) determines an embedding of X into a projective space. More precisely, the projective scheme X and ample line bundle L, determine a finitely generated graded k-algebra

$$R(X, L) := \bigoplus_{r \geq 0} H^0\left(X, L^{\otimes r}\right)$$

We can choose generators of this k-algebra: $s_i \in H^0\left(X, L^{\otimes r_i}\right)$ for $i = 0, \ldots n$, where $r_i \geq 1$. Then these sections determine a closed immersion

$$X \hookrightarrow \mathbb{P}\left(r_0, \ldots, r_n\right)$$

into a weighted projective space, by evaluating each point of X at the sections s_i. In fact, if we replace L by $L^{\otimes m}$ for m sufficiently large, then we can assume that the generators s_i of the finitely generated k-algebra

$$R\left(X, L^{\otimes m}\right) = \bigoplus_{r \geq 0} H^0\left(X, L^{\otimes m r}\right)$$

all lie in degree 1. In this case, the sections s_i of the line bundle $L^{\otimes m}$ determine a closed immersion

$$X \hookrightarrow \mathbb{P}^n = \mathbb{P}\left(H^0\left(X, L^{\otimes m}\right)^*\right)$$

given by evaluation $x \mapsto (s \mapsto s(x))$.

2.2 Linearisation of line bundles

Idea: If we have an affine algebraic group G acting on X, then we would like to lift the G-action to L linear on the fibers in a way that the bundle map becomes equivariant. This idea is made precise by the notion of a linearisation:

Definition 2.1. *Let X be a scheme and G be an affine algebraic group acting on X via a morphism $\sigma : G \times X \to X$. Then a **linearisation of the G-action on X** is a line bundle $\pi : L \to X$ over X with an isomorphism of line bundles*

$$\pi_X^* L = G \times L \cong \sigma^* L$$

where $\pi_X : G \times X \to X$ is the projection, such that the induced bundle homomorphism $\tilde{\sigma} : G \times L \to L$ defined by

$$\begin{tikzcd}
G \times X \arrow[dr, "\cong"] \arrow[drr, bend left, "\tilde{\sigma}"] \arrow[ddr, bend right, "id_G \times \pi"'] & & \\
& \sigma^* L \arrow[r] \arrow[d] & L \arrow[d, "\pi"] \\
& G \times X \arrow[r, "\sigma"'] & X
\end{tikzcd}$$

induces an action of G on L; that is, we have a commutative square of bundle homomorphisms

$$\begin{tikzcd}
G \times G \times L \arrow[r, "id_G \times \tilde{\sigma}"] \arrow[d, "\mu_G \times id_L"'] & G \times L \arrow[d, "\tilde{\sigma}"] \\
G \times L \arrow[r, "\tilde{\sigma}"'] & L
\end{tikzcd}$$

We say that a linearisation is (very) ample if the underlying line bundle is (very) ample.

Remark 2.2. *Notice that since the action on the line bundle $\tilde{\sigma} : G \times L \to L$ is a morphism of vector bundles we have that:*

- *the projection $\pi : L \to X$ is G-equivariant*
- *the action of G on the fibres of L is linear: for $g \in G$ and $x \in X$, the map on the fibres $L_x \to L_{g \cdot x}$ is linear.*

Moreover, we are developing the theory using the notion of vector bundle but we can do all the same procedure by talking about invertible sheafs.

Notice that this notion of linearization is just a generalization of the concept of G acting linearly on $X \subset \mathbb{P}^n$:

Remark 2.3. *Suppose that X is a projective scheme and L is a very ample linearisation. Then the natural evaluation map*

$$H^0(X, L) \otimes_k \mathcal{O}_X \to L$$

is G-equivariant. Moreover, this evaluation map induces a G-equivariant closed embedding

$$X \hookrightarrow \mathbb{P}\left(H^0(X, L)^*\right)$$

such that L is isomorphic to the pullback of the Serre twisting sheaf $\mathcal{O}(1)$ on this projective space. In this case, we see that we have an embedding of X as a closed subscheme of a projective space $\mathbb{P}(H^0(X, L)^*)$ such that the action of G on X comes from a linear action of G on $H^0(X, L)^*$.

2.3 Examples

We start with a really trivial bundle:

Example 2.4. *Let us consider $X = \operatorname{Spec} k$ with necessarily the trivial G-action. Then there is only one line bundle $\pi : \mathbb{A}^1 \to \operatorname{Spec} k$ over $\operatorname{Spec} k$, but there are many linearisations. In fact, the group of linearisations of X is the character group of G. If $\chi : G \to \mathbb{G}_m$ is a character of G, then we define an action of G on $\mathbb{A}^1$ by acting by $G \times \mathbb{A}^1 \to \mathbb{A}^1$. Conversely, any linearisation is given by a linear action of G on $\mathbb{A}^1$; that is, by a group homomorphism $\chi : G \to \mathrm{GL}_1 = \mathbb{G}_m$.*

In a similar way we can twist any linearization:

Example 2.5. *For any scheme X with an action of an affine algebraic group G and any character $\chi : G \to \mathbb{G}_m$, we can construct a linearisation on the trivial line bundle $X \times \mathbb{A}^1 \to X$ by*

$$g \cdot (x, z) = (g \cdot x, \chi(g)z)$$

More generally, for any linearisation $\tilde{\sigma}$ on $L \to X$, we can twist the linearisation by a character $\chi : G \to \mathbb{G}_m$ to obtain a linearisation $\tilde{\sigma}^\chi$.

Not every linearisation on a trivial line bundle comes from a character:

Example 2.6. *Consider $G = \mu_2 = \{\pm 1\}$ acting on $X = \mathbb{A}^1 - \{0\}$ by $(-1) \cdot x = x^{-1}$. Then the linearisation on $X \times \mathbb{A}^1 \to X$ given by $(-1) \cdot (x, z) = (x^{-1}, xz)$ is not isomorphic to a linearisation given by a character, as over the fixed points +1 and -1 in X, the action of $-1 \in \mu_2$ on the fibres is given by $z \mapsto z$ and $z \mapsto -z$ respectively.*

Lastly, probably the most useful example of action:

Example 2.7. *The natural actions of* GL_{n+1} *and* SL_{n+1} *on* $\mathbb{P}^n$ *inherited from the action of* GL_{n+1} *on* $\mathbb{A}^{n+1}$ *by matrix multiplication can be naturally linearised on the line bundle* $\mathcal{O}_{\mathbb{P}^n}(1)$*. To see why, we note that the trivial rank* $n+1$*-vector bundle on* $\mathbb{P}^n$ *has a natural linearisation of* GL_{n+1} *(and also* SL_{n+1} *). The tautological line bundle* $\mathcal{O}_{\mathbb{P}^n}(-1) \subset \mathbb{P}^n \times \mathbb{A}^{n+1}$ *is preserved by this action and so we obtain natural linearisations of these actions on* $\mathcal{O}_{\mathbb{P}^n}(\pm 1)$*. However, the action of* PGL_{n+1} *on* $\mathbb{P}^n$ *does not admit a linearisation on* $\mathcal{O}_{\mathbb{P}^n}(1)$ *(see Exercise Sheet 9), but we can always linearise any* G*-action on* $\mathbb{P}^n$ *to* $\mathcal{O}_{\mathbb{P}^n}(n+1)$ *as this is isomorphic to the nth exterior power of the cotangent bundle, and we can lift any action on* $\mathbb{P}^n$ *to its cotangent bundle via differentials.*

2.4 Projective GIT with respect to an ample linearisation

Let G be a reductive group acting on a projective scheme X and let L be an ample linearisation of the G-action on X. Then consider the graded finitely generated k-algebra

$$R := R(X, L) := \bigoplus_{r \geq 0} H^0\left(X, L^{\otimes^r}\right)$$

of sections of powers of L. Since each line bundle $L^{\otimes r}$ has an induced linearisation, there is an induced action of G on the space of sections $H^0\left(X, L^{\otimes^r}\right)$ (intuitively: since G acts on X and on L in an equivariant way, it acts on the global sections). We consider the graded algebra of G-invariant sections

$$R^G = \bigoplus_{r \geq 0} H^0\left(X, L^{\otimes^r}\right)^G$$

The subalgebra of invariant sections R^G is a finitely generated k-algebra and $\operatorname{Proj} R^G$ is projective over $R_0^G = k^G = k$ following a similar argument to above.

Definition 2.8. *For a reductive group* G *acting on a projective scheme* X *with respect to an ample line bundle, we make the following definitions.*

1. ***A point*** $\mathbf{x} \in \mathbf{X}$ ***is semistable*** *with respect to* L *if there is an invariant section* $\sigma \in H^0\left(X, L^{\otimes^r}\right)^G$ *for some* $r > 0$ *such that* $\sigma(x) \neq 0$*.*

2. ***A point*** $\mathbf{x} \in \mathbf{X}$ ***is stable*** *with respect to* L *if* $\dim G \cdot x = \dim G$ *and there is an invariant section* $\sigma \in H^0\left(X, L^{\otimes^r}\right)^G$ *for some* $r > 0$ *such that* $\sigma(x) \neq 0$ *and the action of* G *on* $X_\sigma := \{x \in X : \sigma(x) \neq 0\}$ *is closed.*

We let $X^{ss}(L)$ and $X^s(L)$ denote the open subset of semistable and stable points in X respectively. Then we define the projective GIT quotient with respect to L to be the morphism

$$X^{ss} \to X/_L G := \operatorname{Proj} R(X, L)^G$$

associated to the inclusion $R(X,L)^G \hookrightarrow R(X,L)$.

Example 2.9. *We have already defined notions of semistability and stability when we have a linear action of G on $X \subset \mathbb{P}^n$. In this case, the action can naturally be linearised using the line bundle $\mathcal{O}_{\mathbb{P}^p}(1)$. Can be checked that the two notions of semistability agree; that is,*

$$X^{(s)s} = X^{(s)s}\left(\mathcal{O}_{\mathbb{P}^n}(1)|_X\right)$$

Theorem 2.10. *Let G be a reductive group acting on a projective scheme X and L be an ample linearisation of this action. Then the GIT quotient*

$$\varphi : X^{ss}(L) \to X /\!\!/ LG = \operatorname{Proj} \bigoplus_{r \geq 0} H^0\left(X, L^{\otimes r}\right)^G$$

is a good quotient and X/L_LG is a projective scheme with a natural ample line bundle L' such that $\varphi^* L' = L^{\otimes n}$ for some $n > 0$. Furthermore, there is an open subset $Y^s \subset X /\!\!/ LG$ such that $\varphi^{-1}(Y^s) = X^s(L)$ and $\varphi : X^s(L) \to Y^s$ is a geometric quotient for the G-action on $X^s(L)$.

The proof is besically the same seen previusly but instead of using X_f we use the open sets X_σ for each $\sigma \in R(X,L)_+^G$.

Remark 2.11. *For an ample linearisation L, we know that some positive power of L is very ample. By definition $X^{ss}(L) = X^{ss}(L^{\otimes n})$ and $X^s(L) = X^s(L^{\otimes n})$ and $X /\!\!/_L G \cong X /\!\!/ L^{\otimes n} G$ (as abstract projective schemes), we can assume without loss of generality that L is very ample and so $X \subset \mathbb{P}^n$ and G acts linearly. However, we note that the induced ample line bundles on $X /\!\!/_L G$ and $X /\!\!/_{L^{\otimes n}} G$ are different, and so these GIT quotients come with different embeddings into (weighted) projective spaces.*

We still have the notion of polystability, and a result similar to the privious case holds for the k-points:

Definition 2.12. *We say two semistable k-points x and x' in X are **S-equivalent** if the orbit closures of x and x' meet in the semistable subset $X^{ss}(L)$. We say a semistable k-point is **polystable** if its orbit is closed in the semistable locus $X^{ss}(L)$.*

Corollary 2.13. *Let x and x' be k-points in $X^{ss}(L)$; then $\varphi(x) = \varphi(x')$ if and only if x and x' are S-equivalent. Moreover, we have a bijection of sets*

$$(X /\!\!/_L G)(k) \cong X^{ps}(L)(k)/G(k) \cong X^{ss}(L)(k)/\sim_{S\text{-equiv.}}$$

where $X^{ps}(L)(k)$ is the set of polystable k-points.

Everythin can be extendent to the case of reductive group actions on quasiprojective schemes with respect to (not necessarily ample) linearisations via the folloqing definition:

Definition 2.14. *Let X be a quasi-projective scheme with an action by a reductive group G and L be a linearisation of this action.*

1. ***A point*** $\mathbf{x} \in \mathbf{X}$ ***is semistable*** *with respect to L if there is an invariant section $\sigma \in H^0\left(X, L^{\otimes r}\right)^G$ for some $r > 0$ such that $\sigma(x) \neq 0$ and $X_\sigma = \{x \in X : \sigma(x) \neq 0\}$ is affine.*
2. ***A point*** $\mathbf{x} \in \mathbf{X}$ ***is stable*** *with respect to L if $\dim G \cdot x = \dim G$ and there is an invariant section $\sigma \in H^0\left(X, L^{\otimes r}\right)^G$ for some $r > 0$ such that $\sigma(x) \neq 0$ and X_σ is affine and the action of G on X_σ is closed.*

The open subsets of stable and semistable points with respect to L are denoted $X^s(L)$ and $X^{ss}(L)$ respectively.

References

[Hos15] Victoria Hoskins. Moduli problems and Geometric Invariant Theory. *Lecture notes*, 2015.