AG for NT First Week 1

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Introduction

What is the aim of this study group?

To have a better understanding of basic Algebraic Geometry. To have developed an intuition for the idea behind Algebraic Geometry

Who is this study group for?

As often quoted "A study group is on subject the organiser knows nothing about and wish to know more". This is the case here. This study group was form due to a group of Number Theorist attending one of the many Number Theory conference at Warwick last year and realising that there was a fair bit of Algebraic Geometry of which we knew nothing. This study group will help to motivate us all to learn Algebraic Geometry, with examples mainly from Number Theory but hopefully broad enough to interest everyone.

Sheaves

Zariski Topology

Let A be a commutative ring with 1 (and not the zero ring). Define $\mathrm{Spec}(A) = \{\text{proper prime ideals } p \subsetneq A\}$. For any ideal I of A let $V(I) := \{p \in \mathrm{Spec}(A) | I \subseteq p\}$, if $f \in A$, let $D(f) := \mathrm{Spec}(A) \setminus V(\langle f \rangle)$

Proposition. *The following holds:*

- $V(I) \cup V(J) = V(I \cap J)$
- $\cap_\lambda V(I_\lambda) = V(\sum I_\lambda)$
- $V(A) = \emptyset$ *and* $V(0) = Spec(A)$

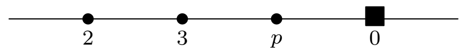
Proof. Exercise □

We define the *Zariski Topology* on $\mathrm{Spec}(A)$ to be defined as its closed sets $V(I)$. $D(f)$ is called a *principal open subset* (and forms a bases of open subset on $\mathrm{Spec}(A)$) while $V(f)$ is called a *principal closed subset.*

Note. Let $p \in \mathrm{Spec}(A)$. Then $\{p\}$ is closed if and only if p is a maximal ideal. (If it is maximal then $\{p\} = V(p)$, if it is not maximal then $\{p\} \subsetneq V(p)$, so not closed)

Definition. In the above context, p is called a *closed point*

Example. Consider Spec($\mathbb{Z}$).



We see that $\langle p \rangle$ where p is prime are closed points, while is the point $\{0\} =: \zeta$ is non-closed. In fact $\overline{\zeta} = \text{Spec}(\mathbb{Z})$, and so we call ζ the *generic point* (of Spec($\mathbb{Z}$))

Example. Let k be a field and let $^1_k := \text{Spec}(k[x])$ be the affine line over k. Again we have a point corresponding to the $\{0\}$ ideal, which we call the *generic point* (of ^1_k). The other closed points correspond to the maximal ideal of $k[x]$ which correspond to the monic irreducible polynomials of $k[x]$. (In particular if $k = \overline{k}$ is algebraic closed, then the closed points corresponds to elements of k)

We show that the proper closed set correspond to finite sets (of points). Let I be an ideal and let $p(x)$ be the polynomial generating it. We can write $p(x) = \prod_{i=1}^{n} p_i(x)$ into irreducible factors. Then $V(I) = \{p_1(x), \dots, p_n(x)\}$. (This should be look like the definition of the Zariski topology on $\mathbb{R}$ you meant in a metric space course)

Example. Let $k = \overline{k}$ and $^2_k := \text{Spec}(k[x, y])$ be the affine plane. Again we have the generic point (of ^2_k) ζ which correspond to the $\{0\}$ ideal ($\overline{\zeta} = {}^2_k$). We have closed points which are in correspondence to pairs of elements of k. ($V(\langle x - a, y - b \rangle)$ as $\langle x - a, y - b \rangle$ is maximal). Given an irreducible polynomial $f(x, y)$, there is a point η whose closure is η and all the closed points (a, b) for which $f(a, b) = 0$. Namely η correspond to the ideal $\langle f(x, y) \rangle$ as $\langle f(x, y) \rangle \subset \langle x - a, y - b \rangle$ where a, b are such that $f(a, b) = 0$. (To see this, by the division algorithm, on any ordering you want, we have $f(x, y) = h_1(x - a) + h_2(y - b) + r$, where h_1, h_2, r are polynomials. Furthermore r must be constant, either because we can put any x term in $(x - a)$ and y terms in $(y - b)$ or because the leading term of r must be less than the leading term of $(x - a)$ and $(y - b)$. Substituting $x = a$ and $y = b$, we see that $r = 0$). We say that η is the *generic point of the curve* $f(x, y) = 0$

See Hartshorne's picture on page 75.

Sheaves

Definition. Let X be a topological space. A presheaf $\mathcal{F}$ (of abelian groups) on X consist of the following data:

- An abelian group $F(U)$ for every open subset U of X
- For every pair of open subset $V \subseteq U$ a group homomorphism (called the *restriction map*) $\rho_{UV} : F(U) \to F(V)$

such that

1. $F(\emptyset) = \{1\}$
2. $\rho_{UU} = \text{id}$
3. $W \subseteq V \subseteq U$ then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$

Note. (For those who know Category Theory, otherwise ignore this). A presheaf is just a contravariant function from the category of Topological spaces (with morphisms being the inclusion map) to the category of Abelian groups.

Notation. An element $s \in F(U)$ is called a *section* of F over U. $s|_V$ denotes the element $\rho_{UV}(s) \in F(V)$ and is called the *restriction* of s to V

Definition. A presheaf $\mathcal{F}$ is a *sheaf* if we have the following properties:

1. Uniqueness: Let U be an open subset of X, $s \in F(U)$ a section and $\{U_i\}_i$ a covering/refinement of U. If $s|_{U_i} = 0 \forall i$ then $s = 0$

2. Gluing local sections (or Gluability): Using the same notation as above. Let $s_i \in F(U_i)$ for all i be sections such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. Then there exists $s \in F(U)$ such that $s|_{U_i} = s_i$. Note that by 4. this is unique.

Definition. We say $\mathcal{F}$ is a *subsheaf* of $\mathcal{F}$ if $F'(U)$ is a subgroup of $F(U)$ for all U and ρ'_{UV} is induced from ρ_{UV}

Example. Let X be a topological space and k a field. For any open subset U of X, let $C(U) = C^0(U, k)$ be the set of continuous functions from U to k. The restrictions ρ_{UV} are the usual restrictions of functions. Then $\mathcal{C}$ is a sheaf on X. $(1-3$ obvious, $4-5$ follows from properties of continuous functions)

Note. Every (pre)sheaf $\mathcal{F}$ on X induces a (pre)sheaf $F|_U$ on $U \subseteq X$ (by setting $F|_U(V) = F(V)$ for all $V \subseteq U$)

Note. We show that a sheaf is completely determined by its sections over a basis of open sets:

Let $\mathcal{B}$ be a basis of open subsets on X. Define $\mathcal{B}$-(pre)sheaf by replacing '$U \subseteq X$ open' by '$U \in \mathcal{B}$'. Then we can extend a $\mathcal{B}$-sheaf $\mathcal{F}_0$ to a sheaf $\mathcal{F}$ on X since foe any $U \subseteq X$ can be written as $\cup_i U_i$ with $U_i \in \mathcal{B}$. So $F(U)$ is the set of elements $(s_i)_i \in \prod_i \mathcal{F}_0(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$

Definition. Let $\mathcal{F}$ be a presheaf on X and let $x \in X$. The *stalk* of $\mathcal{F}$ at x is the group

$$F_x := \varinjlim_{x \in U} F(U)$$

Let $s \in F(U)$ be a section. For any $x \in U$ we denote the image of s in F_x by s_x. We call s_x the *germ* of s at x. The map $F(U) \to F_x$ defined by $s \mapsto s_x$ is a group homomorphism.

Example. Back to the sheaf $\mathcal{C}$ on X. Then C_x is the set of continuous functions at x. Another way to see the stalk, is that elements in F_x are represented by a pair (U, f) where U is an open subset of X containing x and $f \in F(U)$, up to the equivalence that $(U, f) \sim (V, g)$ if $f|_{U \cap V} = g|_{U \cap V}$.

Lemma. *Let $\mathcal{F}$ be a sheaf on X. Let $x, t \in F(X)$ be sections such that $s_x = t_x \forall x \in X$. Then $s = t$, i.e., sections are determined by their germs.*

Proof. WLOG assume $t = 0$. For all $x \in X$, there exists open U_x of x such that $s|_{U_x} = 0$, since $s_x = 0$. As U_x cover X as x varies, we have (by gluability) $s = 0$ □

Definition. Let $\mathcal{F}$ and $\mathcal{G}$ be two presheaves on X. A *morphism* of presheaf $\alpha : \mathcal{F} \to \mathcal{G}$ consist of group morphism $\alpha(U) : F(U) \to G(U)$ for all open $U \subseteq X$, which is compatible with the restriction ρ_{UV}. That is the following diagram commutes

$$\begin{array}{ccc} F(U) & \xrightarrow{\alpha(U)} & G(U) \\ \downarrow{\scriptstyle \rho_{UV}} & & \downarrow{\scriptstyle \rho'_{UV}} \\ F(V) & \xrightarrow[\alpha(V)]{} & G(V) \end{array}$$

α is *injective* if every $\alpha(U)$ is injective.

An *isomorphism* is an invertible morphism α, i.e., $\alpha(U)$ is an isomorphism for all U

For any $x \in X$, α induces a group homomorphism $\alpha_x : F_x \to G_x$ such that $(\alpha(U)(s))_x = \alpha_x(s_x)$. We say that α is *surjective* if α_x is surjective for all $x \in X$

Example. We can define a morphism between the sheaf of differential functions to the sheaf of continuous function (by forgetting we are differentiable)

Proposition. *Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Then α is an isomorphism if and only if α_x is an isomorphism for every $x \in X$.*

Proof. $\Rightarrow$) Clear

$\Leftarrow$) Let $s \in F(U)$ be a section. If $\alpha(U)(s) = 0$, then for every $x \in U$, we have $\alpha_x(s_x) = (\alpha(U)(s))_x = 0$. As α_x is an isomorphism it follows that $s_x = 0$ for all $x \in X$. Hence $s = 0$

Let $t \in G(U)$. We can find a covering/refinement of U by open subset U_i and $s_i \in F(U_i)$ such that $\alpha(U_i)(s_i) = t|_{U_i}$. As α is injective, s_i and s_j coincide on $U_i \cap U_j$. Th s_i therefore glue to a section $s \in F(U)$ such that $s|_{U_i} = s_i$. By construction $\alpha(U)(s)$ and t coincide on every U_i and hence are equal. So $\alpha(U)$ is surjective $\square$

Similarly we can prove α is injective if and only if α_x is.

There is a method to construct a sheaf associated to a presheaf by preserving stalks (sometimes called sheafification) Such a construction is unique. This is a very technical point and I don't think we will use it in this study group, so I cover it today. See either Hartshorne or Qin-Liu book (pg 36)

Definition. Let $f : X \to Y$ be a continuous map of topological spaces. $\mathcal{F}$ a sheaf on X and $\mathcal{G}$ a sheaf on Y.

For every $V \subseteq Y$, $V \mapsto F(f^{-1}(V))$ defines a sheaf $f_*\mathcal{F}$ on Y which is called *the direct image* or *pushforward* of $\mathcal{F}$

We also have the inverse image of $\mathcal{G}$ denoted by $f^{-1}\mathcal{G}$ which is the sheaf associated to the presheaf $U \mapsto \varinjlim_{f(U)\subseteq V} G(V)$. This construction is complicated, but we have the nice property that $(f^{-1}\mathcal{G})_x = G_{f(x)}$

Note. When I defined the constant presheaf on the board, part of the definition which I forgot is that $F(\emptyset) = 0$ (which is a property of presheaf anyway)

Ringed Topological Spaces (Not covered in week 1)

Definition. A *ringed topological space* consist of a topological space X with a sheaf of rings $\mathcal{O}_X$ on X, such that $\mathcal{O}_{X,x}$ is a local ring for every $x \in X$. We denote it $(X, \mathcal{O}_X)$ or X if $\mathcal{O}_{\mathcal{X}}$is obvious.

Let m_x be the maximal ideal of $\mathcal{O}_{X,x}$, we call $\mathcal{O}_{X,x}/m_x$ the *residue field* of X at x and denote it $k(x)$

Example. Let $X = {}^n_k$ where k is a field.

For any open subset U, let $\mathcal{O}_X(U)$ be the set of regular functions on U, i.e., $f = \frac{g}{h}$ with g, h polynomials in $k[x_1, \ldots, x_n]$ and h non-zero on U

We show that $\mathcal{O}_{X,x}$ are local (and hence $(X, \mathcal{O}_X)$ is a ringed topological space). Let $x \in X$, then $\mathcal{O}_{X,x}$ can be identified with the regular functions defined on a neighbourhood of x, and let m_x be the set of regular functions which vanished on x. We see that $\mathcal{O}_{X,x}/m_x \cong k$ (since $\frac{g}{h} = \underbrace{\frac{f}{g} - \frac{f(x)}{g(x)}}_{\in m_x} + \frac{f(x)}{g(x)}$). It is the only maximal ideal, because if f does not vanish on x, then g does not vanish on x and $\frac{1}{f} = \frac{h}{g}$ is still regular. (So any ideal which contains f contains 1)