AG for NT 10

Goal:

To understand the theory behind blowdown for arithmetic surfaces, learn minimal model and maybe canonical model.

Give intuition and where hypothesis (normality, regularity)

Everything here will be Noetherian and of finite type

1 Prerequisites

1.1 Sheaves of differentials

Start with rings: $f : A \to B$, write $B = A[x_1, \dots, x_n]/(f_1, \dots f_m)$. Let $\Omega^1_{B/A} = \sum B dx_i / \langle df_j, da \rangle$ (for all $a \in A$ and where d is evaluated using Leibniz rule). This is a B-module.

Example. $A = k[y] = k[x^n] \hookrightarrow B = k[x]$, so $B = A[t]/(t^n - y)$. Then

$$\begin{aligned} \Omega^1_{B/A} &= Bdt/d(t^n - y) \\ &= Bdt/(nt^{n-1}dt - dy) \\ &= Bdt/nt^{n-1}dt \\ &= B/nt^{n-1} \end{aligned}$$

This B-module correspond to a sheaf on $\operatorname{Spec} B$ supported at 0 only. This is exactly where the map $x \mapsto x^n$ ramifies. Taking $B' = k[x, x^{-1}]$ then $\Omega^1_{B'/A} = 0$

The idea is that $\Omega^1_{B/A}$ detects smoothness and ramification.

Let $f : X \to Y$ be a morphism of schemes. Then this construction sheafifies and gives $\Omega^1_{X/Y}$, a sheaf on X.

Properties

- $f : X \to Y$ equidimensional fiber of dimension n and $x \in X$ a point. Then f is smooth at x if and only if $\Omega^1_{X/Y}$ is locally free of dimension n around x.

  So f is smooth if and only if $\Omega^1_{X/Y}$ is locally free of rank n. (Note that f is smooth if and only if the fibers of f are all smooth)

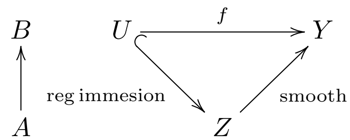
- $i : Z \hookrightarrow X$ a closed immersion with defining sheaf of ideal $\mathcal{I} \subset \mathcal{O}_X$. In general, there is a sequence

  $$0 \to \underbrace{i^*(\mathcal{I}/\mathcal{I}^2)}_{=:C_{X/Y} \text{ the conormal sheaf}} \to \Omega^1_{X/Y} \otimes_{\mathcal{O}_x} \mathcal{O}_Z \to \Omega^1_{Z/Y} \to 0$$

  X is smooth over Y, then Z is smooth if and only if this sequence is left exact.

1.2 Local complete intersections

$f : X \to Y$ morphism of schemes. f is a *local complete intersection* if it for every $x \in X$ there is a an open neighborhood $x \in U$ such that there exists Z and



Where a regular immersion is defined as follows: On rings this corresponds to $B = A/(x_1, \ldots, x_d)$ where x_i is not a zero divisor in $A/(x_1, \ldots, x_{i-1})$ for all $i \le d$. "Successive quotient by non-zero divisors".

Geometrically, this means that U is defined by a number of equations equal to its codimension in Z.

Intuition: $Y = \operatorname{Spec} k$, then Z is a smooth variety K (such as $\mathbb{A}^n_k$) and X is locally defined by the appropriate number of equations

Complete intersection: Same definition as above except with $U = X$. This is much more restrictive

Example. of a local complete intersection that is not a complete intersection is the twisted cubic, i.e, $\operatorname{Proj}(k[x, y, z, w]/\langle xz - y^2, y$
While this is a curve, it needs 3 equations to define it and not 2.

Example.

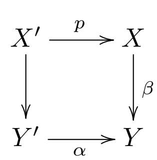
1. Curves over a field are local complete intersections except if they have embedded point. An example of non local complete intersection: $k[x, y]/(x^2, xy)$.
2. Let R be a Dedekind ring, $F \in R[x, y]$. Then $R[x, y]/\langle F\rangle$ is a local complete intersections.
3. $f : X \to Y$ morphism of regular schemes is an local complete intersection.

Definition 1.1. The *canonical sheaf* of a local complete intersection $X \to Y$ is $\omega_{X/Y} = \det\left(C^\vee_{X/Z}\right) \otimes_{\mathcal{O}_X} i^*\left(\det \Omega^1_{Z/Y}\right)$. This is locally free of rank 1.

Example. X is a curve smooth over $Y = \operatorname{Spec} k$, then $\omega_{X/Y} = \Omega^1_{X/k}$

Properties:

- $\omega_{X/Y}$ is stable under flat base change

  (either α or β is flat), then $\omega_{X'/Y} = p^*\omega_{X/Y}$

  Additionally $f : X \to Y$ is a local complete intersection if and only if it is so fiberwise
- For composition: $f : X \to Y$, $g : Y \to Z$. $\omega_{X/Z} = \omega_{X/Y} \otimes_{\mathcal{O}_X} f^*(\omega_{Y/Z})$. (This is the Riemann-Hurwitz formula in disguise)

The sheaf $\omega_{X/Y}$ gives Serre duality: $H^0(X, \omega_{X/Y}) = H^d(X, \mathcal{O}_X)^\vee$.

2 Arithmetic surfaces

Definition 2.1. A *fibered surface* is an integral surface X with a projective flat map $\pi : X \to S$ where S is a one dimension Dedekin scheme (like $\operatorname{Spec}\mathbb{Z}$).

See diagram on slides for intuition

Divisors on X comes in two flavors:

- Vertical one (components of special fibers X_S)
- Horizontal ones (closures of points in X_η).

Properties:

- X_η is geometrically integral (like a smooth curve), hence $\mathcal{O}_S \xrightarrow{\sim} \pi_*\mathcal{O}_X$
- If X_η is smooth, then there are only finitely many non-smooth fibers (proof: Smooth locus is open, use $\Omega^1_{X/S}$, and non-empty. Its complement is closed hence so is its image under π (proper), therefore this image is finite)
- $\omega_{X/S}|_{X_S} = \omega_{X_S|k(S)}$which follows from base change
- $p_s(X_S) = p_a(X_\eta)$, where p_a of a curve is $1 - \dim H^0(C, \mathcal{O}_C) + \dim H^1(C, \mathcal{O}_C)$ (and in the case the curve is smooth, we have $= \dim H^0(C, \omega_C) = g(C)$ the usual genus of C)

Definition 2.2. $\pi : X \to S$ is called *normal* if X is, and *regular* (or an *arithmetic surface*) if X is regular.

2.1 Desingularisation

Process of finding $Y \dashrightarrow X$ birational (isomorphism $Y_\eta \to X_\eta$) such that Y is regular.

If X_η is smooth, one can do:

$$\begin{array}{cl} X & \\ \uparrow & \\ X_1 & \text{normalisation of } X \\ \uparrow & \\ X_1' & \text{blowup of remaining sing in } X_1 \\ \uparrow & \\ X_2 & \\ \uparrow & \\ X_2' & \end{array}$$

Fact. *This stops and gives a regular surface at some points. After further blowup, all special fibers can be taken to have normal crossings.*

2.2 Contraction

$X \to S$ arithmetic surface, E component of special fibers X_S. We want typically to construct a morphism:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow & \downarrow \\ & & S \end{array}$$

contracting E means we want $f(E)$ to be a point and f is an isomorphism outside E.

This is done by using invertible sheaves. $\mathcal{L}$ an invertible sheaf on X: $H^0(X, \mathcal{L}) = R_{s_0} \oplus \cdots \oplus R_{s_n}$. This gives a morphism $f_{\mathcal{L}} : X \to \mathbb{P}^n$ defined by $x \mapsto (s_0(x) : \cdots : s_n(x))$. This is well defined as long as $\mathcal{L}_x = \sum_i s_i \mathcal{O}_{X,x}$, i.e., $\mathcal{L}$ should be generated by its global sections.

$f = f_{\mathcal{L}}$, suppose $Z \subset X_S$ is a projective component of fibers, then $f(Z)$ is a point if and only if $\mathcal{L}|_Z = \mathcal{O}_Z$.

$\Rightarrow$: suppose the point is $(1 : 0 : \cdots : 0)$, then by construction s_0 generates on all points above y.

$\Leftarrow$: Restricting to Z the space $H^0(Z, \mathcal{L})$ is finite (Z is projective), so we get $f|_Z : Z \to H^0(Z, \mathcal{L}) \to \mathbb{P}^n_A$.

Fact. *Birational maps $X \to Y$ between normal fibered surfaces are sequences $X \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \cdots \dashrightarrow Y$ (where the map is either a blowup of a point or a contraction of a curve to a point).*

Contraction Criteria

Let $\mathcal{E}$ be a set of vertical divisors: Contraction of $\mathcal{E}$ exists if and only if there exists a carties divisor D on X such that

- $\deg(D|_{X_\eta}) > 0$
- $\mathcal{O}_X(D)$ generated by global sections
- $\mathcal{O}_X(D)|_E \cong \mathcal{O}_E$ for E vertical if and only if $E \in \mathcal{E}$

Over affine S, for any effective horizontal Cartier divisors D, the sheaf $\mathcal{O}_X(nD)$ is generated by its global sections if $n \gg 0$.

$D + E$ may not be generated by global sections even if D and E are.

2.3 Intersection Theory

Let X be a fibered surface, D, E divisors on X. Suppose that D and E have no common component. D, E then intersect in finitely many points. Suppose $x \in X$ is a point of intersection, we set $i_x(D, E) = \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/(\mathcal{O}_{X,x}(-D) + \mathcal{O}_{X,x}(-E))$ and let $i(D, E) = \sum_{x \in X} i_x(D, E)$. (With the convention that if x is not an intersection point the $i_x(D, E) = 0$)

Alternatively: $D|_E = j^*(D)$ where $j : E \hookrightarrow X$, then $i_x(C, D)$ =multiplicity of x on $D|_E$.

On a fibered surface we get for $s \in S$, $i_s : \text{Div}(X) \times \text{Div}_s(X) \to \mathbb{Z}$. If E is a component of X_S then $i_s(D, E) = \deg_{k(S)} \mathcal{O}_X(D)|_E$.

Properties

- X_s fiber of $X \to S$ then i_s is negative definite and $x \cdot x = 0$ implies $x \in \mathbb{Z}X_s$.
- $X \to Y$ a contraction, $\Gamma_i \to y$. Look at $\sum n_i n_j \Gamma_i \Gamma_j \leq 0$, with equality if and only if $n_i = 0$.
- Hodge index theorem for ordinary surfaces $\text{Pic}(X) \times \text{Pic}(X) \to \mathbb{Z}$, has signature $(1, -1, \ldots, -1)$.
- P a point of X_η. Consider $\overline{\{P\}} \cdot X_s = [K(P) : K(S)]$, hence it is 1 if P is a rational point
- Take$K$ such that $\omega_{X|S} = \mathcal{O}_X(K)$, then

$$\begin{aligned} 2p_a(X_\eta) - 2 &= \deg \omega_{X_\eta}|_{k(\eta)} \\ &= -2\chi_{k(\eta)}(\mathcal{O}_{X_\eta}) \\ &= 2X_{k_{(S)}}(\mathcal{O}_{X_S}) \\ &= \deg(\omega_{X_s}|_{k(s)}) \\ &= \deg(\mathcal{O}_X(K)|_{X_S}) \\ &= K \cdot X_S \\ &= \sum d_i(K_{X|S} \cdot \Gamma_i) \end{aligned}$$

where Γ_i are components of X_S and d_i are the length of $\mathcal{O}_{X,\Gamma_i}$.