AG for NT First Week 2

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Recap Sheaves

Let X be a topological space.

- A presheaf $\mathcal{F}$ is some collection $F(U)$ of abelian group (rings, modules) for each open sets $U \subseteq X$, with some compatibility.
- A presheaf $\mathcal{F}$ is a sheaf if it satisfies additional local conditions.
- Define the stalk F_x for each $x \in X$.
- Local conditions, we have some nice "local to global" properties. (i.e., a morphism of sheaf is an isomorphism if and only if it is an isomorphism of stalks)

Example. (The constant sheaf) Let A be a group/ring and define the constant presheaf to be the presheaf $F(U) = A$ for each open $U \subseteq X$. This is not a sheaf. Let $X = \{x_1, x_2\}$ with discrete topology, and $A = \mathbb{Z}$. Then $F(\{x_1\}) = \mathbb{Z}, F(\{x_2\}) = \mathbb{Z}$. Take $2 \in F(\{x_1\})$ and $3 \in F(\{x_2\})$. There does not exists $s \in F(X)$ such that $s|_{\{x_1\}} = 2$ and $s|_{\{x_2\}} = 3$.

We fix this by instead setting $F(U) = \bigoplus_{\text{connect component of } U} A$.

Schemes

Motivation: Let X to some irreducible affine variety over an algebraically closed field K. This gives some $K[X]$ a regular function field. Hilbert's Nullstellensatz $\Rightarrow$ there exists a bijection between

$$\{K\text{−pts of } X\} \leftrightarrow \{\text{maximal ideal in } K[X]\}$$

by $p \mapsto m_p = \{f(p) = 0\}$

Definition. Let A be a (commutative). Define a space $X := \mathrm{Spec}(A) = \{p \subsetneq A : p \,\text{prime}\}$. We give X the Zariski topology, i.e., the closed sets are $V(f) = \{p \in X : \langle f \rangle \subset p\}$ (plus the intersections of some collection of $V(f)$), giving rise to the open sets $D(f) = X \setminus V(f)$. The set $D(f)$ form a basis for the open sets of X.

We put a sheaf $\mathcal{O}$ of rings on $\mathrm{Spec}(A)$. For each $D(f)$ we define $\mathcal{O}(D(f)) = A\left[\frac{1}{f}\right]$. (Here $A\left[\frac{1}{f}\right] = \left\{\frac{a}{f^n} : a \in A, n \in \mathbb{Z}\right\}$, e.g., $A = \mathbb{Z}$ then $\mathbb{Z}\left[\frac{1}{5}\right] = \left\{\frac{q}{5^n} : q \in \mathbb{Z}\right\}$). This actually defines a $\mathcal{B}$-sheaf, where $\mathcal{B}$ is the basis consisting of all the $D(f)$. We can then use the lemma from last week to extend this to a sheaf $\mathcal{O}$ on the whole of X.

Proposition (Q-L page 42 proposition 3.1)**.** *$\mathcal{O}$ is a sheaf.*

Proof. By last week lemma, we just need to show that $\mathcal{O}$ is a $\mathcal{B}$-sheaf.

First, need compatibility maps. If $D(g) \subset D(f)$, then $g \in \sqrt{\langle f \rangle} = \{a \in A : \exists n \in \mathbb{N} \text{ s.t. } a^n \in \langle f \rangle\}$. Hence $g^m = af$ for some $a \in A, m \in \mathbb{N}$. Defined a map $A\left[\frac{1}{f}\right] \to A\left[\frac{1}{g}\right]$ by $bf^{-n} \mapsto ba^n g^{-mn}$. (Check that if $D(f) = D(g)$ then this map is an isomorphism)

So now let $\{U_i\}$ be a covering of X.

Claim. There is a finite subcover

Let $U_i = D(f_i)$, then $X = \cup D(f_i) \Rightarrow \cap V(f_i) = \emptyset$. Hence $V(\sum \langle f_i \rangle) = \emptyset \Rightarrow \sum \langle f_i \rangle = A$. Hence there exists some finite set I such that $1 = \sum_I a_i f_i$. Then $\sum_I (f_i) = A \Rightarrow X = \cup_I D(f_i)$.

To prove the local conditions:

4. Suppose $s \in \mathcal{O}(X) = \mathcal{O}(D(1)) = A$, with $s|_{U_i} = 0$ for all i. We want to show $s = 0$. For each $i \in I$, $s|_{U_i} = 0 \Rightarrow \exists m_i \in \mathbb{Z}$ such that $sf_i^{m_i} = 0$. Then as $\cup D(f_i^m) = \cup D(f_i) = X$. In particular $\sum (f_i^m) = A$, we can write $1 = \sum_{i \in I} a_i f_i^m$. So $sf_i^m = 0$ for all $i \in I$. Hence $\sum_I s a_i f_i^m = s = 0$.

5. Let $s_i \in \mathcal{O}(D(f_i))$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. We want some $s \in \mathcal{O}(X) = A$ such that $s|_{U_i} = s_i$. Now, $D(f_i) \cap D(f_j) = D(f_i f_j)$ so $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ means that there exists some $r \in \mathbb{Z}$ such that $(s_i - s_j)(f_i f_j)^r = 0$. Each $s_i \in A\left[\frac{1}{f_i}\right]$, so $s_i = c_i f_i^{-m_i}$ for some $m_i \in \mathbb{N}$. Take $m = \max_{i \in I}\{m_i\}$, then $s_i = b_i f_i^{-m}$ for some $b_i \in A$. Combined with the above, we get $(b_i f_i^{-m} - b_j f_j^{-m}) f_i^r f_j^r = 0 \Rightarrow b_i f_j^{m+r} f_i^r = b_j f_i^{m+r} f_j^r$. We still have some $a_i \in A$ such that $1 = \sum a_i f_i^{m+r}$. Define $s := \sum_{i \in I} a_i b_i f_i^r$, then $s f_j^{m+r} = \sum_I a_i b_i f_i^r f_j^{m+r} = \sum_I a_i b_j f_j^r f_i^{m+r} = b_i f_j^r = s|_{D(f_i)}$ □

Fact. *The stalk of $\mathcal{O}$ at $p \in X$ is the local ring A_p.*

Definition. An *affine scheme* is topological space X with a sheaf of rings $\mathcal{O}_X$, such that $(X, \mathcal{O}_X)$ is isomorphic to $(\mathrm{Spec}(A), \mathcal{O})$ for some ring A.

Where isomorphism is an isomorphism of Ringed topological space. A morphism of $(X, \mathcal{O}_X) = (Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ such that $f : X \to Y$ is continuous and $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is a morphism of sheaves, such that the map $f_x^\#$ are local homomorphism.)

Example. (Affine line) Let $A = k[x]$, where k is a field. Then define the affine line over k, $^1_k = \mathrm{Spec}(k[x])$.

Let $A = k[x_1, \dots, x_n]$ we get the affine n-space ^n_k. $\mathcal{O}(D(f_i)) = \left\{ \frac{g}{f_i^n} : g \in k[x_1, \dots, x_n] \right\}$

Let $f \in k[x_1, \dots, x_n]$ be irreducible and let $A := k[x_1, \dots, x_n] / \langle f \rangle$ then $\mathrm{Spec}(A)$ correspond to $V(f)$.

Summary.

- A a commutative ring
- $\mathrm{Spec}(A) = \{$ prime ideal$\}$
- $U = D(f_i) \subset \mathrm{Spec}(A)$, $\mathcal{O}(U) = A[\frac{1}{f}]$, gives a sheaf.
- Affine scheme, something that is isomorphism to $(\mathrm{Spec} A, \mathcal{O})$.

Exercise. Let X be a topological space, $p \in X$, A an abelian group. Define a sheaf $i_p(A)$ as follows: $i_p(A)(U) = \begin{cases} A & p \in U \\ 0 & p \notin U \end{cases}$.

Show that $i_p(A)$ is a sheaf.

Show that $i_p(A)_q = \begin{cases} A & q = p \\ 0 & q \neq p \end{cases}$

Show also that $i_p(A) = i_*(A)$, A constant sheaf $i : \{p\} \to X$ inclusion.