AG for NT Week 3

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Last Time

- Defined a sheaf of rings $\mathcal{O}$ on $\mathrm{Spec}(A)$. $U \mapsto \mathcal{O}(U)$ where $U \subseteq \mathrm{Spec}(A)$ is open and $\mathcal{O}(U)$ is the ring of functions $s : U \to \prod_{p\in U} A_p$ such that

  1. $s(p) \in A_p$
  2. For all $p \in U$, there is a neighbourhood $V \subset U$ of p and elements $a, f \in A$ such that $f \notin q$ for any $q \in V$ and $s(q) = \frac{a}{f}$, $\forall q \in V$

- $(\mathrm{Spec}(A), \mathcal{O})$ the spectrum of A.
- $\mathcal{O}(D(f)) \cong A_f$
- $\mathcal{O}_p \cong A_p$

Today

- Affine schemes and schemes
- $\mathrm{Proj} S$
- Relation between varieties and schemes.

Definition. Let A and B be two local rings with maximal ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$. A ring homomorphism $\phi : A \to B$ is called *local homomorphism* if $\phi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$

Definition. A *ringed space* is a pair $(X, \mathcal{O}_X)$ such that X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X.

A *locally ringed space* (LRS) is a ringed space $(X, \mathcal{O}_X)$ such that $\mathcal{O}_{X,p}$ is a local ring, for all $p \in X$.

For example, $(\mathrm{Spec}\, A, \mathcal{O})$ is a locally ringed space for any ring A.

Definition. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces. A *morphism* of ringed spaces from X to Y is a pair $(f, f^{\#})$ where $f : X \to Y$ is a continuous map and $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a sheaf or rings.

Remark. If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally ringed spaces and a morphism of ring space $(f, f^{\#})$ between them, then $f^{\#}$ induces a ring homomorphism $f^{\#}_{[} : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$.

Indeed if $p \in X$, $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$, so we have a lots of homomorphism of the form $\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$. We get

$$\begin{array}{ccc} \mathcal{O}_{Y,f(p)} & \longrightarrow & \varinjlim_{f(p)\in U} \mathcal{O}_X(f^{-1}(U)) \\ & & \downarrow \\ & & \varinjlim_{p\in V} \mathcal{O}_X(V) = \mathcal{O}_{X,p} \end{array}$$

Definition. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be locally ringed spaces. A *morphism of locally ringed spaces* is a morphism of ringed space $(f, f^\#)$ such that the induced map $f_p^\#$ is a local homomorphism for all $p \in X$

Proposition. 1. *Let A and B be rings and $\phi : A \to B$ a ring homomorphism. Then ϕ induces a morphism of locally ringed spaces $(f, f^\#) : \operatorname{Spec} B \to \operatorname{Spec} A$*

2. *If we have a morphism of locally ringed spaces $(f, f^\#) : \operatorname{Spec} B \to \operatorname{Spec} A$, for some rings A, B then $(f, f^\#)$ is induced by ring homomorphims $\phi : A \to B$*

Proof. 1. We have a ring homomorphism $\phi : A \to B$ and we want a continuous map $f : \operatorname{Spec} B \to \operatorname{Spec} A$. Just defined $f(p) = \phi^{-1}(p)$. We know that every closed subset of $\operatorname{Spec} A$ is of the form $V(a)$ for some ideal $a \lhd A$. One can verify that $f^-(V(a)) = V(\langle\phi(a)\rangle)$

Now we want a sheaf of rings $f^\# : \mathcal{O}_A \to f_*\mathcal{O}_B$. So we will define ring homomorphism $\mathcal{O}_A(V) \to \mathcal{O}_B(f^{-1}(V))$. The elements of $\mathcal{O}_A$ are are functions $s : V \to \sqcup_{p\in V} A_p$

$$\begin{array}{ccc} f(f^{-1}(V)) & \longrightarrow & \sqcup_{p\in f(f^{-1}(V))} A_p = \sqcup_{q\in f^{-1}(V)} A_{f(q)} \\ f\uparrow & & \downarrow * \\ f^{-1}(V) & & \sqcup_{q\in f^{-1}(V)} B_q \end{array}$$

We have a ring homomorphism $\phi : A \to B$, which induces $\phi_p : A_{\phi^{-1}(p)} \to B_p$ giving the map $*$. Just check that this gives us what we wanted.

2. Read in Hartshorn

□

Definition. • An *affine scheme* is a locally ringed space which is isomorphic to the spectrum of some ring

• A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ in which every point has an open neighbourhood $U \subset X$ such that $(U, \mathcal{O}_X|_U)$ is an affine scheme.

Example. 1. Let k be a field. $\operatorname{Spec} k = \{0\}$. Associate a sheaf to it by $\mathcal{O}(\emptyset) = 0$ and $\mathcal{O}(0) = \{s : \{0\} \to k = k_0)$

Definition. Let X be a topological space and Z an irreducible closed subset of X. A *generic point* for Z is a point $p \in Z$ such that $Z = \overline{\{p\}}$.

Proposition. *If X is a scheme, every irreducible closed subset of X has a unique generic point.*

Proof. Exercise □

2. If k is a field, we define the affine line as $\mathbb{A}^1_k = \operatorname{Spec} k[x]$. The generic point is (0). If k is algebraically closed field, each closed point in $\operatorname{Spec} k[x]$ corresponds to a point in the line.

3. Let k be a field, be algebraically closed. Let $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$. The only generic point of $\mathbb{A}^2_k$ is (0). If $f(x,y) \in k[x,y]$ is irreducible, then (f) is a prime ideals and it is a generic point for the closure of $\{(x-a, x-b) : f(a,b) = 0\}$.

4. In general for any ring A, we define $\mathbb{A}^n_A = \operatorname{Spec}[x_1, \dots, x_n]$

Proj S

Let S be a graded ring, i.e., $S = \oplus_{i \geq 0} S_i$ and $S_i \cdot S_j \subseteq S_{i+j}$. We will denote the ideal $\oplus_{i>0} S_i$ by S_+. We define $\operatorname{Proj} S = \{p \lhd S :$ homogeneous prime ideals which do not contain the whole of $S_+\}$ Let $V(a) = \{p \in \operatorname{Proj} S : p \supseteq a\}$ where a is a homogeneous ideal of S.

Lemma. 1. *Let a, b in homogeneous ideals, $V(ab) = V(a) \cup V(b)$*

2. *$\{a_i\}$ a family of homogeneous ideals then $V(\sum a_i) = \cap V(a_i)$*

We can define a topology of $\operatorname{Proj}(S)$ by setting the closed sets to be the sets of the form $V(a)$, where a is a homogeneous ideal of S. We also define a sheaf of rings $\mathcal{O}$ in $\operatorname{Proj}(S)$ using the following tools.

Notation. p homogeneous prime ideal. Let $T_p = \{$homogeneous elements of S not in $p\}$. This is a multiplicatively closed subset. We localise S with respect to T_p. We define $S_{(p)}$ to be the set of elements of degree 0 of $T_p^{-1}S$. (The elements of $T_p^{-1}S$ look like $\frac{a}{b}$. The degree of $\frac{a}{b}$ is just $\deg a - \deg b$)

For any $U \subseteq \operatorname{Proj} S$ open, we define $\mathcal{O}(U)$ to be the set of functions $s : U \to \sqcup_{p \in U} S_{(p)}$ such

1. $s(p) \in S_{(p)}$

2. For each $p \in U$, there is an open neighbourhood $V \subset U$ and homogeneous elements $f, g \in S$ of the same degree such that $g \notin q$ for any $q \in V$ and $s(q) = \frac{f}{g}, \forall q \in V$.

Proposition. *Let S be a graded ring.*

1. *$p \in Proj(S)$ then $\mathcal{O}_p \cong S_{(p)}$*

2. *Defined $D_+(f) = \{p \in Proj(S) : f \notin p\}$. This is open and the set of $D_+(f)$ cover $ProjS$. Also $(D_+(f), \mathcal{O}|_{D_+(f)}) \cong \operatorname{Spec} S_{(f)}$*

3. *ProjS is a scheme.*

Proof. See Hartshorn □

Example. Let us define $\mathbb{P}^m_A = \operatorname{Proj} A[x_1, \dots, x_n]$ for any ring A.

Relation between Varieties and Schemes.

Let X be a topological space. Let $t(X)$ be the set of irreducible closed subset of X.

Some properties:

1. If Y is close in X, then $t(Y) \subset t(X)$

2. $t(Y_1 \cup Y_2) = t(Y_1) \cup t(Y_2)$ for Y_1, Y_2 closed in X.

3. $t(\cap Y_i) = \cap t(Y_i)$ for a family of $\{Y_i\}$ closed in X.

Define a topology of $t(X)$ by setting the closed subsets to be the sets $t(Y)$, where Y is closed in X.

We define the map, $\alpha : X \to t(X)$ by $p \mapsto \overline{\{p\}}$. The map α is easily seen to be continuous. Also if $f : X_1 \to X_2$ is continuous, then we get induced map $t(f) : t(X_1) \to f(X_2)$.

Note. If you know category theory, you will notice that t looks like a functor and we will show that it is the functor between the category of Schemes and the category of Varieties.

Proposition. *Let V be an affine variety over an algebraic closed field k. Then $(t(V), \alpha_* \mathcal{O}_V)$ is isomorphic to* $\operatorname{Spec} A$*, where A is the affine coordinate ring of V.*

Proof. See Hartshorn □

Definition. A *scheme X over a scheme S* is just a scheme X with morphism $X \to S$.

Let $\mathfrak{Var}(k)$ be the category of varieties over k and $\mathfrak{Sch}(k)$ be the category of schemes over $\operatorname{Spec} k$.

Proposition. *Let k be an algebraically closed field. Then the map $t : \mathfrak{Var}(k) \to \mathfrak{Sch}(k)$ is a functor. Also any variety V is homeomorphic to the subset of closed points of $t(V)$ and its associated sheaf is given by restricting $\alpha_* \mathcal{O}_V$ with respect to the homeomorphism.*

Example. Let V be an affine variety. We have $t(V) \cong \operatorname{Spec} A$.