AG for NT Week 7
Divisor

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Introduction

Let C be a non singular projective curve in $\mathbb{P}^2_k$ (k algebraically closed). For any line in $\mathbb{P}^2_k$, $L \cap C$ has exactly d points (where d is the degree of C). Exercise I.5.4

$L \cap C \leftrightarrow \sum n_i P_i$ where n_i is the multiplicity of $P_i \in L \cap C$. Call $\sum n_i P_i$ a *divisor* on C. By varying L, we get a family of divisors on C, parametrized by the set of lines in $\mathbb{P}^2$. This set of divisors is called a *linear system* of divisors of C.

Remark. Knowing the linear system of divisors on C, one can recover the embeddings of C in $\mathbb{P}^2_k$. Given a point on C, say P. Given a point on C, say P. Consider the set of divisors on $P \Rightarrow$ set of lines passing through P, which gives a unique characterization of P in $\mathbb{P}^2_k$.

Consider two lines L and L' in $\mathbb{P}^2_k$ given by $f = 0$ and $f' = 0$ respectively. Then f/f' is a rational function on $\mathbb{P}^2_k$ which restricts to a rational function g on C. Let $D \leftrightarrow L \cap C$ and $D' \leftrightarrow L' \cap C$. By construction g has 0 at points on D and poles at points on D'. If this happens we say D and D' are equivalent.

Group of divisors modulo linear equivalence is called the *Picard Group.* This is an invariant of the variety we are considering.

Weil Divisors

Let X be a Noetherian, regular in codimension 1, Integral, Separated, scheme. We will denote this as NCIS.

Definition. A scheme X is *regular in Codimension 1* if every local ring $\mathcal{O}_X$ of X of dimension 1 is regular.

So for us, it means that $\mathcal{O}_X$ will a a discrete valuation ring.

Example. Nonsignular Variety over a filed
Noetherian normal scheme

Definition. Let X be NCIS. A *prime divisor* on X is a closed integral subscheme Y of codimension 1.

A *Weil divisor* is an element of the free abelian group denoted $\text{Div}(X)$, generated by the prime divisors.

We write $D = \sum n_i Y_i$ where Y_i are prime divisors on X, n_i are integers and all but finitely many are zeros.

A divisors is *effective* if $n_i \geq 0$ for all n_i.

If Y is a divisors on X, let $\eta \in Y$ be its generic point. The local ring $\mathcal{O}_{\eta,X}$ is a discrete valuation ring with the quotient field K. Call the corresponding valuation v_Y.

Let $f \in K^*$ be a non-zero rational function on X. If $v_Y(f)$ is strictly positive, we say f has a *zero* along Y of order $v_Y(f)$. If $v_Y(f)$ is strictly negative, we say f has a *pole* along Y of order $-v_Y(f)$.

Lemma. *Let X be NCIS, $f \in K^*$ then $v_Y(f) = 0$ for all by finitely many prime divisors Y of X.*

Proof. Let $U = \operatorname{Spec} A$ be an open affine subset of X on which f is regular. Let $Z = X \setminus U$, Z is a proper closed subset of X. As X is Noetherian, Z must contain finitely many prime divisors on X. In particular, all other prime divisors must meet U. So we need to show that U contains finitely many divisors with $v_Y(f) \neq 0$. But f is regular on U, in particular $v_Y(f) \geq 0$. If $v_Y(f) > 0$ then Y is contained in the closed subset U defined by the ideal $Af \subset A$. Since $f \neq 0$, this is a proper closed subset. In particular it contains finitely many closed irreducible subsets of codimension 1 of U (which are the divisors) □

Definition. Let X be a NCIS, $f \in K^*$. We define the *divisor of f*, denote $(f) = \sum_Y v_Y(f)Y$ where the sum is taken over all prime divisors of X.

Any divisor in $\operatorname{Div}(X)$ is called *principal* if it is the divisor of a function $f \in K^*$

Remark. Let $f, g \in K^*$, then $(f/g) = (f) - (g)$

This allows us to define $\phi : f \mapsto (f)$ is a homomorphism from the multiplicative group of K^* to the additive group $\operatorname{Div}(X)$.

Definition. Two divisors $D, D' \in \operatorname{Div}(X)$ are *linearly equivalent*, denoted $D \sim D'$, if $D - D'$ is a principal divisor.

$\operatorname{Div}(X)/\operatorname{im}(\phi) = \operatorname{Div}(X)/\sim =$ *divisor class group of X*. This is denoted $\operatorname{Cl}(X)$.

Divisors on Curves

Nice reference: Silverman, *The Arithmetic of Elliptic Curves*, II.3

Definition. Let k be algebraic closed. A curve over k is an integral separated, (complete, proper), scheme X of finite type over k of dimension 1.

If X is a nonsingular curve, then X is NCIS

A prime divisor on X is a closed point. $D = \sum_{P \subset X} n_i P_i$ where $n_i \in \mathbb{Z}$.

Definition. The degree of $D = \sum n_i P_i$ is $\deg(D) = \sum n_i$.

If $f : X \to Y$ is a finite morphism of non-singular curves, we define $f^* : \operatorname{Div}(Y) \to \operatorname{Div}(X)$ to a homomorphism, as follows: Let $Q \in Y$ be given, $t \in \mathcal{O}_Q$ be a local parameter at Q, $t \in K(Y)$. Hence $v_Q(t) = 1$. Then $f^*Q = \sum_{f(P)=Q} v_p(t)P$. Since f is a finite morphism, we have finitely many $P \in X$ such that $f(P) = Q$.

Note. f^* preserves linear equivalence.

Hence f induces $f^* : \operatorname{Cl}Y \to \operatorname{Cl}X$.

Remark. A principal divisor on a complete non singular curve had degree 0. The degree of a divisor on X depends only on the its linear equivalence class.

Proposition. *Let $f : X \to Y$ be a finite morphism. Let $\deg : Div(X) \to \mathbb{Z}$ be defined by $f^*D \mapsto \deg f \cdot \deg D$. The degree map is surjective. Let $Cl^0(X) = \ker(\deg)$.*

There is a natural 1-1 correspondence between the set of closed points of X and $\operatorname{Cl}^0(X)$.

For elliptic curves:

Let $P_0 \in X$, $(P_0 = (0 : 1 : 0))$, The tangent $z = 0$, meets the curve in $3P_0$. Given any line passing through P, R, Q, $P + Q + R \sim 3P_0$. Now to any point $P \in X$, construct $P \mapsto P - P_0 \in \operatorname{Cl}^0(X)$.

Injective: If $P - P_0 \sim Q - Q_0 \iff P \sim Q \Rightarrow$(exercise p139) X is rational. This is a contradiction since X is not birationaly equivalent to $\mathbb{P}^1$ (its an elliptic curve)

Surjective: Let $D \in \operatorname{Cl}^0(X)$, $D = \sum n_i P_i$ with $\sum n_i = 0$. In particular, $D = \sum n_i(P_i - P_0)$. Now for any point $R \in X$, there exists $T \in X$ such that $P_0 + T + R \sim 3P_0$. So $R - P_0 \sim -(T - P_0)$ in $D = \sum n_i(P_i - P_0)$. If $n_i < 0$, we can replace by some $m_i > 0$. Complete proof p139 Hartshorne

Hence we have $\operatorname{Cl}^0(X) \leftrightarrow$set of closed points on X.

Remark. The divisor class group of a variety has a discrete component ($\mathbb{Z}$), a continuous component ($\mathrm{Cl}^0(X)$) which has itself the structure of an algebraic variety. If X is any curve, $\mathrm{Cl}^0(X) \cong$group of closed points of an abelian variety called the *Jacobian Variety of* X. The dimension of the Jacobian variety, $J(X)$ is the *genus* of the curve X.

For genus 2, you can look at Cassels/Flynn: *Prolegomena to a Middlebrow Arithmetic of Curves of Genus 2,* for the construction of the Jacobian.