AG for NT Week 8

We will use the language of schemes to study varieties

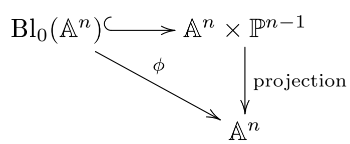
1 Blowing Up Varieties

We will construct the *blow up* of a variety with respect to a non-singular closed subvariety. This tool/technique is the main method to resolve singularities of algebraic variety.

Definition 1.1. The *blowup* of $\mathbb{A}^n$ at 0 is constructed as follows: Take the product $\mathbb{A}^n \times \mathbb{P}^{n-1}$. If $\{x_1, \ldots, x_n\}$ are affine coordinated for $\mathbb{A}^n$, $\{y_1, \ldots, y_n\}$ are the homogeneous coordinates of $\mathbb{P}^{n-1}$, the *blowup of* $\mathbb{A}^n$ $\mathrm{Bl}_0(\mathbb{A}^n)$ is the closed subset defined by

$$\mathrm{Bl}_0(\mathbb{A}^n) := \{x_i y_j = x_j y_i | 1 \leq i, j \leq n, i \neq j\}$$

We have the following commutative diagram



For the next few pages, we let ϕ be the morphism defined as above.

Lemma 1.2.

1. *If $p \in \mathbb{A}^n$, $p \neq 0$ then $\phi^{-1}(p)$ consist of one point. In fact phi gives an isomorphism of $Bl_0(\mathbb{A}^n) \setminus \phi^{-1}(0) \cong \mathbb{A}^n \setminus \{0\}$*
2. *$\phi^{-1}(0) \cong \mathbb{P}^{n-1}$*
3. *The points of $\phi^{-1}(0)$ are in 1-1 correspondence with the lines of $\mathbb{A}^n$ through the origin*
4. *$Bl_0(\mathbb{A}^n)$ is irreducible*

Proof.

1. Let $p = (a_1, \ldots, a_n) \in \mathbb{A}^n$, assume $a_i \neq 0$. So if $p \times (y_1, \ldots, y_n) \in \phi^{-1}(p)$ then for each j, $y_j = \left(\frac{a_j}{a_i}\right) y_i$. So $(y_1 : \cdots : y_n)$ is uniquely determined as a point in $\mathbb{P}^{n-1}$. By setting $y_i = a_i$, we have $(y_1 : \cdots : y_n) = (a_1 : \cdots : a_n)$. Moreover setting $\psi(p) = (a_1, \ldots, a_n) \times (a_1 : \cdots : a_n)$ defines an inverse morphism to ϕ. $\mathbb{A}^n \setminus \{0\} \to \mathrm{Bl}_0\mathbb{A}^n \setminus \phi^{-1}(0)$
2. $\phi^{-1}(0)$ consist of all points $0 \times Q$ for $Q \in \mathbb{P}^{n-1}$ with no restrictions
3. Follows from 2.
4. $\mathrm{Bl}_0(\mathbb{A}^n) = \left(\mathrm{Bl}_0(\mathbb{A}^n) \setminus \phi^{-1}\{0\}\right) \cup \phi^{-1}\{0\}$. The first component, by part 1. is irreducible, and each point in $\phi^{-1}(0)$ is contained in the closure of some line L in $\mathrm{Bl}_0\mathbb{A}^n \setminus \phi^{-1}\{0\}$. Hence $\mathrm{Bl}_0\mathbb{A}^n \setminus \phi^{-1}(0)$ is dense in $\mathrm{Bl}_0(\mathbb{A}^n)$ and hence $\mathrm{Bl}_0(\mathbb{A}^n)$ is irreducible

$\square$

Definition 1.3. If $Y \subset \mathbb{A}^n \setminus 0$ we define $\mathrm{Bl}_0 Y$ to be $\widetilde{Y}$ is $\overline{\phi^{-1}(Y \setminus 0)}$

We see from Lemma 1.2 ϕ induces a birational morphism of $\widetilde{Y}$ to Y.

Fact 1.4. *Blowing up is independent of your choice of embedding.*

Example 1.5. (Node)
Let x, y be coordinates in $\mathbb{A}^2$, and define $X : (y^2 = x^2(x+1))$. Let t, u be homogeneous coordinates for $\mathbb{P}^1$. Then $\mathrm{Bl}_0 X = \{y^2 = x^2(x+1), ty = ux\} \subset \mathbb{A}^2 \times \mathbb{P}^1$. On the affine piece $t \neq 0$, we have $y^2 = x^2(x+1)$ and $y = ux$, hence $u^2x^2 = x^2(x+1)$. This factors, hence we get a variety $\{x = 0\} = E$ (this is the preimage of 0 under ϕ and is called the "Exceptional Divisors") and the variety $\{u^2 = x+1\} = \widetilde{X}$ (This is called "the proper transform of X").

Note that $\widetilde{X} \cap E$ consists of two points, $u = \pm 1$. Notice that this values for u are precisely the values of the slopes of X through the origin. "Blowups separates points and tangent vectors")

Exercise 1.6. (Tacnode)
Let $T : (y^2 = x^4(x+1))$. Blow this up at the origin and see what you get.

Definition 1.7. Blowing up with respect to a subvariety. Let $X \subset \mathbb{A}^n$ be an affine variety. Let $Z \subset X$ be a closed non-singular subvariety, Z defined by the vanishing of the polynomials $\{f_1, \ldots, f_k\}$ in $\mathbb{A}^n$. Let $(y_1 : \cdots : y_k)$ be homogenous coordinates for $\mathbb{P}^{k-1}$. Define $\mathrm{Bl}_Z(\mathbb{A}^n) = \{y_i f_j = y_j f_i | 1 \leq i, j \leq k, i \neq j\}$. As before, we get a birational map

$$\begin{array}{ccc} \mathrm{Bl}_Z(\mathbb{A}^n) & \hookrightarrow & \mathbb{A}^n \times \mathbb{P}^{k-1} \\ & \searrow^{\phi} & \downarrow \text{projection} \\ & & \mathbb{A}^n \end{array}$$

It has a birational inverse, $p = (a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_n) \times (f_1(p) : \cdots : f_k(p))$. Also define $\mathrm{Bl}_Z(X) = \overline{\phi^{-1}(X \setminus 0)}$.

Exercise 1.8. Compare blowing up $y^2 = x^2(x+1)$ in $\mathbb{A}^3_{[x:y:z]}$ with respect to the z-axis.

[Note: 0 the subvariety defined by the vanishing of polynomials $f_i = x_i$]
For most purposes/"classifying all surfaces" only need to know about blowing up a point.

Example 1.9. Let X be the double cone defined by $x^2 + y^2 = z^2 \subset \mathbb{A}^3_{[x:y:z]}$ and let Z be the line defined by $\{y = z, x = 0\}$. Let t, u be coordinates for $\mathbb{P}^1$, hence $\mathrm{Bl}_Z X = \{x^2 + y^2 = z^2, xt = (y - z)u\}$. So on the affine piece $u \neq 0$, we get $xt = y - z$ hence $x^2 = xt(y + z)$. This factorises, so we get two pieces: $\{x = 0, y = z, t \text{ arbitrary}\} = E$ (the exceptional curve); $\{xt = y - z\} := \widetilde{X}$ (this should be nonsingular)

2 Invertible Sheaves

Let X be a variety.

Definition 2.1.

- An *invertible sheaf* $\mathcal{F}$ on X is a locally free $\mathcal{O}_X$-module of rank 1. (That is, there exists an open covering $\{U_i\}$ of X so that $\mathcal{F}(U_i) \cong \mathcal{O}_X(U_i)$)
- We will see soon that the *Picard group* is the group of isomorphism classes of invertible sheaves on X.
- On varieties: Weil divisors are "the same" as Cartier divisors. A *Cartier Divisor* $D = \{(U_i, f_i)\}$ with $\{U_i\}$ an open covering of X, and f_i on U_i is an element of $\mathcal{O}_X(U_i)$ (think of a rational function). Also on $U_i \cap U_j$, we have $\frac{f_i}{f_j}$ is invertible.

Notation 2.2. Let D be a divisors (Weil/Cartier), define $\mathcal{L}(D)$ to be the sub-$\mathcal{O}_X$-module which is generated by f_i^{-1} on U_i. This is well defined since $\frac{f_i}{f_j}$ is invertible on $U_i \cap U_j$, so f_i^{-1} and f_j^{-1} differs by a unit. This $\mathcal{L}(D)$ is called the *sheaf* associated to $D = \{(U_i, f_i)\}$.

Proposition 2.3.

1. *For any divisors D, $\mathcal{L}(D)$ is an invertible sheaf on X and the map $D \mapsto \mathcal{L}(D)$ gives a* 1-1 *correspondence $Pic(X)$ $\leftrightarrow$Invertible sheaves on X.*
2. $\mathcal{L}(D_1 - D_2) \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$
3. *$D_1 \sim D_2$ (linearly equivalence) if and only if $\mathcal{L}(D_1) \cong \mathcal{L}(D_2)$*

Proof.

1. The map $\mathcal{O}_{U_i} \to \mathcal{L}(D)|_{U_i}$ defined by $1 \mapsto f_i^{-1}$ is the isomorphism, so $\mathcal{L}(D)$ is an invertible sheaf. Conversely, D can be recovered from $\mathcal{L}(D)$ by f_i on U_i to be the inverse of a generator for $\mathcal{L}(D)(U_i)$.
2. If $D_1 = \{(U_i, f_i)\}$ and $D_2 = \{(V_i, g_i)\}$, then $\mathcal{L}(D_1 - D_2)$ on $U_i \cap V_j$ is generated by $f_i^{-1} g_j$. So $\mathcal{L}(D_1 - D_2) \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$.
3. By part 2. it is sufficient to show that $D = D_1 - D_2$ is principal if and only if $\mathcal{L}(D) \cong \mathcal{O}_X$. If D is principal, defined by $f \in \Gamma(X, \mathcal{O}_X^*)$, then $\mathcal{L}(D)$ is globally generated by f^{-1}, so $1 \to f^{-1}$ is the isomorphism $\mathcal{O}_X \cong \mathcal{L}(D)$.

□

So we have a 1-1 correspondence from Pic(X) $\to$isomorphism classes of invertible sheaves.

3 Morphisms to $\mathbb{P}^n$

On $\mathbb{P}^n$, the homogeneous coordinates $x_0 : \cdots : x_n$ give the standard cover $\{U_i := (x_i \neq 0)\}$ and on U_i, x_i^{-1} is a local generator for the sheaf $\mathcal{O}(1)$. For any (projective) variety X, let $\phi : X \to \mathbb{P}^n$. Then $\mathcal{L} = \phi^*(\mathcal{O}(1))$ is an invertible sheaf on X. The global sections $s_0, \ldots, s_n$ ($s_i := \phi^*(x_i)$), $s_i \in \Gamma(X, \mathcal{L})$ "generate" the sheaf $\mathcal{L}$. Conversely, $\mathcal{L}$ and s_i determines ϕ.

Theorem 3.1.

1. *If $\phi : X \to \mathbb{P}^n$ is a morphism, then $\phi^*(\mathcal{O}(1))$ is an invertible sheaf generated by global sections $s_i = \phi^*(x_i)$*
2. *Any invertible sheaf $\mathcal{L}$ on X determines a unique morphism $\phi : X \to \mathbb{P}^n$*

Proof.

1. From Above
2. Lengthy argument in Hartshorne, pg 150

□

Proposition 3.2. *Let k be an algebraically closed field. Let X be a variety, and $\phi : X \to \mathbb{P}^n$ be a morphism corresponding to $\mathcal{L}$ and $s_0, \ldots, s_n$ be as above. Let $V \subset \Gamma(X, \mathcal{L})$ be a subspace spanned by $s_i = \phi^*(x_i)$. Then ϕ is a* <u>*closed immersion*</u> *if and only if:*

1. *Elements of V "separate points", i.e., for any $P \neq Q$ on X, exists $s \in V$ with $s \in m_P \mathcal{L}_P$ but $s \notin m_Q \mathcal{L}_Q$.*
2. *Elements of V "separate tangent vectors", i.e., for each points $P \in X$, the set of $\{s \in V : s_P \in m_p \mathcal{L}_p\}$ span the vector space $m_p \mathcal{L}_p / m_p^2 \mathcal{L}_p$.*

Proof. (Only proving $\Rightarrow$) If ϕ is a closed immersion, think of X as a closed subvariety of $\mathbb{P}^n$. So $\mathcal{L} = \mathcal{O}_X(1)$ and the vector space $V \subset \Gamma(X, \mathcal{O}_X(1))$ is spanned by the images of $x_0, \ldots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$. Given $P \neq Q$ in X, we can find a hyperplane H containing P and not Q. If $H = (\sum a_i x_i = 0)$ for $a_i \in k$, then $s = \sum a_i x_i|_X$ satisfies the first property. For the second, each hyperplane passing through P gives rise to sections which generate $m_P\mathcal{L}_P/m_p^2\mathcal{L}_p$.

Example. If $P = (1 : 0 : \cdots : 0)$, then U_0 has local coordinates $y_i = \frac{x_i}{x_0}$, so $P = (0, \ldots, 0) \in U_i$ and m_p/m_p^2 is the vector space spanned by y_i.

□

So we have a 1-1 correspondences $\text{Pic}(X) \leftrightarrow$isomorphism classes of invertible sheaves $\leftrightarrow$ morphisms to $\mathbb{P}^n$.

4 Linear systems of Divisors

Definition 4.1. A *complete linear system* $|D_0|$ on a non-singular projective variety is the set of all effective divisors linearly equivalent to D_0.

That is $|D_0|$ is in 1-1 correspondence to this set: $\Gamma(X, \mathcal{L}(D_0)) \setminus \{0\}/k^*$, i.e., $|D_0|$ "is" a projective space.

Definition 4.2. A *linear system* δ on X is a subset of a complete linear system $|D_0|$ which is a linear subspace for $|D_0|$

That is, δ is a sub-vector space of $\Gamma(X, \mathcal{L}(D_0))$

Definition 4.3. A point $P \in X$ is a *base point* for a linear subsystem δ is $P \in \text{Supp}(D)$ for every $D \in \delta$. (Where $\text{Supp}(D)$ is the set of all prime divisors whose coefficient is non-zero)

Lemma 4.4. *Let δ be a linear system on X corresponding to the subspace $V \subset \Gamma(X, \mathcal{L}(D_0))$. Then a point $P \in X$ is a base point of δ if and only if $s_p \in m_P\mathcal{L}_p$ for all $s \in V$. In particular, δ is base point-fee if and only if $\mathcal{L}(D_0)$ is generated by global sections in V.*

Proof. This follows from the fact that for every $s \in \Gamma(X, \mathcal{L}(D_0))$, $s \mapsto D(U_i, \phi_i(s))$, (where $\phi_i : \mathcal{L}(D_0)(U_i) \overset{\cong}{\to} \mathcal{O}_X(U_i))$ and D is an effective divisors on X. So the support of D is the complement of the open set $X_s := \{x \in X | s_x \notin m_p\mathcal{L}(D_0)\}$. □

Remark. We can use this to rephrase Prop 3.2 in terms of linear systems (without base points). $\phi : X \to \mathbb{P}^n$ is a closed immersion if and only if

1. δ "separates points", i.e., for all $P \neq Q$, $\exists D \in \delta$ with $P \in \text{Supp}D$ and $Q \notin \text{Supp}D$
2. δ "separates tangent vectors", i.e., if $P \in X$ and $t \in m_p/m_p^2$ (is a tangent vector) then there exists $D \in \delta$ such that $P \in \text{Supp}D$ but $t \in (m_{p,D}/m_{p,D}^2)$ where we consider $D \subset X$ as a closed subvariety.