AG for NT 9

1 Recalling

Let X be a scheme, it is *reduced at* $x \in X$ if the stalk $\mathcal{O}_{X,x}$ is a reduced ring (it has not nilpotent elements)

A scheme is *reduced* if it is reduces at all points

Example. $\operatorname{Spec}(k[x]/(x^2))$ is not reduced.

Varieties are always reduced.

A scheme is *irreducible* if it is irreducible as topological spaces.

A scheme is *integral* if it is reduced and irreducible.

Example. $\operatorname{Spec}(k[t]/f(t))$ is integral where f is a irreducible polynomial.

$\operatorname{Spec}(A \times B)$ is not integral. $A, B \neq 0$.

A scheme is called *normal at* $x \in X$ if $\mathcal{O}_{X,x}$ is a normal domain (i.e, it is integrally closed in its fraction field)

A scheme is *normal* if it is normal at all points in X

Remark. (Easy to prove) A normal scheme is connected

Example. The scheme $y^2 = x^3 - x^2$ (a loop) is not normal.

A scheme is *Dedekind* if it is normal and locally Noetherian of dimension 1. (By dimension, we mean the Krull dimension, i.e., the maximal length of a chain of irreducible closed subschemes)

A scheme is *Regular at* $x \in X$, if $\mathcal{O}_{X,x}$ is regular, i.e. $\mathcal{O}_{X,x}$ is a local ring, $\mathfrak{m}_x$ its maximal ideal, $\mathcal{O}_{X,x}/\mathfrak{m}_x = k$, then $\dim \mathcal{O}_{X,x} = \dim_k \left(\mathfrak{m}_x/\mathfrak{m}_x^2\right)$

A scheme is *regular* if it is regular at all points

Example. The above example is not regular.

Let $f : X \to Y$ be any morphism of schemes. Let $V \subseteq Y$ be affine open, $U \subseteq f^{-1}(V)$ to be affine open, then $\mathcal{O}_X(U)$ is an $\mathcal{O}_Y(V)$-algebra. If f is quasi-compact and for all U, V as before, $\mathcal{O}_X(U)$ is finitely generated as $\mathcal{O}_Y(V)$-algebra then f is of finite type.

A morphism is called *finite* if for all $V \subset Y$ open affine, $f^{-1}(V) \subset X$ is affine and of finite type as modules.

A morphism is called *flat* if $f^{\#} : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a flat morphism of rings, i.e., $\mathcal{O}_{X,x}$ is flat as $\mathcal{O}_{Y,f(x)}$-module.

Definition 1.1. Let k be a filed, and X a k-scheme of finite type. Let $\overline{k}$ be an algebraic closure of k. X is *smooth at* $x \in X$ if the points lying above it in $X_{\overline{k}}$ are regular points.

X is *smooth* if it is smooth at every points.

Definition 1.2. Let $f : X \to Y$ be a morphism of finite type, suppose the rings are locally Noetherian, f is *smooth at* $x \in X$ if it is flat and $X_{f(x)} \to \operatorname{Spec} k(f(x))$ is smooth at x.

2 Models

The following definition varies from author to author, but this is the "most general" definition (with the least assumption made, e.g. connected irreducible etc)

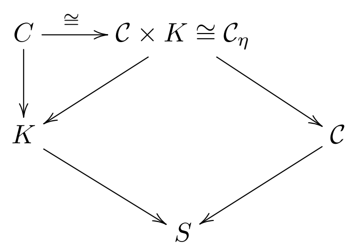
Definition 2.1. Let k be a field. A *curve* over k is a k-scheme of finite type, whose irreducible components have dimension 1.

Definition 2.2. Let S be a scheme, a *curve* over S is a flat S-scheme whose fibers are curves over the corresponding residue fields.

From now on: let S be a Dedekind scheme. Let $K = K(S)$ be its field of rational functions.

Definition 2.3. A *fibred surface* over S is an integral projective flat scheme over S, $X \to S$, of dimension 2.

Definition 2.4. Let C be a smooth, projective, connected curve over K. A *model* $\mathcal{C}$ over S of C is a normal fibred surface, $\mathcal{C} \to S$, together with an isomorphism $\mathcal{C}_\eta \cong C$ (where η is the general point on S)



Definition 2.5. A rational map $Y \dashrightarrow X$ is an equivalence class of maps $(U, f_U : U \dashrightarrow X)$ where U is open, f is a morphism. Two maps are equivalent if they agree on a non-empty open intersection of their domain

Definition 2.6. A regular fibred surface $X \to S$ is *minimal* if every birational map $Y \dashrightarrow X$ of regular fibred surfaces is a birational morphism.

Minimal regular model

Theorem 2.7 (Liu, 9.3.21)**.** *Let $X \to S$ be a regular fibred surface with generic fiber X_η of arithmetic genus ≥ 1. Then X admits a unique minimal model over S up to unique isomorphism.*

The arithmetic genus is $1 - \chi_k(\mathcal{O}_X)$. (where $\mathcal{X}$ is the Euler characteristic)

Jacobian Criterion. *Let k be a field, X an affine variety, closed. $X \subset \mathbb{A}_k^n$ (with local coordinated $T_1, \ldots, T_n$), $x \in X(k)$, $\alpha = V(I)$. Let $F_1, \ldots, F_r$ to be generators for I. The Jacobian $J = \left(\frac{\partial F_i}{\partial T_j}\right)_{1\leq i\leq r, 1\leq j\leq n} \in M_{r\times n}(k)$. Then X is regular at x if and only if* $\operatorname{rk} J_x = n - \dim \mathcal{O}_{X,x}$.

3 Examples

Let $C = \operatorname{Spec}(\mathbb{Q}[x,y]/(y^2 - x^3 + 49))$, this is a curve. Construct a regular model over $\mathbb{Z}$.

Let us try$X = \operatorname{Spec}(\mathbb{Z}[x,y]/(y^2 - x^3 + 49))$. Reduce $y^2 - x^3 + 49$ modulo 7, we have $y^2 = x^3$ which has singular point. Let $\mathfrak{m} = (x, y, 7)$ then $\dim_{\mathbb{F}_7}(\mathfrak{m}/\mathfrak{m}^2) = 3 > 2$. Hence the scheme X is not regular at $\mathfrak{m}$.

Consider

$$\widetilde{X} := \operatorname{Bl}_{\mathfrak{m}}(X) = \begin{cases} y^2 = x^3 - 7^2 \\ 7u = xw \\ 7v = yw \\ uy = xv \end{cases}$$

where $u : v : w$ is projective coordinated. Is $\widetilde{X}$ regular? Regularity is local

$u = 1$

$$X_1 = \begin{cases} y^2 = x^3 - 7^2 \\ 7 = xw \\ 7w = yw \\ y = xv \end{cases}$$

This gives $7v = yw = xzw = vxw = 7v$, $X_1 = \begin{cases} x^2v^2 = x^3 - x^2w^2 \\ 7 = xw \end{cases}$ or in factorisation form $X_1 = \begin{cases} x^2(v^2 - x + w^2) = 0 \\ 7 - xw = 0 \end{cases}$. We use Jacobian criterion $J(x, v, w) = \begin{pmatrix} -1 & 2v & 2w \\ -w & 0 & -x \end{pmatrix}$. X is regular if and only if for all $x \in X$, $\operatorname{rk} J = 2$. Hence we try to solve the following system

$$\begin{cases} x^2(v^2 - x + w^2) \\ 7 - xw \\ -2vw \\ -2vx \\ x + 2w^2 \end{cases}$$

and see there are no solutions. Hence X_1 is regular

$v = 1$

$$X_2 = \begin{cases} y^2 = x^3 - 7 \\ 7u = xw \\ 7 = yw \\ uy = x \end{cases}$$

We also see this is smooth regular

$w = 1$

$$X_3 = \begin{cases} y^2 = x^3 - 7 \\ 7u = x \\ 7v = y \\ uy = xv \end{cases}$$

Again this is regular

Hence we have that $\widetilde{X}$ is a regular model of C over $\mathbb{Z}$.

We now consider a second example: Let $C = \operatorname{Proj}(\mathbb{Q}[x, y, z]/(y^2z - x^3 + 49z^3)$ be a projective curve. We want to find a regular model over $\mathbb{Z}$.

We try $Y = \operatorname{Proj}(\mathbb{Z}[x, y, z]/(y^2z - x^3 + 49z^3)$. Let us cover Y with three charts Y_1, Y_2 and Y_3 which correspond respectively to $z = 1, y = 1$ and $x = 1$.

Look at Y_1, this is X of the previous example. So again, blow it up to get $\widetilde{X}$.

$Y_2 = \operatorname{Spec}(\mathbb{Z}[x, z]/(z - x^3 + 49z^3))$. If we look at the Jacobian, we get $J(x, z) = (-3x^2, 1 + 3 \cdot 49z^2)$, so we try to solve the simultaneous equations

$$\begin{cases} z - x^3 + 49z^3 = z(1 + 49z^2) - x^3 = 0 \\ -3x^2 = 0 \\ 1 + 3 \cdot 49z^2 = 0 \end{cases}$$

Hence $Y_2 \to \mathbb{Z}$ is smooth and Y_2 is regular

The same calculation for Y_3 shows that Y_3 is regular

$\widetilde{Y}$ regular model is obtained by blowing up Y_1 as in the first example.

4 Elliptic Curves

Definition 4.1. An *elliptic curve* over a field K is a pair (E, O) where E is a smooth projective curve of genus 1 over K, and $O \in E(K)$.

Let $T = \operatorname{Spec} A$ be an integral affine scheme.
K be the field of rational functions, $K = \operatorname{Frac}(\mathcal{O}_X(V)) \cong \mathcal{O}_{X,\zeta}$ where ζ is generic point

Definition 4.2. A *Weierstrass model* of (E, O) elliptic curve over T is a triple (f, W, ϕ) where

- $f \in A[x, y, z]$ homogeneous polynomial (called Weierstrass polynomial). $f(x, y, z) = y^2z + a_1xyz + a_3yz^2 - x^3 - a_2x^2z - a_4xz^2 - a_6z^3$
- $W = \operatorname{Proj}\left(A[x, y, z]/(f(x, y, z))\right)$
- ϕ is an isomorphism $\phi : E \xrightarrow{\sim} W \times_{\operatorname{Spec} T} \operatorname{Spec} K$ with $O \mapsto (0 : 1 : 0)$

The Weierstrass model over K is defined similarly

Theorem 4.3. *If (E, O) is an elliptic curve and has a Weierstrass module over K, then it has a Weierstrass model over T.*

Proof. Idea: make f integral. Take the affine chart $z = 1$, $f(x, y) = \dots$, $K = \operatorname{Frac}(A)$, there exists $0 \neq l \in A$, such that $f_1 = l^6 f \in A[x, y]$. Take change or coordinates $l^2x = X$ and $l^3y = Y$ □

Example. Take $y^2 = x^3 + x$ over $\mathbb{Q}$, this is the same as $v^2 = u^3 + 16u$.

Given a Weierstrass polynomial, we can define $\Delta = \operatorname{disc}(f)$ (it is also the discriminant of the model). The *minimal Weierstrass model* is a Weierstrass model which has minimal discriminant. Over $\mathbb{Q}$ there exists a minimal Weierstrass model.

Remark. The minimal Weierstrass model do not need to coincide with the minimal regular model

Example. Let p be a prime in $\mathbb{Z}$ and consider $\mathbb{Q}_p$. Consider $y^2z = x^3 + 2x^2z + 4z^3$ is integral. $\Delta = -2^8 \cdot 29$. Recall the valuation $v_p(n) = \operatorname{ord}_p(n) = \max\{m \in \mathbb{N} : p^m | n\}$, so

$$v_p(\Delta) = \begin{cases} 8 & p = 2 \\ 1 & p = 29 \\ 0 & \text{else} \end{cases}$$

There is a theorem which say if for all $p \in \operatorname{Spec} \mathbb{Z}$ we have $0 \leq v_p(\Delta) < 12$, then it is minimal. So $X = \operatorname{Proj}(\mathbb{Z}[x, y, z,]/(yz^2 - x^3 - 2x^2z - 4z^3))$ is a minimal Weierstrass model. We show that it is <u>not</u> regular. Look at the affine chart $z = 1$, $\mathfrak{m} = (x, y, 2)$ then $\dim(\mathfrak{m}/\mathfrak{m}^2) = 3 > 2$. Hence not regular.