1. Recallings
2. Models
3. Examples (Blowup)
4. Elliptic Curves

(1)

Let X be a scheme

REDUCED at $x \in X$ if the stalk $\mathcal{O}_{X,x}$ is a reduced ring (it has no nilpotents elements)

REDUCED if it is reduced at all pts.

Ex. · $\mathrm{Spec}\left(k[x]/(x^2) \right)$ not reduced.

Varieties are always reduced.

IRREDUCIBLE if it is irreducible as topological space

INTEGRAL = REDUCED + IRREDUCIBLE

Ex. · $\left.\begin{array}{l} \mathrm{Spec}\left(k[t]/f(t) \right) \\ f \text{ irreducible poly} \end{array}\right\}$ INTEGRAL

· $\mathrm{Spec}(A \times B)$ is not integral $A, B \neq 0$

NORMAL at $x \in X$ if $\mathcal{O}_{X,x}$ is a normal domain (i.e. it is integrally closed in its fraction field)

NORMAL if $\forall x \in X$ ....

Remark: NORMAL $\Rightarrow$ CONNECTED

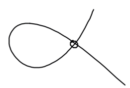
Ex.  $y^2 = x^2(x-1)$ is not normal

DEDEKIND = NORMAL + locally Noetherian of $\underbrace{\text{dimension } 1}_{\text{Krull dimension}}$.

maximal lenght of a chain of irreducible closed subschemes

REGULAR at $x \in X$ if $\mathcal{O}_{X,x}$ is regular i.e. $\mathcal{O}_{X,x}$ is a local ring, $\mathfrak{m}_x$ its maximal id, $\mathcal{O}_{X,x}/\mathfrak{m}_x = k$

$\underbrace{\dim \mathcal{O}_{X,x}}_{\text{KRULL dim}} = \dim_k \underbrace{\left(\mathfrak{m}_x / \mathfrak{m}_x^2 \right)}_{k\text{-vector space}}$

Ex.  not regular

$f: X \to Y$ morphism of schemes
$V \subseteq Y$ affine open, $U \subseteq f^{-1}(V)$ affine open
then $\mathcal{O}_X(U)$ is an $\mathcal{O}_Y(V)$-algebra.

$$\begin{array}{c} f: X \longrightarrow Y \\ f^{-1}(V) \longrightarrow V \\ \cup \\ U \end{array}$$

if f is quasi-compact and for all U, V as before $\mathcal{O}_X(U)$ is finitely generated as $\mathcal{O}_Y(V)$-algebra then f is of FINITE TYPE.

<u>FINITE</u> if $\forall V \subset Y$ open affine $f^{-1}(V) \subset X$ is affine and of finite type as module.

<u>FLAT</u> at $x \in X$ if $f^{\#}: \mathcal{O}_{Y, f(x)} \to \mathcal{O}_{X,x}$ is a flat morph. of rings i.e. $\mathcal{O}_{X,x}$ is flat as $\mathcal{O}_{Y, f(x)}$-module

<u>Def</u> k field, X be a k-scheme of finite type. Let $\overline{k}$ be an algebraic closure of k. X is <u>smooth</u> <u>at $x \in X$</u> if the points lying above it in $X_{\overline{k}}$ are regular pts.
(SMOOTH ___ $\forall x \in X$ ...)

<u>Def</u>. $f: X \to Y$ morphism of finite type, suppose the rings locally Noetherians, f is <u>smooth</u> at $x \in X$ if it is FLAT and $X_{f(x)} \longrightarrow \operatorname{Spec} k(f(x))$ is smooth at x.

2.

Def. k field. A <u>CURVE</u> over k is a k-scheme of finite type whose irreducible components have dim 1.

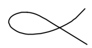
Def S scheme, a CURVE over S is a <u>flat</u> S-scheme whose fibres are curves over the corresponding residue fields.

From now on:
let S be a Dedekind scheme
$K = K(S)$ rational functions of S

Def A FIBRED SURFACE over S is an <u>integral</u> <u>projective</u> <u>flat</u> scheme over S
$X \longrightarrow S$ of dimension 2.

Def Let C be a smooth projective connected over K. A MODEL $\mathcal{C}$ over S of C is a normal fibred surface $\mathcal{C} \longrightarrow S$ togheter with an iso: $\mathcal{C}_\eta \cong C$

$$\begin{tikzcd} C \arrow[r, "\cong"] \arrow[dr] & \mathcal{C} \times_S K \cong \mathcal{C}_\eta \arrow[d] \arrow[r] & \mathcal{C} \arrow[d] \\ & K \arrow[r] & S \end{tikzcd}$$



Def A regular fibred surface $X \to S$ is MINIMAL if every Birational map $Y \dashrightarrow X$ of regular fibred surfaces is a birational morphism.

> A rational map $Y \dashrightarrow X$ is an equivalence class of maps $(U, f_U : U \dashrightarrow X)$
> U open f morph.
> two maps equivalent if they agree on a non-empty open intersection of their dom.

MINIMAL REGULAR MODEL.

Theorem (Liu, 9.3.21) $X \to S$ regular fibred surface with generic fiber X_η of arithmetic genus ≥ 1. Then X admits a unique minimal model over S up to unique iso.

$1 - \chi_k(\mathcal{O}_{X_\eta})$

X affine variety over k, k field, char$(k) = 0$, $\bar{k}$ alg clos.

$X_{\bar{k}}$ $\circlearrowleft$ triangulation

$$\chi = V - E + F$$
$$\chi = 2 - 2g \qquad g \text{ genus.}$$

Jacobian criterion

k field

X affine variety, closed $X \subseteq \mathbb{A}^n_k$ ($T_1 \dots T_n$)

$X = V(I)$

$x \in X(k)$

I: $F_1 \dots F_r$ generators for I

$$J_x = \left(\frac{\partial F_i}{\partial T_j}(x) \right)_{\substack{1 \leq i \leq r \\ 1 \leq j \leq n}} \in M_{r \times n}(k)$$

Then X is regular at x <u>iff</u> $\operatorname{rank} J_x = n - \dim \mathcal{O}_{X,x}$

③. Example

$$C = \operatorname{Spec}\left(\mathbb{Q}[x,y] \big/ (y^2 - x^3 + 49) \right)$$

curve

Construct a regular model over $\mathbb{Z}$.

$$X = \operatorname{Spec}\left(\mathbb{Z}[x,y] \big/ (y^2 - x^3 + 49) \right)$$

$y^2 - x^3 + 49$

mod 7 : $y^2 = x^3$

$\underline{m} = (x, y, 7)$

$\dim_{\mathbb{F}_7} \left(\underline{m} / \underline{m}^2 \right) = 3 > 2$

the scheme X is not reg. at $\underline{m}$

$$\underset{\tilde{X}}{\overset{\|}{Bl_{\underline{m}} X}} = \begin{cases} y^2 = x^3 - 7^2 \\ 7u = xw \\ 7v = yw \\ uy = xv \end{cases}$$

$u : v : w$ projective

is $\tilde{X}$ regular? regularity is <u>local</u>

$\boxed{u=1}$ $X_1 = \begin{cases} y^2 = x^3 - 7^2 \\ 7 = xw \\ 7v = yw \\ y = xv \end{cases}$ → $yw = xvw$, $yw = 7v$, $xvw = vxw = 7v$

$X_1 = \begin{cases} x^2 v^2 = x^3 - x^2 w^2 \\ 7 = xw \end{cases}$

$X_1 = \begin{cases} x^2 (v^2 - x + w^2) = 0 \\ 7 - xw = 0 \end{cases}$

Jacobian crit $J(x, v, w) = \begin{pmatrix} -1 & 2v & 2w \\ -w & 0 & -x \end{pmatrix}$

X is reg. <u>iff</u> $\forall x \in X$ $\operatorname{rk} J = 2$

$$\begin{cases} v - x + w^2 = 0 \\ 7 - xw = 0 \\ \text{minors of } J = 0 \end{cases}$$

X_1 reg.

$\boxed{v=1}$ $X_2 = \begin{cases} y^2 = x^3 - 7^2 \\ yw = 7 \\ uy = x \\ 7u = xw \end{cases}$

smooth
reg.

$\boxed{w=1}$ $X_3 \begin{cases} 7^2 (v^2 - 7u^2 - 1) = 0 \\ 7u = x \\ 7v = y \end{cases}$ regular.

$\Rightarrow \tilde{X}$ is a regular model of C over $\mathbb{Z}$.

EX.2

$$C = \mathrm{Proj}\left(\mathbb{Q}[x,y,z] \big/ (y^2 z - x^3 + 49 z^3) \right)$$

reg. model over $\mathbb{Z}$

$$Y = \mathrm{Proj}\left(\mathbb{Z}[x,y,z] \big/ \quad \right)$$

$\underbrace{Y_1}_{z=1}, \underbrace{Y_2}_{y=1}, \underbrace{Y_3}_{x=1}$

Y_1 is X of EX 1.

$\rightsquigarrow \tilde{X}$

$$Y_2 = \mathrm{Spec}\left(\mathbb{Z}[x,z] \big/ (z - x^3 + 49 z^3) \right)$$

$$J(x,z) = (-3x^2, 1 + 3 \cdot 49 z^2)$$

$$\begin{cases} -3x^2 = 0 \\ 1 + 3 \cdot 49 z^2 = 0 \\ z^2 - x^3 + 49 z^3 = z(1 + 49 z^2) - x^3 = 0 \end{cases}$$

$Y_2 \to \mathrm{Spec}\, \mathbb{Z}$ is smooth

Y_2 reg.

Y_3. regular

$\tilde{Y}$ regular model is obtained by blowing up Y_1 as in EX 1.

4. Elliptic curves.

Def An ELLIPTIC CURVE over a field K is a pair (E, O) where E is a smooth projective connected curve of genus 1 over K and $O \in E(K)$.

$T = \operatorname{Spec} A$ integral affine scheme
K = field of rat. functions
$K = \operatorname{Frac}(\mathcal{O}_X(V)) \simeq \mathcal{O}_{X,\eta}$ η generic.

Def. A WEIERSTRASS MODEL (W-Mod.) of (E, O) ell. curve over T is a triple (f, W, φ)

- $f \in A[x, y, z]$ hom. poly, Weierstrass poly

$$f(x,y,z) = y^2 z + a_1 xyz + a_3 yz^2 - x^3 - a_2 x^2 z + - a_4 x z^2 - a_6 z^3$$

- $W = \operatorname{Proj}\left(A[x,y,z] / (f(x,y,z)) \right)$
- φ is an iso

$$\varphi : E \xrightarrow{\sim} W \times_{\operatorname{Spec} T} \operatorname{Spec} K$$
$$O \longmapsto (0:1:0)$$

W-mod over K (similar way)

Thm if (E, O) is an ell. curve and it has a W-mod over K then it has a W-mod over T.

pf. Idea make f integral. Affine chart ($z=1$)

$f(x,y) = \text{---}$
$K = \operatorname{Frac}(A)$ $\exists\, 0 \neq \ell \in A$
s.t. $f_1 = \ell^6 f \in A[x,y]$
take change coord $\ell^2 x = X$
$\ell^3 y = Y$

EX. $y^2 = x^3 + x$ $/\mathbb{Q}$
$v^2 = u^3 + u \cdot 16$

Weierstrass poly
$\Delta = \operatorname{disc}(f) = \Delta_W$

MINIMAL W-mod
(minimal discrim.)
over $\mathbb{Q}$ $\exists$ minimal Weierstrass mod

Remark The minimal W-mod do not need to coincide with the minimal regular model.

EX.

p prime in $\mathbb{Z}$, $\mathbb{Q}_p$

$$y^2 z = x^3 + 2x^2 z + 4z^3$$

integral $\qquad \Delta = -2^8 \cdot 29$

$$v_p(n) = \mathrm{ord}_p(n) = \max\{m \in \mathbb{N} \text{ s.t. } p^m \mid n\}$$

$$v_p(\Delta) = \begin{cases} 8 & p=2 \\ 1 & p=29 \\ 0 & \text{otherwise} \end{cases}$$

if $\forall p \in \operatorname{Spec} \mathbb{Z}$ $\quad 0 \le v_p(\Delta) < 12$

then MINIMALITY.

$$X = \operatorname{Proj}\left(\mathbb{Z}[x,y,z] \big/ (y^2 z - x^3 - 2x^2 z - 4z^3)\right)$$

minimal W-mod.

it IS NOT REGULAR

affine chart $z=1$, $\quad \underline{m} = (x, y, 2)$

$\dim \underline{m}/\underline{m}^2 = 3 > 2$

not reg.

$$y^2 = x^3 - 2x^2 - 4 =$$
$$= x^2(x-2) - 4$$