Goal:

Understand the theory behind blowdown for arithmetic surfaces.

$\rightsquigarrow$ minimal model

canonical model

Give intuition + where hypotheses (normality, regularity)

0. Prerequisites

0.1 Sheaves of differentials

Start with rings: $A \xrightarrow{f} B$

Write $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$

Let $\Omega^1_{B/A} = \sum B\,dx_i \Big/ \langle df_i, da \rangle$, $a \in A$, d : d is Leibniz

This is a B-module

<u>Ex</u> $A = k[y] = k[x^n] \hookrightarrow B = k[x]$

So $B = A[t]/(t^n - y)$

$\Omega^1_{B/A} = Bdt/d(t^n - y)$

$= Bdt/nt^{n-1}dt - dy$

$= Bdt/nt^{n-1}dt$

$= B/nt^{n-1}$

This B-module corresponds to a sheaf on Spec B supported at 0 only. This is exactly where the map $x \longmapsto x^n$ ramifies. $*$

$B' = k[x, x^{-1}]$: $\Omega^1_{B'/A} = 0$.

Ω^1 detects smoothness and ramification

Let $X \xrightarrow{f} Y$ be a morphism of schemes. Then this construction sheafifies and gives $\Omega^1_{X/Y}$, a sheaf on X.

<u>Properties</u>

- $f: X \to Y$ equidimensional fibres: dim n

$x \in X$ pt

f is smooth at $x \iff \Omega^1_{X/Y}$ is locally free of dim n around x

f smooth $\iff \Omega^1_{X/Y}$ locally free of rank n

$\Updownarrow$

fibers of f all smooth

- $Z \overset{i}{\hookrightarrow} X$ closed with defining sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$.

In general, there is a seq

$$\underbrace{i^*(\mathcal{I}/\mathcal{I}^2)}_{C_{X|Y} \text{ conormal sheaf}} \to \Omega^1_{X|Y} \otimes_{\mathcal{O}_X} \mathcal{O}_Z \longrightarrow \Omega^1_{Z|Y} \longrightarrow 0$$

- X smooth $|Y$: then Z is smooth $\Longleftrightarrow$ this sequence is left exact

0.2 Local complete intersections

$f: X \longrightarrow Y$ morphism of schemes
f is an lci if for every $x \in X$ there is $x \in U$ open neigh. s.t.

$$\begin{tikzcd} B & U \arrow[rr, "f"] & & Y \\ A \arrow[u] & & Z \arrow[ul, hook', "\text{reg. immersion}"] \arrow[ur, "\text{smooth}"'] & \end{tikzcd}$$

Reg. immersion: on rings this coresps to $B = A/(x_1, \dots, x_d)$ where x_i is not a zero div in $A/(x_1, \dots, x_{i-1})$ for all $i \le d$.
"Successive quotient by non-zero divisors"

Geometrically: U defined by a number of equations equal to its codim. in Z.

Intuition: $Y = \operatorname{Spec} k$.
Then Z is a smooth variety $/k$ (such as $\mathbb{A}^n_k$)
and X is locally defined by an appropriate # of eqs.
Complete intersection: same with $U = X$.

More restrictive.

Ex LCI non CI: twisted cubic

$$\operatorname{Proj} k[x, y, z, w] / \underbrace{(xz - y^2, yw - z^2, xw - yz)}_{3 \neq 2}$$

Ex (1) Curves over a field are lcis except if they have embedded

Nonlci: $k[x, y]/(x^2, xy)$

lci $k[x, y]/(xy)$

(2) R Dedekind ring: $F \in R[x,y]$

$R[x,y]/(F)$ is an lci.

(3) $X \xrightarrow{f} Y$ morphism of

$\underline{\text{reg.}}$ schemes is an lci.

$\underline{\text{Def}}$ The $\underline{\text{canonical sheaf}}$ of an lci

$X \longrightarrow Y$ is

$$\omega_{X|Y} = \det\left(C_{X|Y}^{\vee}\right) \otimes_{\mathcal{O}_X} f^*(\det \Omega_{Y|})$$

This is locally free of rank 1.

$\underline{\text{Ex}}$ $X =$ curve smooth over

$Y = \operatorname{Spec} k$,

then $\omega_{X|Y} = \Omega^1_{X|k}$.

$\underline{\text{Properties}}$:

- $\omega_{X|Y}$ is stable under flat base change

$$\begin{tikzcd}
X' \arrow[r, "p"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\
Y' \arrow[r] & Y
\end{tikzcd}$$

either flat

$\omega_{X'|Y'} = p^* \omega_{X|Y}$.

Additionally,

$f: X \longrightarrow Y$ is an lci iff

it is so fiberwise.

- For composition:

$f: X \longrightarrow Y$, $g: Y \longrightarrow Z$

$$\omega_{X|Z} = \omega_{X|Y} \otimes_{\mathcal{O}_X} f^*(\omega_{Y|Z})$$

$\rightsquigarrow$ Riemann–Hurwitz

The sheaf $\omega_{X|Y}$ gives Serre

duality: $H^0(X, \omega_X) = H^d(X, \mathcal{O}_X)$

Grothendieck: there is a map

$$R^r f_* \omega_f \xrightarrow{tr} \mathcal{O}_Y$$

induces

$$f_* \underline{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_{X|Y})$$

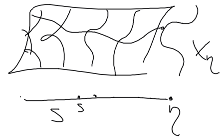
$$\cong \underline{Hom}_{\mathcal{O}_Y}(R^r f_* \mathcal{F}, \mathcal{O}_Y)$$

1. Arithmetic surfaces

Def A <u>fibered surface</u> is an integral surface X with a projective, flat map

$$\pi : X \longrightarrow S$$

where S is a one-dimensional Dedekind scheme (like $\mathrm{Spec}\,\mathbb{Z}$)



- Divisors on X come in two flavors:
  - vertical ones (components of special fibers X_s)
  - horizontal ones (closure of points in X_η)

Properties:

- X_η geometrically integral (like a smooth curve)

  $\Longrightarrow \mathcal{O}_S \xrightarrow{\sim} \pi_* \mathcal{O}_X$
- If X_η is smooth, then there are only finitely many non-smooth fibers.

  (Pf: Smooth locus is open (use $\Omega^1_{X/S}$) and non-empty. Its complement is closed, hence so is its image under π (proper). Therefore this image is finite)
- $\omega_{X/S}\big|_{X_s} = \omega_{X_s/k(s)}$

  follows from base change
- $p_a(X_s) = p_a(X_\eta)$

$$p_a(C) = 1 - h^0(C, \mathcal{O}_C) + h^1(C, \mathcal{O}_C)$$
$$= h^0(C, \omega_C) = g(C)$$

if C is smooth

$\pi : X \longrightarrow S$ is called <u>normal</u> if X is, and <u>regular</u> (or an <u>arithmetic surface</u>) if X is regular.

1.1 Desingularization.

Process of finding $Y \dashrightarrow X$ birational (isomorphism $Y_\eta \longrightarrow X_\eta$) such that Y is regular.

If X_η smooth, one can do:

X

$\uparrow$

X_1 = normalization of X

$\uparrow$

X_1' = blowup of remaining sing in X_1.

<u>Fact</u>: This stops and gives a regular surface at some points.

$\uparrow$

X_2

$\uparrow$

X_2'

$\uparrow$

<u>Fact</u>: After further blowup, all special fibers can be taken to have normal crossings:

1.2 Contraction

$X \longrightarrow S$ arithmetic surface

E component of a special fiber X_s.

We want typically to construct a morphism

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \ \swarrow & \\ & S = \operatorname{Spec} R & \end{array}$$

contracting E:

$$\begin{cases} f(E) = pt \\ f \text{ is an isomorphism outside } E. \end{cases}$$

This is done by using invertible sheaves.

$\mathcal{L}$ inv on X: $H^0(X, \mathcal{L}) = Rs_0 \oplus \dots \oplus Rs_n$.

This gives a morphism

$$f_{\mathcal{L}} : X \longrightarrow \mathbb{P}^n_R$$
$$x \longmapsto (s_0(x) : \dots : s_n(x))$$

which is well-defined as long as

$$\mathcal{L}_x = \sum_i s_i \mathcal{O}_{X,x}$$

($\mathcal{L}$ should be <u>generated by its global sections</u>)

$f = f_{\mathcal{L}}$. $Z < X_s$ proj comp of fiber.

$$f(Z) = pt \iff \mathcal{L}|_Z = \mathcal{O}_Z$$

$\Longrightarrow$: suppose $pt = (1 ; 0 \dots ; 0)$. Then by construction s_0 generates on all points above y.

$\Longleftarrow$: Restricting to Z, the space $H^0(Z, \mathcal{L})$ is finite (Z proj). So we get

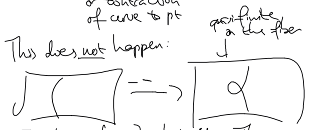
$$f|_Z : Z \longrightarrow H^0(Z, \mathcal{L}) \longrightarrow \mathbb{P}^n_A$$

<u>Fact</u>: birational maps $X \dashrightarrow Y$ between <u>normal</u> fibered surfaces are sequences

$$X \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow Y.$$

blowup of a pt or contraction of curve to pt

This does <u>not</u> happen:



Excluded by <u>Zariski's Main Theorem</u>.

<u>Contraction criteria</u>

- $\mathcal{E}$ set of vertical divisors: Contraction of $\mathcal{E}$ exists
  $\Longleftrightarrow$
  $\exists D$ Cartier div on X st

$$\begin{cases} \deg(D|_{X_\eta}) > 0 \\ \mathcal{O}_X(D) \text{ gen. by its global sections} \\ \mathcal{O}_X(D)|_E \cong \mathcal{O}_E \text{ for } E \text{ vert.} \\ \qquad \Longleftrightarrow E \in \mathcal{E}. \end{cases}$$

- Over affine S, for any effective horizontal Cartier divisor D the sheaf $\mathcal{O}_X(nD)$ is gen by its global sections if $n \gg 0$.
- $D+E$ may not be gen by glob seq even if D and E are!

<u>1.3</u> Intersection theory

X fibered surface, D, E divs on X: suppose that D, E no common comp: D, E then intersect in finitely many points

Suppose $x \in X$ point of intersection: we set

$$i_x(D,E) = \operatorname{length}_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x} / (\mathcal{O}_{X,x}(-D) + \mathcal{O}_{X,x}(-E))$$

and let

$$i(D,E) = \sum_{x \in X} i_x(D,E)$$

Alternatively:

$D|_E = j^*(D)$ then $i_x(D,E) = \operatorname{mult}_x(D|_E)$

$j : E \hookrightarrow X$

On a fibered surface we get for $s \in S$

$$i_s : \operatorname{Div}(X) \times \operatorname{Div}_s(X) \longrightarrow \mathbb{Z}$$

If E is a component of X_s then

$$i_s(D,E) = \deg_{k(s)} \mathcal{O}_X(D)|_E$$

Properties:

- X_s fiber of $X \to S$: $\cdot_s$ is neg. def, and $x \cdot x = 0$ implies $x \in \mathbb{Z} X_s$.
- $X \longrightarrow Y$ contraction
  $\Gamma_i \longmapsto y$

$$\sum n_i n_j \Gamma_i \cdot \Gamma_j \leq 0.$$

equality $\iff n_i = 0$.

- Hodge index theorem for ordinary surfaces

$$\mathrm{Pic}(X) \times \mathrm{Pic}(X) \longrightarrow \mathbb{Z}$$

has signature $(1, -1, \ldots, -1)$

- P point of X_η:

$$\overline{\{P\}} \cdot X_s = [K(P) : K(S)]$$

$= 1$ if P is a rational point.

- Take K st $\omega_{X/S} = \mathcal{O}_X(K)$.

Then

$$\begin{aligned}
2p_a(X_\eta) - 2 &= \deg \omega_{X_\eta / k(\eta)} \\
&= -2\chi_{k(\eta)}(\mathcal{O}_{X_\eta}) \\
&= -2\chi_{k(s)}(\mathcal{O}_{X_s}) \\
&= \deg(\omega_{X_s/k(s)}) \\
&= \deg(\mathcal{O}_X(K)|_{X_s}) \\
&= K \cdot X_s \\
&= \sum d_i (K_{X/S} \cdot \Gamma_i)
\end{aligned}$$

Γ_i comps of X_s

$d_i = \mathrm{length}(\mathcal{O}_{X, \Gamma_i})$