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Zariski Topology

A be a commutative ring

$\operatorname{Spec} A = \{$ proper prime ideal $\}$

For any ideal I of A, let

$V(I) = \{ p \in \operatorname{Spec} A \mid I \subseteq p \}$

$D(f) = \operatorname{Spec} A \setminus V(\langle f \rangle)$

Prop

- $V(I) \cup V(J) = V(I \cap J)$
- $\bigcap_\lambda V(I_\lambda) = V(\sum_\lambda I_\lambda)$
- $V(A) = \emptyset$, $V(0) = \operatorname{Spec} A$

$D(f)$ is called principal open subset

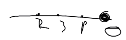
$V(f)$ " " " closed "

Note: $p \in \operatorname{Spec} A$, $\{p\}$ is closed iff p is maximal.

Def^n p is a closed pt

Example

$\operatorname{spec} \mathbb{Z}$

$\exists$ a generic pt 

2) let k be a field, $\mathbb{A}^1_k = \operatorname{Spec} k[x]$

$\xi \sim (0)$ generic pt

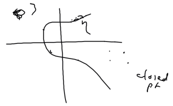
The closed pts correspond to maximal ideals of $k[x]$.

Let I be an ideal, let $I = \langle p(x) \rangle$

let $p(x) = \prod_i p_i(x)$

$V(I) = \{p_1(x), \dots, p_n(x)\}$

3) $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$ $\quad k = \bar{k}$



$\xi \sim \{0\}$, $\langle x-a, y-b \rangle \sim (a,b) \in k^2$

$f(x,y) \sim$ $\underline{\eta \text{ a generic}}$ pt of $f(x,y)$ whose closure is η & all pts (a,b) s.t. $f(a,b) = 0$. $\quad V(\langle f(x,y) \rangle) = \{a,b\} + \eta$

<u>Def^n</u> A generic pt of X is a point ξ s.t. $\bar{\xi} = X$.

<u>Sheafs</u>

<u>Def^n</u> let X be a topo space. A presheaf $\mathcal{F}$ (of ab gp) on X consist of the following data:

- An ab gp $F(U)$ for every $U \underset{\text{open}}{\subseteq} X$
- For every pair $V \subseteq U \subseteq X$ a gp homo

  $\rho_{UV} : F(U) \to F(V)$

  called restriction map

s.t.

1) $F(\phi) = \{1\}$
2.) $\rho_{UU} = id$
3.) $W \subseteq V \subseteq U$, $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$

<u>Notation</u>: An element $s \in F(U)$ is called a <u>section</u> of $F(U)$ on U. $s|_V$ denote $\rho_{UV}(s) \in F(V)$, called the <u>restriction</u> of s on V.

<u>Def^n</u>: A presheaf $\mathcal{F}$ is a sheaf if it satisfy

4) Uniqueness: let $U \underset{\text{open}}{\subseteq} X$, let $s \in F(U)$ & let $\{U_i\}$ be a covering of U. If $s|_{U_i} = 0 \ \forall i$, then $s = 0$

5.) Notation as above: let $s_i \in F(U_i)$ be sections s.t. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ then $\exists s \in F(U)$ s.t. $s|_{U_i} = s_i$.

__Def^n__: __Subsheaf__ $\mathcal{F}'$ of $\mathcal{F}$

$F'(U)$ is a subgp of $F(U)$ & ρ_{UV} is induced from ρ_{UV}.

__Example__: Let k be a field & X a topo space. For any $U \underset{\text{open}}{\subseteq} X$ let $C(U) = C^0(U, k)$ i.e. cont^n functions from U to k. Let ρ_{UV} be the usual restriction of funct^n.

Then C is a sheaf.

__Note__: Every (pre) sheaf $\mathcal{F}$ on X induces a (pre) sheaf $\mathcal{F}|_U$ on $U \underset{\text{open}}{\subseteq} X$.

Let $\mathcal{B}$ be a basis of X. Define $\mathcal{B}$-presheaf & $\mathcal{B}$-sheaf by replacing $U \underset{\text{open}}{\subseteq} X$ by $U \in \mathcal{B}$.

Let F_0 be a $\mathcal{B}$-sheaf. Extend this to a sheaf $\mathcal{F}$ on X since $U \subseteq X$ $U = \bigcup U_i$, $U_i \in \mathcal{B}$. $F(U)$ is the set of elements $(s_i)_i \in \prod_i F_0(U_i)$

s.t. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$

__Def^n__ Let $\mathcal{F}$ be a presheaf on X. Let $x \in X$. The __stalk__ of $\mathcal{F}$ at x is the gp

$$F_x = \varinjlim_{x \in U} F(U) \qquad \text{D}$$

Let $s \in F(U)$, for any $x \in U$ we denote the image of s in F_x by s_x. This, s_x, is called the __germ__ of s at x.

The map $F(U) \to F_x$ defined by $s \mapsto s_x$ is a gp homo.

__Example__: Back to C on X. Then C_x is the set of funct^n which are cont^n on x.

__Lemma__ Let F be a sheaf on X. Let $s, t \in F(X)$ be section s.t. $s_x = t_x$ $\forall x \in X$. Then $s = t$.

__Pf__: WLOG assume $t = 0$.

$\forall x \in X$, $\exists$ open U_x of x s.t. $s|_{U_x} = 0$ since $s_x = 0$.

As U_x cover X when x varies, we have $s = 0$. $\boxtimes$

__Def^n__ Let $\mathcal{F}$ & $\mathcal{G}$ be two presheaf on X. A __morphism__ of presheaf $\alpha : \mathcal{F} \to \mathcal{G}$ consist of gp homo $\alpha(U) : F(U) \to G(U)$ $\forall U \subseteq X$ which makes the following diagram commutes

$$\begin{tikzcd} F(U) \arrow[r, "\alpha(U)"] \arrow[d, "\rho_{UV}"'] \arrow[dr, phantom, "\circlearrowleft"] & G(U) \arrow[d, "\rho'_{UV}"] \\ F(V) \arrow[r, "\alpha(V)"] & G(V) \end{tikzcd}$$

α is __injective__ if $\alpha(U)$ $\forall U$ is.

An __isomorphism__ is an invertible morphism, i.e. $\alpha(U)$ $\forall U$ is an isomorphism

For any $x \in X$, α induces a gp homo $\alpha_x : F_x \to G_x$ s.t. $(\alpha(U)(s))|_x = \alpha_x(s_x)$

We say α is __surjective__ if α_x is $\forall x$

__Ex__ We can define a morphism between the sheaf of diff funct^n to the sheaf of cont^n function.

__Prop__: Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Then α is iso iff α_x is iso $\forall x \in X$.

__Pf__: $\Rightarrow$) Clear

$\Leftarrow$) Let $s \in F(U)$. If $\alpha(U)(s) = 0$

$\forall x \in U$, we have $\alpha_x(s_x) = (\alpha(U)(s))_x = 0$

As α_x is iso, $s_x = 0$ $\forall x$ $\Rightarrow s = 0$

Let $t \in G(U)$. Then we can find refinement $\{U_i\}$ & $s_i \in F(U_i)$ s.t. $\alpha(U_i)(s_i) = t|_{U_i}$. As α injective s_i & s_j coincide on $U_i \cap U_j$.

So glue s_i to get $s \in F(U)$

$s|_{U_i} = s_i$

$(\alpha(U_i)(s_i))_x = t_x$

$\varinjlim_{x \in U} F(U)$

__Def^n__: Let $f : X \to Y$ be a ct^n map of topo space, $\mathcal{F}$ a sheaf on X, $\mathcal{G}$ a sheaf on Y.

Then $V \subseteq Y \mapsto F(f^{-1}(V))$ defines a sheaf $f_* F$ on Y called the __direct image__ or __pushforward__ of $\mathcal{F}$

Inverse image of $\mathcal{G}$ denote $f^{-1}\mathcal{G}$ which is the sheaf associated to the presheaf $U \mapsto \lim_{f(U) \subseteq V} G(V)$

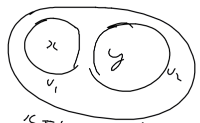
$$(f^{-1}G)_x = G_{f(x)}$$

Constant presheaf

Let A be an Ab gp. Let X be a topological space.

$F(U) = A$

$\rho_{UV} = \mathrm{id}$



$x = y$ ↑ cannot define U_3 s.t. $U_3 \cap U_2 = x$, $U_3 \cap U_1 = y$

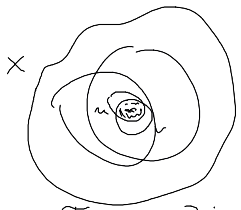
$U_i' \subseteq U_i'$

$x \in U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n \subseteq \cdots$

$F_x \leftarrow F(U_1) \leftarrow F(U_2) \leftarrow \cdots \leftarrow F(U_n)$

$C_x := \{\langle f, U\rangle \mid x \in U \subseteq X,\ f \text{ cont}\} / \sim$

$\langle f, U\rangle \sim \langle g, V\rangle$ if $f = g$ on $U \cap V$



$\mathcal{F}$

$\mathcal{F}(U), \mathcal{F}(V)$

$\mathcal{F}_x$ ab. gp

$z = \{(U, z_U)\}$

$z_V \in \mathcal{F}(V)$ $z_U \in \mathcal{F}(U)$

$(U, z_U) \sim (V, z_V) \iff$

$\rho_{U,U\cap V}(z_U) = z_U|_{U\cap V} = z_V|_{U\cap V} = \rho_{V,U\cap V}(z_V)$

$\rho_{U,U\cap V} : \mathcal{F}(U) \longrightarrow \mathcal{F}(U\cap V)$

$\rho_{V,U\cap V} : \mathcal{F}(V) \longrightarrow \mathcal{F}(U\cap V)$