Recap: Sheaves

Let X be a top. space.

- A **presheaf** $\mathcal{F}$ is some collection $\mathcal{F}(\mathcal{U})$ of abelian gps (rings, modules) for each open $\mathcal{U} \subset X$

- A presheaf $\mathcal{F}$ is a **sheaf** if it satisfies additional **local** conditions.

- define the stalk $\mathcal{F}_x$, for each $x \in X$.

- local conditions, we have some nice "local to global" properties

example: (the constant sheaf).

Let A be a group/ring, and define the **constant presheaf** to be the presheaf $\mathcal{F}(\mathcal{U}) = A$.

This is **not** a sheaf: Let $X = \{x_1, x_2\}$, and $A = \mathbb{Z}$. Then

$$\mathcal{F}(\{x_1\}) = \mathbb{Z},$$
$$\mathcal{F}(\{x_2\}) = \mathbb{Z},$$

take $2 \in \mathcal{F}(\{x_1\})$, $3 \in \mathcal{F}(\{x_2\})$

$\nexists \; s \in \mathcal{F}(X)$ st.

$$s\big|_{\{x_1\}} = 2, \quad s\big|_{\{x_2\}} = 3$$

We fix this by instead setting

$$\mathcal{F}(\mathcal{U}) = \bigoplus_{\substack{\text{connected} \\ \text{cpts of } \mathcal{U}}} A$$

$$\begin{tikzcd} & X \\ & \mathbb{Z} \times \mathbb{Z} \arrow[dl] \arrow[dr] & \\ \{x_2\} \; \mathbb{Z} & & \mathbb{Z} \; \{x_1\} \end{tikzcd}$$

Schemes

<u>Motivation</u>: Let X be some irreducible affine variety over an a.c. field K.

$\longrightarrow K[X]$ regular function ring.

Hilbert's Nullstellensatz $\Rightarrow$ $\exists$ bijection

$$\{K\text{-pts of } X\} \longleftrightarrow \{\text{prime ideals in } K[X]\},$$

$$p \longmapsto \mathfrak{m}_p = \{f(p) = 0\}.$$

<u>Definition</u>: Let A be a (commutative) ring. Define a space

$$X := \operatorname{Spec}(A) = \{\mathfrak{p} \lhd A : \mathfrak{p} \text{ prime}\}$$

We give X the <u>Zariski topology</u>, i.e. the closed sets are

$$V(f) = \{\mathfrak{p} \in X : (f) \subset \mathfrak{p}\},$$

f irreducible in A, giving rise to the principal open sets

$$D(f) = X \setminus V(f).$$

We put a sheaf of rings $\mathcal{O}$ on $\operatorname{Spec}(A)$. For each $D(f)$, define

$$\mathcal{O}(D(f)) = A\left[\tfrac{1}{f}\right]$$

(Here $A[\frac{1}{f}] = \{\frac{a}{f^n} : a \in A, n \in \mathbb{Z}\}$

e.g. $\mathbb{Z} = A$, then

$$\mathbb{Z}\left[\tfrac{1}{5}\right] = \{\tfrac{z}{5^n}, z \in \mathbb{Z}\}).$$

<u>Proposition</u>: $\mathcal{O}$ is a sheaf.

Proof: First, we need compatibility maps

If $D(f) \supset D(g)$, then

$g \in \sqrt{(f)} = \{ a \in A : \exists n \in \mathbb{N}$ s.t. $a^n \in (f) \}$.

e.g. $\sqrt{(x^2)} = (x)$

$\Rightarrow g^m = af$, for some $a \in A, m \in \mathbb{N}$.

Define a map

$A[\frac{1}{f}] \longrightarrow A[\frac{1}{g}]$ by

$bf^{-n} \longmapsto ba^n g^{-mn}$

So now let $\{\mathcal{U}_i\}$ be a covering of X.

<u>Claim:</u> $\exists$ a finite subcover.

Pf: Let $\mathcal{U}_i = D(f_i)$

Then $X = \bigcup_i D(f_i)$

$\Rightarrow \bigcap_i V(f_i) = \emptyset$

$= V(\sum(f_i)) = \{p \in X : p \supset \sum(f_i)\}$

$\Rightarrow \sum(f_i) = A$.

$\therefore \exists$ some finite set F s.t.

$1 = \sum_F a_i f_i$.

Then $\sum_F (f_i) = A$

$\Rightarrow X = \bigcup_F D(f_i)$

To prove the local conditions:

(4) Suppose $s \in \underset{= A}{\mathcal{O}(X)}$, with

$s|_{\mathcal{U}_i} = 0 \ \forall i$. We want $s = 0$.

For each $i \in F$,

$s|_{\mathcal{U}_i} = 0 \Rightarrow \exists m_i \in \mathbb{Z}$ s.t.

$sf_i^{m_i} = 0$.

Since F finite, $\exists m \in \mathbb{Z}$

s.t. $sf_i^m = 0 \quad \forall i \in F$.

Then as $\bigcup D(f_i^m)$

$$= \bigcup D(f_i) = X.$$

In particular $\sum (f_i^m) = A$

$\Rightarrow$ we can write $1 = \sum_{i \in F} a_i f_i^m$.

$s f_i^m = 0 \;\; \forall i \in F,$

$\sum_{i \in F} s a_i f_i^m = s = 0. \;\; \checkmark$

(5) Let $s_i \in \mathcal{O}(D(f_i))$

s.t. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$.

We want: some $s \in \mathcal{O}(X) = A$

s.t. $s|_{U_i} = s_i$.

Now, $D(f_i) \cap D(f_j) = D(f_i f_j)$

$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ means that

$\exists r \in \mathbb{Z}$ s.t. $(s_i - s_j)(f_i f_j)^r = 0$

Each $s_i \in A\left[\frac{1}{f_i}\right]$

$\Rightarrow s_i = c_i f_i^{-m_i}$, some m_i.

take $m = \max\{m_i : i \in F\}$.

Then $s_i = b_i f_i^{-m}$ for some $b_i \in A$.

Combined with the above, we get

$$(b_i f_i^{-m} - b_j f_j^{-m}) f_i^r f_j^r = 0$$

$$\Rightarrow b_i f_j^{m+r} f_i^r = b_j f_i^{m+r} f_j^r.$$

We still have some $a_i \in A$ s.t.

$$1 = \sum a_i f_i^{m+r}.$$

Define $s := \sum_{i \in F} a_i b_i f_i^r$

Then $s f_j^{m+r} = \sum_{i \in F} a_i b_i f_i^r f_j^{m+r}$

$$= \sum_{i \in F} a_i \, b_i f_i^{r} f_i^{m-r}$$

$$(s_i)\Big|_{D(f_i)} = b_i f_i^{r}$$

$\square$

Fact: the stalk of $\mathcal{O}$ at $\mathfrak{p} \in X$ is the local ring $A_{\mathfrak{p}}$.

<u>Def'n:</u> an affine scheme is a topological space X with a sheaf of rings s.t. $(X, \mathcal{O}_X)$ is isomorphic to $(\mathrm{Spec}(A), \mathcal{O})$ for some ring A.

(a morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^{\#})$

$f : X \to Y$ cts,

$f^{\#} : \mathcal{O}_Y \to f_* \mathcal{O}_X$ morphism of sheaves

s.t. the maps $f^{\#}_x$ are local homomorphisms

Example: (affine line).

Let $A = K[x]$, where K is a field.

Then define $\mathbb{A}^1_K = \mathrm{Spec}(K[x])$

① Let $A = K[x_1, \dots, x_n]$,

we get $\mathbb{A}^n_K$.

$$\mathcal{O}(D(f_i)) = \left\{ \frac{g}{f_i^n} : g \in K[x_1, \dots, x_n] \right\}$$

③ Let $f \in K[x_1, \dots, x_n]$ be irreducible, then $A := K[x_1, \dots, x_n]/(f)$.

Then $\mathrm{Spec}(A) \sim V((f))$

Summary:

- A comm. ring.
- $\mathrm{Spec}(A) = \{\text{prime ideals}\}$
- $\mathcal{U} \subset \mathrm{Spec}(A)$, $\mathcal{O}(\mathcal{U}) = A\left[\frac{1}{f}\right]$
  $\overset{\shortparallel}{D(f)}$
  gives a sheaf,
- Affine scheme: something that is iso to $(\mathrm{Spec}\, A, \mathcal{O})$

Ex: X topological spaces
$p \in X$, A an abelian gp.
Define a sheaf $i_p(A)$ as follows:

$$i_p(A)(\mathcal{U}) = \begin{cases} A : & p \in \mathcal{U} \\ 0 : & p \notin \mathcal{U}. \\ \overset{\shortparallel}{\text{trivial gp}} & \end{cases}$$

(show that it's a sheaf

Show that $i_p(A)_Q = \begin{cases} A & : Q = p \\ 0 & : Q \neq p \end{cases}$

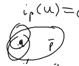
Show also that $i_p(A) = i_*(A)$, A constant sheaf on $\{p\}$
$i : \{p\} \hookrightarrow X$ inclusion

Sol^n:

It is a sheaf
- 1) $i_p(A)(\emptyset) = 0$ ✓
  2) What are the restriction map $\rho_{UV} : \begin{cases} id & \text{if } i_p(U) = i_p(V) = A \\ 0 & \text{otherwise} \end{cases}$
  Hence $\rho_{UU} = id$ ✓
  3.) Composition ✓
  4.) Let $s \in i_p(U)$ s.t. $s|_{U_j} = 0 \ \forall j$
  Suppose $s \neq 0$, take $U_j \ni p$
  then $s|_{U_j} = s \neq 0$ #
  since $\rho_{UU_j} = id$
  5.) let $s_j \in i_p(U_j)$ s.t. $s_j|_{U_j \cap U_k} = s_k|\ldots$
  If $p \notin U$ clearly $s = 0$
  If $p \in U$. Look at U_j s.t. $p \in U_j$
  & take $s = s|_{U_j}$ (since they agree on intersection they must be the same) ✓

Show $i_p(A)_q = \begin{cases} A & q \in \bar{p} \\ 0 & q \notin \bar{p} \end{cases}$

If $q \notin \bar{p}$ then $\exists$ U with $q \in U$, $p \notin U$. Hence $i_p(U) = 0$
For any V open
$U \cap V$ has only 0



in, hence the pair $\langle V, s \rangle$ must be $\langle V, 0 \rangle$