Recap

Last Time: 1) A ring, $X = \operatorname{Spec} A$

M A-module $\rightsquigarrow \widetilde{M}$ sheaf of $\mathcal{O}_X$-modules

Have shown: Any q-c sheaf on $\operatorname{Spec} A$ has this form

2) S graded ring, $X = \operatorname{Proj} S$

M graded S-module

$\rightsquigarrow \widetilde{M}$ q-c sheaf.

$\widetilde{M}(D_+(f)) = \{\text{homog elts deg 0 in } M_f\}$

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Want to show: Given any q-c sheaf on $X = \mathrm{Proj}\, S$, have $\mathcal{F} \cong \widetilde{M}$ some graded S-module M.

We won't (can't?) say this in general, but

Prop 1: S graded ring, finitely generated by S_1 as an S_0-alg $\left(S = \bigoplus_{d \geq 0} S_d\right)$ let $X = \operatorname{Proj} S$, and $\mathcal{F}$ q-c sheaf on X. Then there is a graded S-module M s.t.

$$\mathcal{F} \cong \widetilde{M}.$$

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Remarks: (1) Applies to $S = A[x_0, \dots, x_n]$ some ring A, i.e. to $\mathbb{P}^n_A$

(2) Don't have an equiv of categories btw graded S-modules and q-c sheaves on Proj S. Indeed, 2 different graded modules can give the same sheaf, e.g. $\bigoplus_{n\geq 0} M_n$ and $\bigoplus_{n \geq n_0} M_n$ some $n_0 \geq 0$

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(3) How to define M given $\mathcal{F}$?
Cannot just set $M = \Gamma(X, \mathcal{F})$
as in affine case. Sol^n is to
use Ext's.

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Example: $S = A[x_0, \ldots, x_r]$, $X = \mathrm{Proj}\, S$
$= \mathbb{P}^r_A$

Claim: $\Gamma(X, \mathcal{O}_X(n)) = S_n = \left\{ \begin{array}{l} \text{homog polys} \\ \text{of deg } n \end{array} \right\}$

In particular, $\Gamma(X, \mathcal{O}_X) \cong A$.

Proof: The sets $D_+(x_i)$ cover X
A section $t \in \Gamma(X, \mathcal{O}_X(n))$ is the same as an $(r+1)$-tuple of sections $(t_0, \ldots, t_r)$ with $t_i \in \mathcal{O}_X(n)(D_+(x_i))$ compatible on $D_+(x_i) \cap D_+(x_j) = D_+(x_i x_j)$

Now, $\mathcal{O}_X(n)(D_+(x_i)) = \{$deg n homog elts in $S_{x_i}\}$

$\mathcal{O}_X(n)(D_+(x_ix_j)) = \{$ " " $S_{x_ix_j}\}$

$\mathcal{O}_X(n)(D_+(x_0 \cdots x_r)) = \{$ " " $S_{x_0 \cdots x_r}\}$

Have $\forall\, i, j$

$$\begin{array}{ccccccc} S & \hookrightarrow & S_{x_i} & \hookrightarrow & S_{x_i x_j} & \hookrightarrow & S_{x_0 \dots x_r} \\ f & \longmapsto & f/1 & & & & \\ & & f/x_i^k & \longmapsto & \dfrac{f\, x_j^k}{(x_i x_j)^k} & & \end{array}$$

These preserve gradings, injective as no x_i is a zero divisor in S, and induce the relevant rest^n maps

So to give a section $t \in \Gamma(X, \mathcal{O}_X(n))$

is to give a homog elt of deg n

in $S_{x_0 \dots x_r}$, lying in $\bigcap_i S_{x_i}$

But easy to see that $\bigcap_i S_{x_i} = S$. $\square$

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So have $S \cong \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{O}_X(n))$

$= \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \tilde{S}(n))$

<u>Def^n</u>: S graded ring. gen by S_1 as S_0-alg. $X = \mathrm{Proj}\, S$, $\mathcal{F}$ sheaf $\mathcal{O}_X$-modules.

Define the graded S-module <u>associated to $\mathcal{F}$</u> to be

$$\Gamma_*(F) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, F(n))$$

<u>Module Structure</u>:

If $s \in S_d$, view this as an elt of $\Gamma(X, \mathcal{O}_X(d))$ (glue the $s/1$ in $\mathcal{O}_X(d)(D_+(f))$)

Then $F(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(d) \cong F(n+d)$

So get hom^m

$$\Gamma(X, \mathcal{F}(n)) \otimes_{\Gamma(X,\mathcal{O}_X)} \Gamma(X, \mathcal{O}_X(d)) \longrightarrow \Gamma(X, \mathcal{F}(n+d))$$

So for $t \in \Gamma(X, \mathcal{F}(n))$ define $s.t$ to be the image of $t \otimes s$ in $\Gamma(X, \mathcal{F}(n+d))$.

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Rmk: For $S = A[x_0, \ldots, x_m]$,

have shown $S \cong \Gamma_*(\mathcal{O}_X)$

as graded S-modules.

Prop^n 1 (again): S graded ring, f.g. by S_1 as S_0-alg. $X = \operatorname{Proj} S$, $\mathcal{F}$ q-c sheaf on X.

Then $\mathcal{F} \cong \widetilde{\Gamma_*(\mathcal{F})}$.

Pf: See Hartshorne. $\square$

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Closed Subschemes

Recall: A closed immersion is a morphism $f : Y \to X$ of schemes s.t. f induces a homeo from Y to a closed subset of X, and s.t.

$$f^{\#} : \mathcal{O}_X \to f_* \mathcal{O}_Y$$

is surjective.

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A <u>closed</u> <u>subscheme</u> of a scheme X

is an equiv. class of closed immersions

where $f: Y \to X$, $f': Y' \to X$

are equiv. if there's an iso

$$i: Y' \to Y$$

s.t. $f' = f \circ i$.

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Example: A ring, $\underline{a} \trianglelefteq A$,

then hom^{m} $A \to A/\underline{a}$ induces

closed immersion

$$\mathrm{Spec}(A/\underline{a}) \longrightarrow \mathrm{Spec}\, A$$

(image $V(\underline{a})$).

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Aim: Show that the kernel of $f^\#$, that is, the sheaf

$$U \longmapsto \ker(\mathcal{O}_X(U) \to (f_*\mathcal{O}_Y)(U))$$

is quasi-coherent when f is closed immersion.

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<u>Def^n</u>: A top space X is <u>quasi-separated</u> if the intersection of any two quasi-compact open subsets is again quasi-compact.

<u>Easy exercise</u>: For any ring A, Spec A is quasi-separated.

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__Defn__: A morphism of schemes $f : X \to Y$ is __quasi-compact__ (resp. __quasi-separated__) if for every affine open $U \subseteq Y$, $f^{-1}(U)$ is quasi-compact (resp. quasi-separated)

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Remarks: 1) If X is Noetherian, then for any scheme Y and morphism $f : X \to Y$, f is both quasi-compact and quasi-separated.

2) If $f : X \to Y$ is separated then it's quasi-separated.

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<u>Lemma</u>: Let $i : Y \to X$ closed immersion. Then i is both quasi-compact and quasi-separated.

<u>Pf</u>: Let $V = i(Y)$ and $U \subseteq X$ affine open. ($U = \operatorname{Spec} A$ say). Then $i^{-1}(U)$ is homeo to $U \cap V$.

As V is closed in X, $U \cap V$ is closed in U.

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So $U \cap V$ is of the form $V(\underline{a})$

some ideal $\underline{a} \triangleleft A$.

Then $i^{-1}(U)$ is homeo to $\mathrm{Spec}(A/\underline{a})$,

hence both quasi-separated

and quasi-compact. $\square$

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<u>Lemma</u>: X scheme. Then the kernel of any morphism of q-c sheaves $\phi: \mathcal{F} \longrightarrow \mathcal{G}$ on X is also quasi-coherent.

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Pf: The question is local

So WMA $X = \operatorname{Spec} A$ is affine.
Then there are A-modules
M, N s.t. $\mathcal{F} \cong \widetilde{M}$, $\mathcal{G} \cong \widetilde{N}$
and ϕ is induced from a hom
$\psi : M \to N$. It easy to show
that $\ker \phi \cong \widetilde{\ker \psi}$. $\square$

Lemma: Let $f: X \to Y$ morphism of schemes. Suppose f is quasi-compact and quasi-separated. Let $\mathcal{F}$ q-c sheaf on X. Then $f_* \mathcal{F}$ is a q-c sheaf on Y.

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Pf: WMA Y is affine (exercise).

Then by assumption, $X = f^{-1}(Y)$ is both quasi-compact and quasi-separated. So we may cover X by finitely many open affine U_i, and for each i,j we may cover $U_i \cap U_j$ by finitely many open, affine U_{ijk}

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Let $V \subseteq Y$ open. To give a section $s \in (f_* \mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$

is the same as giving sections s_i over each $f^{-1}(V) \cap U_i$ agreeing on each $f^{-1}(V) \cap U_{ijk}$.

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So $f_*\mathcal{F}$ fits into an exact seq.

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow \bigoplus_i f_*(\mathcal{F}|_{U_i}) \longrightarrow \bigoplus_{i,j,k} f_*(\mathcal{F}|_{U_{ijk}})$$

of sheaves on Y.

As each U_i, U_{ijk} is affine, both $\bigoplus_i f_*(\mathcal{F}|_{U_i})$ and $\bigoplus_{i,j,k} f_*(\mathcal{F}|_{U_{ijk}})$

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are quasi-coherent (1st lemma last week)

Thus, as the kernel of a morphism of q-c sheaves, $f_* \mathcal{F}$ is quasi-coherent. $\square$

Corollary: Let $i: Y \to X$ closed immersion of schemes. Then $i_* \mathcal{O}_Y$ is quasi-coherent.

Defn: $(X, \mathcal{O}_X)$ scheme. A sheaf of ideals on X is a sheaf $\mathcal{F}$ on X s.t. for each $U \subseteq X$, $\mathcal{F}(U)$ is an ideal of $\mathcal{O}_X(U)$. These are clearly $\mathcal{O}_X$-modules.

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Def^{n}: let Y closed subscheme of X

and $i : Y \to X$ corresponding closed immersion.

Define the ideal sheaf of Y, $\mathcal{I}_Y$,

to be the kernel of

$$i^{\#} : \mathcal{O}_X \to i_* \mathcal{O}_Y$$

It depends only on the equiv class of the closed immersion.

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Note: $\mathcal{I}_Y$, as defined above is quasi-coherent.

e.g. $X = \mathrm{Spec}\, A$ affine, $Y = \mathrm{Spec}(A/\mathfrak{a})$

$i : Y \longrightarrow X$ obvious closed immersion

then $\Gamma(X, \mathcal{I}_Y) = \mathfrak{a}$.

In particular, $\mathcal{I}_Y \cong \widetilde{\mathfrak{a}}$.

Prop^n: $X = \operatorname{Spec} A$ affine scheme.

Then any closed subscheme Y of X is of the form $(\operatorname{Spec} A/\mathfrak{a}, i)$ where $\mathfrak{a} \lhd A$ and $i : \operatorname{Spec} A/\mathfrak{a} \to \operatorname{Spec} A$ is induced from $A \to A/\mathfrak{a}$.

This gives a 1–1 correspondence btw ideals of A and closed subschemes of $\operatorname{Spec} A$. In particular, any closed subscheme of an affine scheme is affine.

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<u>Pf</u>: Note that distinct ideals give non-equiv closed subschemes as can recover $\underline{a}$ as $\Gamma(X, \mathcal{I}_Y)$ as in example above.

Now need a lemma.

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Lemma: X top space, $i: Y \to X$ cnts & induces homeo. from Y to a closed subset of X. Let $\mathcal{F}$ any sheaf on Y. Then $i^{-1} i_* \mathcal{F}$ is canonically isomorphic to $\mathcal{F}$.

Pf. Exercise. $\square$

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and a def^n...

Def^n: X top space, $\mathcal{F}$ sheaf on X.

Define the support of $\mathcal{F}$,

$\mathrm{Supp}(\mathcal{F})$ to be the set

$$\{x \in X \mid \mathcal{F}_x \neq 0\}.$$

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Pf (cont'd): let $i: Y \to X = \operatorname{Spec} A$ closed immersion. $\mathcal{I}_Y$ be the corresponding ideal sheaf.

Then $\mathcal{I}_Y$ is quasi-coherent so $\mathcal{I}_Y \cong \tilde{\mathfrak{a}}$ where $\mathfrak{a} = \Gamma(X, \mathcal{I}_Y) \triangleleft A$.

Now $i^{\#}: \mathcal{O}_X \longrightarrow i_* \mathcal{O}_Y$ is surjective, so have a s.e.s. of sheaves

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$$0 \longrightarrow \tilde{\underline{a}} \longrightarrow \mathcal{O}_X \xrightarrow{i^\#} i_*\mathcal{O}_Y \longrightarrow 0$$

But $\mathcal{O}_X/\tilde{\underline{a}}$ is iso to $\widetilde{(A/\underline{a})}$

Canonically, so $i^\#$ induces

iso $\widetilde{(A/\underline{a})} \xrightarrow{\sim} i_*\mathcal{O}_Y$.

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Now for $\mathfrak{p} \in \operatorname{Spec} A$ we have

$$(i_* \mathcal{O}_Y)_{\mathfrak{p}} \cong \begin{cases} 0 & \mathfrak{p} \notin i(Y) \\ \mathcal{O}_{Y, i^{-1}(\mathfrak{p})} \neq 0 & \mathfrak{p} \in i(Y) \end{cases}$$

So $i(Y) = \operatorname{Supp}(i_* \mathcal{O}_Y)$
$= \operatorname{Supp}(\widetilde{(A/\mathfrak{q})})$
$= V(\mathfrak{q})$.

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Moreover, have iso of sheaves of rings on Y

$$i^{-1}(\widetilde{A/a}) \xrightarrow{\sim} i^{-1} i_* \mathcal{O}_Y \xrightarrow{\sim} \mathcal{O}_Y$$

Lemma

It is an exercise to use this to define a morphism $Y \to \operatorname{Spec} A/a$ s.t.

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$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[dr] & X = \operatorname{Spec} A \\ & \operatorname{Spec} A/\mathfrak{a} \arrow[u] \end{tikzcd}$$

Commutes. $\Box$

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<u>Prop</u>: X scheme. Then $Y \mapsto \mathcal{I}_Y$ is a bijection btw quasi-coherent sheaves of ideals on X and closed subschemes. If X is Noeth. Can replace q-c by coherent.

Pf: let $\mathcal{I}$ q-c sheaf of ideals on X.

Set $Y = \mathrm{Supp}(\mathcal{O}_X/\mathcal{I})$ and

let $i : Y \longrightarrow X$ inclusion of top spaces.

then $i^{-1}(\mathcal{O}_X/\mathcal{I})$ is a sheaf of rings on Y. Need to show Y is closed in X and that

$(Y, i^{-1}(\mathcal{O}_X/\mathcal{I}))$ is a scheme.

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We can check this locally, so WMA
X affine. Then follows from
previous proof. Details an exercise. $\square$

Let S graded ring, $I \trianglelefteq S$ homog ideal.
Ring hom $S \to S/I$ induces a
closed immersion $\mathrm{Proj}(S/I) \to \mathrm{Proj}\, S$.

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<u>Prop^n</u>; A ring

(1) If Y is a closed subscheme of $\mathbb{P}^r_A$, then $\exists$ homog ideal $I \subseteq S = A[x_0, \ldots, x_r]$ s.t. Y is the closed subscheme determined by I.

<u>Pf</u>: Uses previous result on $\mathcal{Q}$-c sheaves on $\mathbb{P}^r_A$. See Hartshorne. $\square$

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