Study Group: (Linear) Algebraic Groups

1 Basic Definitions and Main Examples (Matt)

Definition 1.1. Let $I \lhd \overline{K}[x]$ (where K is some field) then $V_I = \{P \in \mathbb{A}^n : f(P) = 0 \forall f \in I\}$ is an *affine algebraic set*. If I is prime, then V_I is an *affine algebraic variety.*

Definition 1.2. A *linear algebraic group, G,* is a variety V/K with a group structure such that the group operations are morphisms and V is affine.

K-rational point $e \in G$ and K-morphisms $\mu : G \times G \to G$ ($\mu(x,y) = xy$) and $i : G \to G$ ($i(x) = x^{-1}$)
$K[G]$ is the K algebra of regular functions in V.
$\Delta : K[G] \to K[G] \otimes_K K[G]$ the *comultiplication*, $\iota : K[G] \to K[G]$ is the *coinverse*

We get the following axioms:
$(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$

Example. $\mathbb{G}_a$: where $\mu(x,y) = x+y$, $a, y \in \mathbb{A}^1$. $K[x]$, $\Delta : x \mapsto x \otimes 1 + 1 \otimes y \in K[x] \otimes K[y] \cong K[x,y]$. $\mathbb{G}_a^{n_2} \cong M_n(K)$.

$\mathbb{G}_m$: $\overline{K}^* \to V = \left\{(x,y) \in \mathbb{A}^2 | xy = 1\right\}$ defined by $t \mapsto (t, 1/t)$. Then $(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2, y_1 y_2)$. So $\Delta : x \mapsto x \otimes x, y \mapsto y \otimes y$. Note that $\mathbb{G}_m \cong \mathrm{GL}_1(\overline{K})$

Let $K = \mathbb{Q}(\sqrt{2})/\mathbb{Q}$, $(x_1 + y_1\sqrt{2})(x_2 + y_2\sqrt{2}) = (x_1x_2 + 2y_1y_2) + (x_1y_2 + x_2y_1)\sqrt{2}$, $(x_1 + y_1\sqrt{2})^{-1} = \frac{x_1 - y_1\sqrt{2}}{x_1^2 - 2y_1^2}$. Let $g = x_1^2 - 2y_1^2$, then we can use $V(g \neq 0)$. $\Delta : x_1 \mapsto x_1 \otimes x_2 + 2y_1 \otimes y_2$, $y_1 \mapsto x_1 \otimes y_2 + x_2 \otimes y_1$. $\iota : x_1 \mapsto x_1/g, y_1 \mapsto -y_1/g$.

$\mathrm{End}(\mathbb{G}_m^n) = M_n(\mathbb{Z})$.

Flags

Let V be a vector space with $\dim V = n$. We define a flag F of type (d) for $0 \le d \le n$: this is a subspace $W \subset V$ with dimension d.

$\{$Flags of type (d) in $V\}$=Grassmannian variety $\mathcal{G}(n,d)$.
Example: $K = \mathbb{Q}(i)/\mathbb{Q}$, take $\{$flags of type $(1)\} \cong \mathbb{P}^1$
The dimension of $\{$flags of type (d) in $V\}$ is $d(n-d)$.
Example: $L = \mathbb{Q}(\sqrt[3]{2})$, take $\{$flags of type $(2)\}$

Flags of type $(d_1, \ldots, d_t)$, where $0 \le d_1 < d_2 < \cdots < d_t \le n$: Series of subspace $\{V_i\}$ with $\dim V_i = d_i$ and $V_1 \subset V_2 \subset \cdots \subset V_t$. Then $\mathcal{F}_{(d_1,\ldots,d_t)} \subseteq \mathcal{G}(n, d_1) \times \cdots \times \mathcal{G}(n, d_t)$.

An algebraic group acting on a variety $G \times V \to V$ by $(g, v) \mapsto gv$. So we can define $\mathrm{Stab}(v) = \{g \in G : gv = v\} \le G$.

For a flag, $\mathrm{Stab}(\mathcal{F})$ is called *the parabolic subgroup.* (I.e., $\mathrm{Stab}(\mathcal{F}) = \{g \in G : gV_i = V_i\}$)
A flag of type $(0, 1, \ldots, n)$ is called a *maximal flag*
Let us take a basis of V to be $\{e_1, \ldots, e_n\}$ such that $e_1, \ldots, e_{d_1}$span V_1, $e_1, \ldots, e_{d_2}$ span V_2, etc, then we get

$$\begin{pmatrix} A_1 & & & * \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_{t+1} \end{pmatrix}$$

where $A_i \in M_{d_i - d_{i-1}}$. For a maximal flag

$$\begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix} = \text{Borel Subgroup}$$

Consider $B : V \times V \to K$, $q : V \to K$ a quadratic form defined as $q(x) = B(x, x)$ if B is symmetric.

We can create the *orthogonal group* $O_B = O_q = \{A \in \mathrm{GL}(V) : B(Ax, Ay) = B(x, y) \forall x, y\}$.

We define the *special orthogonal group* $SO_B = \mathrm{SL}(V) \cap O_B$.

Definition 1.3. A *Clifford algebra* is a unital associative algebra that contains and is generated by a vector space V, where V has a quadratic form q subject to $v^2 = q(v) \cdot 1$. This is equivalent to saying $uv + vu = 2B(u, v)$. We denote this by $\mathrm{Cliff}(V, q)$.

2 Algebraic groups (Vandita)

2.1 Motivation

In this study group we are doing linear algebraic groups: Affine variety V over field k, with group operation such that groups operations are morphism. Lie algebra have rich structure. V is a closed subgroup of $\mathrm{GL}_n(k)$ for some n, hence called *linear.*

We knows about Abelian varieties: Trivial linear representations and commutative lie algebras. Interesting Galois representations.

There is also something in the middle called Semi-abelian varieties to merge the theory of the two.

Continuation from last week:

Example.

1. (Example 3.3) Let GL_n be the general linear group. If you have a field k, we call consider GL_n over k. This is an affine variety in $\mathbb{A}^{n^2}$ (or can view it as $(n \times n)$ matrices $X = (x_{ij})_{n\times n}$), which is the complement of matrices with det $= 0$. (The matrices with det $= 0$ are closed subvariety). This is an affine variety because it is isomorphic to $\mathbb{A}^{n^2+1}$ with coordinates x_{ij} and y such that $\det(x_{ij}) - u - 1 = 0$.

   We have a group law, i.e., matrix multiplication.

   Group homomorphism: $\mathrm{GL}_n \to \mathbb{G}_m$ defined by $(x_{ij}) \mapsto \det(x_{ij})$

   Closed subgroups of GL_n are:

   (a) Torus: $T = \mathrm{diag}(x_1, \ldots, x_n) : x_1 \ldots x_n \neq 0\}$
   (b) Borel subgroups of upper triangular matrices, $B = \{(x_{ij}) \in \mathrm{GL}_n : x_{ij} = 0 \,\text{if}\, i < j\}$
   (c) Groups of unipotent matrices

2. (Example 3.5) The sympletic group. Let V be an even dimensional vector space over a field k with characteristic not equal 2. Sympletic forms on V is an alternating perfect (bilinear, non-degenerate and maybe symmetric?) pairing $\langle \cdot, \cdot \rangle : V \times V \to k$. There is always a basis $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ of V such that the basis is $\langle y_i, y_j \rangle = \langle x_i, x_j \rangle = 0$ and $\langle x_i, y_j \rangle = \delta_{ij}$. Therefore by suitable change of basis we can define equivalent pairs of bases.

   We have a form given by $2n \times 2n$ matrix, $J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ or with change of basis $\begin{pmatrix} 0 & & & & 1 \\ & & & / & \\ & & & 1 & \\ & & -1 & & \\ & / & & & \\ -1 & & & & 0 \end{pmatrix}$

   We define the *sympletic group* $Sp(2n, k)$ linear transformations T of V preserving pairings, i.e., the set of matrices $\{g \in \mathrm{GL}_{2n} \,|g^t J g = J\}$

3. Note that $Sp_2 = \mathrm{SL}_2$. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $J = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, then $g^t J g = \begin{pmatrix} 0 & ad-bc \\ bc-ad & 0 \end{pmatrix}$.

2.2 Constructible sets (Algebraic groups part II, chapter 4)

Definition 2.1. Let X be a quasi-projective variety, equipped with its Zariski topology. We call a subset C of X *locally closed* if $C = V \cap Z$, where V is open and Z is closed.

We call a subset of X *constructible* if it is a finite disjoint union of locally closed set. I.e., $Y \subseteq X$ with $Y = \coprod C_i$ with C_i locally closed.

Example (4.0.1)**.** Let $T = (\mathbb{A}^2 \setminus \{x = 0\}) \cup \{(0,0)\}$. Note that T is not locally closed (as T is dense in $\mathbb{A}^2$). But T is constructible since T is the disjoint union of 2 locally closed sets, i.e., $T = (\mathbb{A}^2 \setminus \{x = 0\}) \coprod \{(0,0)\}$ (mainly because the first component is open and the second one is closed, so intersect each with $\mathbb{A}^2$ as it is both open and closed)

Lemma 2.2 (Lemma 4.02)**.**

1. *An open set (or closed set) is constructible*
2. *Finite intersection of constructible sets is constructible*
3. *The complement of constructible set is constructible*

Corollary 2.3. *The finite union of constructible sets is constructible*

Theorem 2.4 (Theorem 4.05)**.** *If $\phi : X \to Y$ is a morphism of varieties, then ϕ maps constructible sets to constructible sets. In particular $\phi(X)$ is constructible and contains a set open and dense in its closure.*

2.3 Connected Components (chapter 5)

Assume that G is an algebraic group over algebraically closed field k. We do not assume that G is linear.

Proposition 2.5 (5.07)**.**

- *There is a unique irreducible component G^0 of G that contains the identity element e.*
- *G_0 is a closed subgroup of finite index*
- *G^0 is also the unique connected component of G that contains and is contained in any closed subgroups of G of finite index.*

Proof. Let X, Y be components containing e, then $xy = \mu(x \times y)$, $\overline{xy}$, x^{-1} and txt^{-1} for any $t \in G$ are irreducible and contains e. Since by definition an irreducible component is a maximal irreducible closed set it follows that $x = xy = \overline{xy}$, $x = xy$ so $x = y$. Since x^{-1} is an irreducible component containing e, $x = x^{-1}$ and since txt^{-1} is an irreducible component containing e, $x = txt^{-1}$. Hence it is a normal subgroup. □

Example (Example 5.08)**.** Consider $\mathbb{G}_a, \mathbb{G}_m$,, they are connected. Hence $\mathbb{G}_a^n$ and the tori are connected. The unipotent group of GL_n is connected and isomorphic to $\mathbb{A}^{n(n-1)/2}$. Therefore the Borel subgroup is connected. GL_n is connected, closed subsets of $\mathbb{A}^{n^2} \times \mathbb{A}^1$ defined by $y \cdot \det(x_{ij})$-1 (an irreducible polynomial). So GL_n is irreducible, because it is an open non-empty subset of $\mathbb{A}^{n^2}$.

2.4 Subgroups (chapter 6)

Read in own time

2.5 Group actions and linearity vs affine (Chapter 7)

Theorem 2.6 (Theorem 7.12). *Let G be an affine algebraic groups, then G is* linear. *That is G is isomorphic to a closed subgroup of* GL_n *for some n.*

3 Jordan decomposition and commutative algebraic groups (Angelos)

3.1 Jordan decomposition

Linear algebra: Let's take k an algebraic closed field and V a finite dimensional vector space over k.

Definition 3.1. Let $a \in \mathrm{End}(V)$. We call a:

1. *semi-simple*, or *diagonalisable*, if there is a basis of V consisting of eigenvectors of a
2. *nilpotent*, if $a^n = 0$ for some $n \in \mathbb{N}$
3. *unipotent*, if $(a-1)^n = 0$ for some $n \in \mathbb{N}$.

Proposition 3.2. *Let $a \in \mathrm{End}(V)$ then there exists $a_n, a_s \in \mathrm{End}(V)$ such that $a = a_s + a_n$ and $a_n \cdot a_s = a_s \cdot a_n$ (and a_s is semi-simple, a_n is nilpotent)*

Furthermore if $a \in \mathrm{GL}(V)$, then there exists $a_u, a_s \in \mathrm{End}(V)$ such that $a = a_s \cdot a_u$ and $a_s \cdot a_u = a_u \cdot a_s$ (and a_s is semi-simple, a_u is unipotent)

We want to generalise this idea for infinite cases. So let V be an infinite dimensional vector space over k

Definition 3.3. Let $a \in \mathrm{End}(V)$. We call *a locally finite* if $V = \cup V_i$ is a union of finite dimensional vector spaces stable under a. Then a is called *locally semi-simple* (respectively *locally nilpotent*) if $a|_{V_i}$ is semi-simple for all i (respectively nilpotent).

Remark. We work with countable dimensional V, so we can find $V_1 \subset V_2 \subset \cdots \subset V = \cup V_i$. Note that in this case, since $a|_{V_i} = a_{s,i} + a_{n,i}$ and $a_s|_{V_i} = a_{s,i}$, $a_n|_{V_i} = a_{n,i}$, we can say that $a = a_s + a_n$.

Similarly analogue can be made with locally unipotent (in terms of definition and lifting elements)

3.1.1 Jordan decomposition for linear algebraic groups

Let G be a linear algebraic group over k, $\rho : G \to \mathrm{GL}(k[G])$ such that $(\rho(g)f)(x) = f(xg)$ for all $x, g \in G$, $f \in \mathrm{GL}$. So $\rho(g) = \rho(g)_s \rho(g)_u$ where $\rho(g)_s$ is semisimple and $\rho(g)_u$ is unipotent.

Theorem 3.4. *Fix $g \in G$ and decompose $\rho(g) = \rho(g)_s\rho(g)_u$ as above.*

1. *There are unique $g_s, g_u \in G$ such that g_s is semisimple, g_u is unipotent, $g_s g_u = g_u g_s$ and $\rho(g)_s = \rho(g_s)$, $\rho(g)_u = \rho(g_u)$.*
2. *If $\phi : G \to H$, then $\phi(g_s) = \phi(g)_s$ and $\phi(g_u) = \phi(g)_u$*
3. *If $G = \mathrm{GL}_n$, the decomposition agrees with the classical Jordan decomposition.*

Theorem 3.5. *Let $G \subseteq \mathrm{GL}_n$ consisting of unipotent elements. Then there exists $x \in G$ such that $xGx^{-1} \subseteq U_n$ where U_n is group of upper unipotent matrices.*

3.2 Commutative algebraic groups

Proposition 3.6. *Let G be a commutative algebraic group over k,, where k is algebraically closed. The set G_s of semi-simple elements of G and G_u the set of unipotent elements are both closed subgroups of G. Moreover, $G \cong G_s \times G_u$.*

We define$X^* = \mathrm{Hom}_k(G, \mathbb{G}_m)$ which is the set of *characters* of G and $(\chi_1 + \chi_2)(g) = \chi_1(g)\chi_2(g)$. We define $X_*(G) = \mathrm{Hom}_k(\mathbb{G}_m, G)$, since G is commutative, X_* is a group.

Definition 3.7. A *k-rational representation* of G is a representation $r : G \to \mathrm{GL}(V)$ where V is a vector space over k.

Denote T_n to be the torus $\mathbb{G}_m^n$

Proposition 3.8. *Let $x \in T_n$ and $x = (\chi_1(x), \dots, \chi_n(x))$. Each χ_i is a character ofT_n and $k[T_n] = k[\chi_1^{\pm 1}, \dots, \chi_n^{\pm 1}]$. Every character of T_n is of the form $\chi_1^{a_1} \dots \chi_n^{a_n}$ with $a_i \in \mathbb{Z}$.*

Definition 3.9. A linear algebraic group is called *diagonalisable* if it is isomorphic to a closed subgroup of T_n (for some n)

Theorem 3.10. *The following are equivalent:*

1. *G is commutative and $G = G_s$*
2. *G is diagonalisable*
3. *$X^*(G)$ is a finitely generated abelian group and its elements form a k basis of $k[G]$*
4. *Any k-rational representations of G is a direct sum of one dimensional representations.*

Theorem 3.11. *Let G be a connected linear algebraic group of dimension 1. Then, either $G \cong \mathbb{G}_m$ or $G \cong \mathbb{G}_a$.*

4 Lie Algebra (associated to algebraic groups) (Heline)

Notation.

- R a commutative ring
- A an R-algebra
- M an A-module
- k is algebraic closed

Definition 4.1. An *M-valued R-derivation of A* is a function $D : A \to M$ such that D is R-linear and $D(ab) = aD(b) + D(a)b$ for all $a, b \in A$.

A collection of all such derivation is $\mathrm{Der}_R(A, M)$.

Example. If k is an algebraic closed field, X an affine k-variety, $\mathcal{O}_{X,x}$ the local ring of x on X ($x \in X$), view k as an $\mathcal{O}_{X,x}$-module via $f \cdot a = f(x)a$ for $f \in \mathcal{O}_{X,x}$, $a \in k$. Then $\mathrm{Der}_R(\mathcal{O}_{X,x}, k)$ are the linear functions $\delta : \mathcal{O}_{X,x} \to k$ such that $\delta(fg) = f\delta(g) + g\delta(f)$.

Definition 4.2. Let X be a variety. *The tangent space* at $x \in X$, $T_{X,x} = \left(m_x/(m_x)^2\right)^*$, i.e. $T_{X,x} = \mathrm{Hom}_k(m_x/(m_x)^2, k)$, where $m_x$is the maximal ideal of the local ring $\mathcal{O}_{X,x}$ and $m_x/(m_x)^2$ is viewed as a k-vector space.

Example. Let $X \subseteq \mathbb{A}^n$ be defined by the ideal I, M the maximal ideal of $k[X]$ comprising the functions vanishing at x. Then $T_{X.x} = (M/M^2)^*$. If $T_1, \ldots, T_n$ are variables of $\mathbb{A}^n$ and f a function vanishing on X. Develop f into a Taylor series

$$d_x f = \sum \frac{\partial f}{\partial T_i}(x)(T_i - x_i) \ \ (\dagger)$$

Define $T_{X,x}^{\text{naive}}$ to be the affine linear variety defined by all ($\dagger$) as f ranges over I. $d_x f = 0$ for all $f \in I$. So $T_{x,x}^{\text{naive}}$ with origin at x is the solution to homogeneous system of equations $I = \langle f_1, \ldots, f_m \rangle$:

$$\left(\frac{\partial f_i}{\partial T_j}\right)_{1 \leq i \leq m, 1 \leq j \leq n}$$

Note. We can define $d_x f$ for any $f \in k[X]$ by ($\dagger$), and new $d_x f$ as a function on $T_{X,x}^{\text{naive}}$ which is linear once we make x the origin. So we can view d_x as a map, $d_x : M \to (T_{X,x}^{\text{naive}})^*$ because $d_x(fg) = f(x)d_x(g) + g(x)d_x(f)$. Now d_x vanishes on M^2, so $d_x : M/M^2 \to (T_{X,x}^{\text{naive}})^*$.

Fact. *This is an isomorphism.*

Definition 4.3. A variety X is called *equi-dimensional* if all the components of X have the same dimension.

From now on we assume that X is equi-dimensional

Definition 4.4. A point $x \in X$ is called *regular* (or *non-singular*, or *simple*) if $\dim_k T_{X,x} = \dim(X)$

Example. $y^2 = x^3$, the point $(0, 0)$ is singular, non-regular.

Definition 4.5. X is called *regular* if all its points are.

Proposition 4.6. *The set of all regular points in X is a dense Zariski open set.*

Proposition 4.7. *If X is an algebraic group then X is regular.*

Definition 4.8. A *Lie algebra* L over a field k is a k-vector space together with a binary operation $[\,,\,] : L \times L \to L$ called the *Lie bracket,* which is

- bilinear: $[ax + by, z] = a[x, z] + b[y, z]$, $[z, ax + by] = a[z, x] + b[z, y]$ for all $a, b \in k$, $x, y, z \in L$
- alternation $[x, x] = 0$ for all $x \in L$
- Jacobi identity $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$

Note. Bilinear and alternating implies that we have anti-commutativity, i.e., $[x, y] = -[y, x]$.
If characteristic is not 2, then anti-commutativity and bilinear implies alternating.

Let G be an algebraic group, $A = k[G]$. Consider $\mathrm{Der}_k(A, A)$. If D_1, D_2 are derivations, then so is $[D_1, D_2] = D_1 D_2 - D_2 D_1$. So $\mathrm{Der}_k(A, A)$ is a Lie algebra

Notation. $(\lambda_x f)(y) = f(x^{-1}y)$.

Definition 4.9. Define $\mathcal{L}(G)$, the *left invariant derivations* of G as $\mathcal{L}(G) = \{D \in \mathrm{Der}_k(A, A) : D \cdot \lambda_x = \lambda_x \cdot D \forall x \in G\}$

This is a Lie algebra, under the same bracket.
Let e be the identity of G, let $\mathfrak{g} = T_{G,e}$ the tangent space at the identity. Then there exists a map $\mathrm{Der}_k(A, A) \to \mathfrak{g}$ defined by $D \mapsto \{f \mapsto (Df)e\}$. This restricts to a k-linear map.

Theorem 4.10. *$\mathcal{L}(G) \to \mathfrak{g}$ is an isomorphism of k-vector space (so $\dim \mathcal{L}(G) = \dim(\mathfrak{g})$).*

So we can think of $\mathfrak{g}$ to be the Lie algebra associated to the algebraic group G. If $\phi : G \to G'$ is a homomorphism of algebraic groups then the induced map $d\phi_e : \mathfrak{g} \to \mathfrak{g}'$ is a homomorphism of Lie algebras, where $\mathfrak{g}$ and $\mathfrak{g}'$ get bracket via isomorphism from $\mathcal{L}(G)$ and $\mathcal{L}(G')$ respectively.

5 Subgroups and Lie Subalgebras (Florian)

Definition 5.1. Give a point derivation δ at e and $f \in A$, we define the *covulation* $*\delta$ as $(f * \delta)(x) := \delta(\lambda_{x^{-1}} f)$

5.1 Subgroups

Let G be an algebraic group and let H be a closed subgroup of G defined by the ideal J (so $H = V(J)$). Then the inclusion $H \hookrightarrow G$ induces a map on the tangent spaces $T_{H,e} \to T_{G,e}$.

Fact. *$\mathcal{O}_{H,e} = \mathcal{O}_{G,e}/J$ and $\mathfrak{m}_{H,e} = \mathfrak{m}_{G,e}/J$*

Therefore:

Theorem 5.2. *The image of $T_{H,e} \to T_{g,e}$ is the subspace of $T_{G,e}$ consisting of δ such that $\delta(f) = 0$ for all $f \in J$*

Theorem 5.3. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebra of G and H respectively. Then $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ and $\mathfrak{h} = \{\delta \in \mathfrak{g} : f * \delta \in J \forall f \in J\}$*

Example.

- Let $G = \mathrm{GL}_n$. Since the inclusion $\mathrm{GL} \hookrightarrow \mathbb{A}^{n^2}$ identifies the tangent spaces, a basis for $T_{\mathrm{GL}_n,\mathrm{id}}$ is $f \mapsto \underbrace{\frac{\partial f}{\partial t_{ij}}(\mathrm{id})}_{=:\delta}$.

  We want to calculate the corresponding left invariant derivation. Let f be a general function defined by $f((t_{ij})) = (t_{kl})$. Consider the function

$$\begin{aligned} g((t_{ab})) &= \lambda_{(x_{ab})^{-1}} f((t_{ab})) \\ &= f((x_{ab})(t_{ab})) \\ &= f((\sum_r x_{ir} t_{rj})_{ij}) \\ &= (\sum_r x_{kr} t_{rl})_{kl} \end{aligned}$$

  Hence this is the function $(t_{ab}) \mapsto (\sum_r x_{kr} t_{rl})$. So we calculate that

$$\begin{aligned} (f * \delta)((x_{ab})) &= \delta(g((t_{ab})) \\ &= \frac{\partial(\sum_r x_{kr} t_{rl})}{\partial t_{ij}}(\mathrm{id}) \\ &= \delta_{jl} \cdot x_{ki} \end{aligned}$$

  where $\delta_{jl} = \begin{cases} 1 & j = l \\ 0 & j \neq l \end{cases}$. Now consider the Derivations $D_{ij} f = \sum_a t_{ai} \frac{\partial f}{\partial t_{aj}}$ (this is a derivation as it is the sum of derivations). For the function $f((t_{ij})) = t_{kl}$, we have $D_{ij} f = \sum_a t_{ai} \frac{\partial t_{kl}}{\partial t_{aj}} = \delta_{jl} \cdot t_{ki}$. Since t_{kl} is a basis, any derivation is determined by its values on the function t_{kl}, so D is the derivation of $*\delta$ and in particular left invariant. Hence $\mathcal{L}(\mathrm{GL}_n) = \{\sum_{ij} m_{ij} D_{ij}\}$.

  Finally, we want to calculate the Lie bracket of the left invariants. Before we do so, associate an element $\sum m_{ij} D_{ij} \in \mathcal{L}(\mathrm{GL}_n)$ with the matrix $(m_{ij}) \in M_n(k)$. Then since the Lie brackets are bilinear, it is enough to calculate them on the elements D_{ij}, D_{kl}, who correspond to the elementary matrices E_{ij} and E_{kl} respectively. Let us calculate $[D_{kl}, D_{ij}]$ on the $f((t_{ab})) = (t_{cd})$

$$\begin{aligned} [D_{kl}, D_{ij}] &= D_{kl} \circ D_{ij} f - D_{ij} \circ D_{kl} f \\ &= D_{kl} \underbrace{[\delta_{jd} \cdot t_{ci}]}_{f((t_{ab}))=(\delta_{jd} t_{ci})} - D_{ij} \underbrace{[\delta_{ld} \cdot t_{ck}]}_{f((t_{ab}))=(\delta_{ld} t_{ck})} \\ &= \delta_{li} \cdot \delta_{jd} \cdot t_{ck} - \delta_{jk} \cdot \delta_{ld} \cdot t_{ci} \\ &= \delta_{li} D_{kj} - \delta_{jk} D_{il} \end{aligned}$$

But we also know that $[E_{kl}, E_{ij}] = \delta_{li}E_{kj} - \delta_{jk}E_{il}$. Hence we conclude that $\mathfrak{gl}_n$ is canonically identified with the k-vector space of all $n \times n$ matrices with Lie algebra $[X, Y] = XY - YX$.

- Subgroups of GL_n

  Let H be a closed subgroup of GL_n with H defined by J. Then $\mathfrak{h}$ is defined by the vanishing of $\sum \frac{\partial f}{\partial t_{ij}}(\mathrm{id})$ for all $f \in J$.

  Fact. *Consider the expansion* $f(\mathrm{id}_n + (t_{ij})) = \underbrace{f(\mathrm{id}_n)}_{=0} + \sum t_{ij}\frac{\partial f}{\partial t_{ij}}(\mathrm{id}_n) + \text{h.o.t}$ *(*$\mathrm{id}_n \in H$, $f \in J$, *so* $f(\mathrm{id}_n) = 0$*). Then* $\mathfrak{h}$ *is determined by looking at* $f(\mathrm{id}_n + (t_{ij}))$ *up to* $\mod (t_{ij}t_{kl})_{ijkl}$.

  - $H = \mathrm{SL}_n$. Then H is defined by $f(M) = \det M - 1$. Consider

$$\begin{aligned} f(\mathrm{id}_n + (t_{ij})) &= \det(\mathrm{id}_n + (t_{ij})) - 1 \\ &= -1 + \det(\mathrm{id}_n) + \sum t_{ij}\frac{\partial \det}{\partial t_{ij}}(\mathrm{id}) \\ &= -1 + 1 + \sum t_{ii} \\ &= \sum t_{ii} \end{aligned}$$

    (up to squares). So $\mathfrak{sl}_n = \{M \in M_n(k) : \mathrm{Tr}(M) = 0\}$

  - $H = O_B$ where B is a symmetric matrix (so that $\langle x, y\rangle = {}^t xBy$). Recall that $O_B = \{M \in \mathrm{GL}_n : {}^tMBM = B\}$, hence H is defined by $f(M) = {}^tMBM - B$. Consider

$$\begin{aligned} f(\mathrm{id}_n + (t_{ij})) &= (\mathrm{id}_n + {}^t(t_{ij}))B(\mathrm{id}_n + (t_{ij})) - B \\ &= B + {}^t(t_{ij})B + B(t_{ij}) + {}^t(t_{ij})B(t_{ij}) - B \\ &= {}^t(t_{ij})B + B(t_{ij}) \end{aligned}$$

    (up to squares). So $\mathfrak{o}_B = \{M \in M_n(k) : {}^tMB = -BM\}$, i.e., the "$B$-skew symmetric matrices"
    Note that

$$\begin{aligned} \mathrm{Tr}(M) &= \mathrm{Tr}(BMB^{-1}) \\ &= \mathrm{Tr}(-{}^tMBB^{-1}) \\ &= -\mathrm{Tr}({}^tM) \\ &= -\mathrm{Tr}(M) \end{aligned}$$

    Hence $\mathrm{Tr}(M) = 0$ and so $\mathfrak{so}_B = \mathfrak{o}_B$.
    For $B = \mathrm{id}_n$ (and hence $q(x) = \sum x_i^1$) we have$\mathfrak{so}_n = \mathfrak{o}_n = \{M \in M_n(k) : {}^tM = -M\}$

Remark. A useful consequence that an algebraic group G over k is non singular and that $\dim_k \mathcal{L}(G) = \dim_k \mathfrak{g} = \dim_k G$ is that we can calculate the dimension of G by the dimension of its Lie algebra. e.g.,

- $\dim(\mathrm{GL}_n) = \dim\{M \in M_n(k)\} = n^2$ (which we already knew)
- $\dim(\mathrm{SL}_n) = \dim\{M \in M_n(k) : \mathrm{Tr}(M) = 0\} = n^2 - 1$ (which we already knew)
- $\dim(O_B) = \dim(SO_B) = \dim(M \in M_n(k) : {}^tMB = -BM\} = \frac{1}{2}n(n-1)$

Exercise. $\mathcal{L}(G_1 \times G_2) \cong \mathcal{L}(G_1) \oplus \mathcal{L}(G_2)$

5.2 The adjoint representation

Let $A = k[G]$. We are going to define two equal actions of G on $\mathfrak{g}$.

- Let G act on $\mathcal{L}(G)$ by $D \mapsto \rho_x \circ D \circ \rho_x^{-1}$ where $(\rho_x f)(y) = f(yx)$ (right multiplication). Since $\lambda_y \rho_x = \rho_x \lambda_y$ we can see that this is an action. This induces an action on $\mathfrak{g}$ via the isomorphism $\mathfrak{g} \to \mathcal{L}(G)$. So x takes a derivation δ to a derivation μ such that $*\mu = \rho_x \circ *\delta \circ \rho_x^{-1}$
- Let $\mathrm{Int}(x)$ be the automorphism of G given by $\mathrm{Int}(x)(y) = xyx^{-1}$. We denote its differential at the identity by $\mathrm{Ad}(x)$, i.e., $\mathrm{Ad}(x) = d\mathrm{Int}(x)_e$

Lemma 5.4. $\mathrm{Ad}(x)(\delta) = \rho_x \circ *\delta \circ \rho_x^{-1}$

Let $G = \mathrm{GL}_n$. The map $\mathrm{Int}(x)$ is a linear map on $\mathbb{A}^{n^2}$, and a differential of a linear map is equal to the map so:

Theorem 5.5. *Let $\delta \in \mathfrak{gl}_n \cong M_n(k)$. Then $\mathrm{Ad}(x)(\delta) = x\delta x^{-1}$*

While for a general algebraic group we have no such formula, since any linear algebraic group is isomorphic to a subgroup of GL_n we can use the following lemma

Lemma 5.6. *Let H be a closed subgroup of GL_n and view $\mathfrak{h}$ as a Lie subalgebra of $\mathfrak{gl}_n$. Let $h \in H$, then $\mathrm{Ad}(h)(\delta) = h\delta h^{-1}$*

Corollary 5.7. *The adjoint representations $\mathrm{Ad} : \mathrm{GL}_n \to \mathrm{GL}(\mathfrak{gl}_n) \cong \mathrm{GL}_{n^2}$ and $\mathrm{Ad} : H \to \mathrm{GL}(\mathfrak{h}) \cong \mathrm{GL}_m$ (where $m = \dim H$) are algebraic representations*

Note that $\mathrm{Ad}(x)$ respect the Lie bracket on $\mathfrak{g}$. So we have an induced map of Lie algebras $\mathfrak{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$

Proposition 5.8. *The homomorphism of Lie algebras*

$$\mathfrak{ad} : \mathfrak{gl}_n \to \mathfrak{gl}(\mathfrak{gl}_n) \cong M_{n^2}$$

is given by $\mathfrak{ad}(X)(Y) = [X, Y] = XY - YX$

Corollary 5.9. *Let G be an algebraic group. The homomorphism of Lie algebras*

$$\mathfrak{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

is given by $\mathfrak{ad}(X)(Y) = [X, Y]$.

6 Quotient of Algebraic Groups (Pedro)

Throughout, k will denote an algebraic closed field

6.1 First results

Definition 6.1. Let X be a k-variety and G an (abstract) group. An *action* of G on X is a homomorphism $\phi : G \to \mathrm{Aut}_k(X)$. If X is a k-algebraic group, then we will usually ask for $\phi(G) \subseteq \mathrm{Hom}_k(X)$.

Example.

1. The group C_4 acts on $\mathbb{A}^1_{\mathbb{C}}$: Choose a generator $\sigma \in C_4$ and define $\sigma z = iz$ for $z \in \mathbb{C}$.
2. The group $\mathbb{C}^*$ acts on $\mathbb{A}^1_{\mathbb{C}}$: If $a \in \mathbb{C}^*$ define $(a, z) \mapsto az$
3. If G is an algebraic group and $H \subseteq G$ is a closed algebraic subgroup. Then H acts on G.

Question: Given a k-variety X and a group G acting on X, is there a quotient variety X/G? That is, is there a k-variety Y, together with a k-morphism $\pi : X \to Y$ such that $\pi(gx) = \pi(x)$ for all $g \in G, x \in X$ and such that for any k-morphism $\psi : X \to Z$ with $\psi(gx) = \psi(x)$ we have that ψ factors uniquely through π?

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "\psi"'] & X/G \arrow[d] \\ & Z \end{tikzcd}$$

We also want π to give us a bijection between $X(k)/G(k)$ and $(X/G)(k)$

Answer: In general, no!

Example. $\mathbb{C}^* = \mathbb{G}_m(\mathbb{C})$ acting on $\mathbb{A}^1_{\mathbb{C}}$ by multiplication. We want to check the existence of $\mathbb{A}^1_{\mathbb{C}}/\mathbb{G}_m$. Suppose that $\mathbb{A}^1_{\mathbb{C}}/\mathbb{G}_m$ exists. $(\mathbb{A}^1_{\mathbb{C}}/\mathbb{G}_m)(\mathbb{C}) = \mathbb{A}^1_{\mathbb{C}}(\mathbb{C})/\mathbb{G}_m(\mathbb{C}) = \{\{0\}, \{\mathbb{C}^*\}\}$, hence both of these points should be closed. Consider $\pi : \mathbb{A}^1_{\mathbb{C}}(\mathbb{C}) \to (\mathbb{A}^1_{\mathbb{C}}/\mathbb{G}_m)(\mathbb{C})$ but $\pi^{-1}(\{\mathbb{C}\}) = \mathbb{C}^*$ so it is not closed hence does not exits.

Theorem 6.2. *Let X be a quasi-projective k-variety and G a finite group action on X. Then there is a quotient variety $\pi : X \to X/G$, where X/G is quasi-projective and such that π gives us a bijection between $(X/G)(k)$ and $X(k)/G$ and π is an open map.*

Furthermore if X is an affine variety with co-ordinate ring $k[X]$, then the coordinate ring of X/G is $k[X]^G$ and π is the morphism induced by $k[X]^G \hookrightarrow k[X]$.

Theorem 6.3. *Let G be a k-algebraic group and $H \le G$ a closed k-algebraic subgroup of G. Then there is a quotient variety $\pi : G \to G/H$ such that G/H is quasi-projective, π gives us a bijection between $(G/H)(k)$ and $G(k)/H(k)$, π is open.*

Furthermore if X is an affine variety with co-ordinate ring $k[X]$, then the coordinate ring of X/G is $k[X]^G$ and π is the morphism induced by $k[X]^G \hookrightarrow k[X]$.

Properties:

- π is projective
- $\dim(G/H) = \dim(G) - \dim(H)$
- If $H_1 \supseteq H$ there is a natural surjective morphism $G/H \to G/H_1$

6.2 Parabolic subgroups

Definition 6.4. A k-variety X is said to be *complete* if for every k-variety Y the projection $X \times Y \to Y$ is closed.

Properties:

1. A quasi-projective variety is complete if and only if it is projective
2. A closed subvariety of a complete variety is complete
3. If X_1 and X_2 are complete, so is $X_1 \times X_2$
4. If X is complete and irreducible, then $k[X] = k$
5. X is complete and affine if and only if X is finite
6. If $\phi : X \to Y$ is a morphism and X is complete, then $\phi(X)$ is closed and complete.

Definition 6.5. A closed subgroup $P \le G$ is said to be *parabolic* if G/P is a complete variety

Lemma 6.6. *If P is parabolic, then G/P is projective*

Lemma 6.7. *Let $\phi : Y_1 \to Y_2$ be a bijective morphism of G-varieties, where G is a linear k-algebraic group. Then for every variety Y, the morphism $\phi \times \mathbb{I}_Y : Y_1 \times Y \to Y_2 \times Y$ is a homeomorphism.*

Lemma 6.8. *Let G be a k-algebraic group acting transitively on a projective k-variety V. Let $v_0 \in V$ and $P = \mathrm{Stab}_G(v_0)$. Then P is parabolic.*

Proof. We have a natural bijective morphism $G/P \to V$, $[g] \mapsto gv_0$. For any Y, the map $G/P \times Y \to V \times Y$ is a homeomorphism and it also commutes with the projection to Y. Therefore, G/P is complete if and only if V is. □

Corollary 6.9. *The subgroup P of GL_n given by*

$$P = \left\{ \begin{pmatrix} A_1 & & & * \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_{t+1} \end{pmatrix} \middle| A_i \in \mathrm{GL}_{d_1 - d_{i-1}}\, i = 1, \dots t+1 \right\}$$

is parabolic.

From now on, G is an algebraic k-group.

Proposition 6.10. *Let $Q \subseteq P \subseteq G$ be closed subgroups. If Q is parabolic in G, then it is parabolic in P.*

Proposition 6.11. *Let $Q \subseteq P \subseteq G$ be closed subgroups. If P is parabolic in G and Q is parabolic in P then Q is parabolic in G.*

Proposition 6.12. *Let $Q \subseteq P \subseteq G$ be closed subgroups. If Q is parabolic in G then P is parabolic in G.*

Proposition 6.13. *Let $P \subseteq G$ be closed subgroup. Then P is parabolic in G if and only if P^0 is parabolic in G^0.*

Theorem 6.14. *A connected linear algebraic group G contains a proper parabolic subgroup if and only if G is non-soluble.*

Borel's fixed point theorem. *Let G be a connected linear k-algebraic group which is soluble and let X be a complete G-variety. Then there exist a fixed point in X.*

Proof. Let $Y \subseteq X$ be a closed orbit and $y \in Y$. The stabilizer of y is a parabolic subgroup. By Theorem 6.14 we know that this can't be a proper subgroup and therefore it must be the whole of G. □

7 Borel Subgroups and Maximal Tori (Alex)

7.1 Borel Subgroups

There are 3 important pieces of Algebraic groups to look at:

- Parabolic subgroups (always exists). We spoke about this in the previous section. Recall that P is closed parabolic if and only if G/P is a complete projective variety
- Borel subgroups. Leads to Weyl groups
- Maximal Tori. Leads to Weyl groups and Cartan subgroups

Recall Theorem 6.14, there is in a sense an obstruction. This motivates the following definition:

Definition 7.1. A subgroup B of an algebraic group B is called *Borel subgroup* if they are maximal with respect to the properties of being closed connected and soluble.

This is well defined as we only need to take finite unions, it certainly always exists and $B = G$ is possible.

Example. Let $G = \mathrm{GL}_n$. We claim that the upper triangular matrices T_n are Borel subgroup. They are clearly closed and connected. They are indeed soluble as $U_n \leq T_n$ with U_n nilpotent (so soluble) and T_n/U_n in $(\mathbb{G}_m)^n$. We want to show that T_n is maximal. Recall that if G acts transitively on a projective variety, the stabilisers are parabolic. Now T_n is the stabiliser of a maximal flag, therefore T_n is parabolic. Suppose $T_n \subsetneq P$ with P connected, then T_n parabolic implies that P is non-soluble, hence T_n is maximal.

Theorem 7.2.

1. *A closed subgroup of G is parabolic if and only if it contains a Borel subgroup*
2. *A Borel subgroup is parabolic*
3. *Any two Borel subgroups are conjugate*

The main technique to prove this depends on the following theorem

Borel's fixed point theorem. *Let G be a connected linear soluble algebraic group, X a complete G-variety. Then X has a fixed point*

Proof of 7.2. Without loss of generality we can assume that G is connected (since P is parabolic in G if and only if P^0 is parabolic in G^0)

1. $\Rightarrow$: Take P parabolic, so G/P is projective. Let B be any Borel subgroup of G. So we have a permutation representation $B \circlearrowright G/P$ defined by $gP \mapsto bgP$. So by Borel's fixed point theorem, there exists g such that $\mathrm{Stab}_B(gP) = B$, i.e., $BgP \subset gP$, $g^{-1}Pg \subset P$. But B is Borel if and only if $g^{-1}Bg$ is.

   $\Leftarrow$: Assuming 2: if it clear
2. We will do induction on $\dim G - \dim B$. Recall that G contains a proper parabolic subgroup if and only if G is soluble. Either $G = B$ (our base case), or there exists $P \subsetneq G$ which is parabolic. Without loss of generality $B \subseteq P$ (by applying Borel's fixed point theorem). So $B \circlearrowright G/P$, therefore there exists g such that $BgP \subseteq gP$, hence $B \subseteq gPg^{-1}$. But as $\dim G > \dim P$ (connected) we have that B is parabolic in P and hence in G.
3. We just use the same technique again given B_1, B_2, we can conclude $g^{-1}B_1g \subset B_2$, $g^{-1}B_2g \subset B_1$ and so $\dim B_1 = \dim B_2$.

□

Theorem. *Let G be closed, connected soluble subgroup of GL_n. Then G can be conjugated into T_n*

Proof. Could do directly by construction, or just use the above theorem. □

7.2 Maximal Tori

Definition 7.3. In a connected soluble group a *Maximal torus* is a maximal subgroup isomorphic to $(\mathbb{G}_m)^k$.

Proposition 7.4. *Let G be connected soluble and G_u the set of unipotent elements. G_u is a closed connected nilpotent normal subgroup of G. Moreover G/G_u is a torus*

In some sense G_u and Tori are orthogonal

Theorem 7.5. *Let G be a connected soluble algebraic group then:*

1. *If $s \in G$ is semisimple (diagonalisable under embedding to GL_n) then s lies in a maximal torus*
2. *The centraliser $Z_G(s)$ of a semisimple element s is connected*
3. *Any two maximal Tori of G are conjugate*
4. *The dimensions of a maximal torus is $\dim G/G_u$*
5. *If T is a maximal torus then the morphism $T \times G_u \to G$ is an isomorphism*

8 Recap and Stuff (Chris B)

Proposition 8.1. *Any two maximal Tori are conjugate.*

Proof. Let T_i be two torus. We know that $T_i \subset B_i$ and $gB_1g^{-1} = B_2$ hence $gT_1g^{-1} = T_2$. $\square$

Definition 8.2. Let G be a linear algebraic group, then let $\mathrm{rk}(G) = \dim(T)$.

Example.

- $\mathrm{rk}(\mathrm{GL}_n) = n$
- $\mathrm{rk}(\mathrm{SL}_n) = n - 1$
- $\mathrm{rk}(D_n) = n$

Theorem 8.3. *Let G be a connected linear algebraic group. Then*

1. *Every element is contained in some Borel*
2. *Every semi-simple element is contained is some maximal Torus*
3. *If X is either a Borel or Parabolic subgroup, then $N_G(X) = X$.*

8.1 Semi-simple and reductive groups

Lemma 8.4. *Let A and B be closed normal subgroup of a connected linear algebra group G. Then*

- *AB is a closed normal subgroup of G.*
- *If A and B are connected, then so is AB*
- *If A and B are soluble, then so is AB*
- *If A and B are unipotent, then so is AB*

By Zorn's Lemma there exists a maximal closed connected soluble normal subgroup of G, called the *radical of* G, denoted $R(G)$.

There exists a maximal closed connected soluble normal unipotent subgroup of G, called the *unipotent radical* of G, denoted $R_u(G)$.

Definition 8.5. A connected linear algebraic group G is called *semi-simple* if $R(G) = \{1\}$. It is called *reductive* if $R_u(G) = \{1\}$.

Fact.

- $R(G)_u = R_u(G)$
- $R(G) = \left(\cap_{\text{Borels } B} B\right)^0$
- $R_u(G) = \left(\cap_{\text{Borels } B} B_u\right)^0$
- $C(G)^0 \subseteq R(G)$

Example. Let $G = \mathrm{GL}_n$. As this is a connected group $G^0 = G$. Hence $C(G) \subseteq R(G)$ and since $C(G) = \mathbb{G}_m \subseteq R(\mathrm{GL}_n)$, we have that GL_n is not semi-simple.

Note that $R(\mathrm{GL}_n) \subseteq \cap_B B = \mathbb{T}_n \cap {}^t\mathbb{T}_n = \mathbb{D}_n$, but as $\mathbb{D}_n$ is not normal, we take its maximal and so $R(\mathrm{GL}_n) = K^* \cdot I_n$. Hence $R_u(\mathrm{GL}_n) = \{1\}$, so GL_n is reductive.

Let us do the same for SL_n. We get $R(\mathrm{SL}_n) \subseteq R(\mathrm{GL}_n) = K^* \cdot I_n$, but SL_n has only elements with determinants one, hence $R(\mathrm{SL}_n) = \{1\}$. So SL_n is semi-simple.

Proposition 8.6. *If G is a connected linear algebraic group, then $G/R(G)$ is semi-simple and $G/R_u(G)$ is reductive.*

Fact.

- *If G is semi-simple then $G = (G, G)$ and has finite centre*
- *If G is connected and semi-simple of rank 1 then $G \cong \mathrm{SL}_2$ or PSL_2.*
- *If G is reductive then $R(G)$ is a maximal Tori.*

8.2 Summary of the Results

8.2.1 Conjugation

- Conjugate of maximal torus is a maximal torus and any two maximal tori are conjugate.
- Every maximal torus is contained in some Borel
- Any two Borels are conjugate
- Any conjugate of a Parabolic is Parabolic

8.2.2 Inclusions

- Every Parabolic contains a Borel, and every Borel is in a Parabolic.
- The centraliser of a maximal Torus T (called Cartan) is connected nilpotent and equal to $N_G(T)^0$ and is contained in every Borel containing T.
- If P is Parabolic then $N_G(P) = P$
- If $C = C(T)$ (Cartan), then $N_G(C) = N_G(T)$.
- If P is Parabolic, then $C(P) = Z_G(P) = C(G)$.
- The collection of Borel's of G is in bijection with the projective variety G/B for some Borel B.
- The collection of maximal Tori is in bijection with $G/N_G(T)$ for some maximal Tori T.