Intersection Theory

1 Introduction (Simon Hampe)

1.1 Some motivational examples: What should intersection theory be?

Example. What is the "intersection" of $C := \{y = x^3\}$ and $l = \{y = 0\}$?

Naive answer: The point $(0, 0)$.

Problem: It's not "continuous". Replace $l_t = \{y = t\}$, then $l_t \cap C = 2$ points. This should somehow be reflected in the "limit" $t \to 0$.

Algebraic approach: Intersection scheme: $X = \operatorname{Spec} K[x, y]/\langle y - x^2, y\rangle = \operatorname{Spec} K[x]/\langle x^2\rangle$, $\dim_K = \mathcal{O}_{X,(0,0)} = 2$. So we get somewhat more informal answer, twice the point $(0, 0)$.

Example. What is the intersection of a line L in $\mathbb{P}^2$ with itself?

Naive: L

Better answer: The equivalence class (?) of a point in $\mathbb{P}^2$.

Example. Numerative geometry

Question: How many lines in $\mathbb{P}^3$ intersect four general lines? (Why four? and why General? should not depend on the choice of lines)

Usual approach: via moduli spaces.

1. Find a suitable *parameter space* M for the objects we want to count. (Here: lines in $\mathbb{P}^3 \cong G(1, \mathbb{P}^3) = G(2, 4)$)
2. Find subscheme (or equivalence classes thereof) that correspond to object fulfilling certain geometric conditions. (Here: $[Z] \cong$ lines meeting a given line)
3. Compute the "intersection product" of these classes. In particular, the product should be 0-dimensional, if we want a finite answer. (Here: $\dim M = 4$, $\dim Z = 3$ therefore $[Z]^4$ is 0-dimensional and $\deg[Z]^4 = 2$)

Insight: We need some equivalence

Definition 1.1. $Z(\lambda)$ =free abelian group on subvarieties on $X = \oplus Z_k(X)$ (where Z_K is k-dimensional). We say $A \sim B$ if and only if there exists a subvariety $X \subseteq \mathbb{P}^1 \times X$ such that A is the fibre over 0 and B is the fibre over 1.

$A(X) = Z(X)/\sim$ is the *Chow group of* X.

Theorem 1.2 (Hartshorne, pg 427)**.** *There is a unique intersection theory on the Chow groups of smooth (quasi-projective) varieties over k $(= \overline{k})$ fulfilling:*

1. *It makes $A(X)$ into a commutative ring with 1, graded by codimension*
2. $\vdots$
3. $\vdots$
4. $\vdots$

5. $\vdots$

6. *If Y, Z intersect properly, then $Y \cdot Z = \sum m_j w_j$ where w_j are components of $Y \cap Z$, m_j depends only on a neighbourhood of w_j.*

1.2 An approach we will not take: Chow's moving lemma

Definition 1.3.

1. Two subvarieties A, B of C are *dimensionally transverse*, if $A \cap B$ only have components of $\mathrm{codim} A + \mathrm{codim} B$
2. A and B are *transverse* at p, if X, A, B are smooth at p and $T_pA + T_pB = T_pX$
3. A and B are *generically transverse*, if every components of $A \cap B$ contains a point p, at which A, B are transverse.

Theorem 1.4 (Strong Chow's Moving Lemma [Chevakey '58, Roberts '70, Eischenbud-Harris, Chapter 5.2])**.** *Let X be smooth, quasi-projective over $k = \overline{k}$.*

1. *If $\alpha \in A(X)$, $B \in Z(X)$, there exists $A \in Z(x)$ such that $[A] = \alpha$ and A, B are generically transverse.*
2. *If $A, B \in Z(X)$ are generically transverse, then $[A \cap B]$ depends only on $[A]$ and $[B]$.*

Corollary 1.5. *For X smooth, quasi-projvective, we can define the intersection product on $A(X)$ by $\alpha \cdot \beta = [A \cap B]$, where $[A] = \alpha$, $[B] = \beta$ and A, B are generically transverse.*

Remark. This is not generalisable! Not really constructive.

Intersection multiplicity

If A, B are only dimensionally transverse, can we write $[A] \cdot [B] = \sum m_i [C_i]$ where $C_i$are components of $A \cap B$ and m_i to be determined?

Easy case: Plane curves. If F, G are plane curves in $\mathbb{A}^2$, $p \in \mathbb{A}^2$,

$$i(p : F \cdot G) = \dim_K \mathcal{O}_{F \cap G, p} = \begin{cases} 0 & \text{if } p \neq F \cap G \\ \infty & \text{if } F, G \text{ have a common component through } p \\ \text{finite} & \text{otherwise} \end{cases}$$

This works!

Generalisation 1: Module length. If M is a finitely generated A-module, then there exists a chain $M = M_0 \supsetneq \cdots \supsetneq M_r = 0$ such that $M_{i-1}/M_i = A/P_i$ where P_i is prime. If all P_i are maximal, then r is independent of our choice and we call the length of M $l_A(M) := r$.

Lemma 1.6. *If A, B are Cohen-Macauly and dimensionally transverse, Z a component of $A \cap B$, then $i(Z, A \cdot B) = l_{\mathcal{O}_{A \cap B, Z}}(\mathcal{O}_{A \cap B, Z})$*

Generalisation 2: Serre's multiplicity formula.

Theorem 1.7 (Serre '57)**.** *On a smooth variety X, the multiplicity of a component Z of a dimensionally transverse intersection $A \cap B$ is*

$$\sum_{i=0}^{\dim X} (-1)^i \mathrm{length}_{\mathcal{O}_{A \cap B, Z}} \left(\mathrm{Tor}_i^{\mathcal{O}_{X,Z}} (\mathcal{O}_{A,Z}, \mathcal{O}_{B,Z}) \right)$$

1.3 Our approach: Following Fulton's book

The standard construction

Given the fibre square

$$\begin{array}{ccc} W = f^{-1}(X) & \xrightarrow{j} & V \\ \downarrow g & & \downarrow f \\ X & \xrightarrow[i]{} & Y \end{array}$$

where f is any morphism, i is closed, regular embedding.

$X \cdot V = S^*([C])$ where $S \cdot W \to g^* N_X Y =: N$ is the zero section, $C = C_W V$ embedded in N (N_X is the normal bundle, C_W the normal cone)

Example. Let $\mathcal{X}$ be smooth, then $X = \mathcal{X}$, $T = X \times X$, $i = \delta : x \mapsto (x, x)$ regular. For A, B subvarieties, set $V = A \times B$, f the inclusion then $W = A \cap B$ and $[A] \cdot [B] = X \cdot V$

Example. Let $H_1, \ldots, H_d$ be effective Cartier divisors on some variety $\mathcal{X}$, let $\mathcal{V} \subseteq \mathcal{X}$ be a subvariety. Let $X = H_1 \times \cdots \times H_d$, $Y = \mathcal{X} \times \cdots \times \mathcal{X}$, i be the product embedding and $V = \mathcal{V}$. Then $W = H_1 \cap \cdots \cap H_d \cap V$ and $X \cdot V$ is a class of this.

Fact. *Can write this in terms of* Chern *and* Segre classes. *Then* $X \cdot V = \{c(N) \cap s(W, V)\}_{\text{expeted dimension}}$.

2 Divisors and Rational Equivalence (Paulo)

2.1 Length of a module

Let R be a commutative ring, M a module. Consider chains $M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_m = (0)$.

Definition 2.1. We say the *length* $l_R(M) =$ maximal among all length of such chains.

Fact (EIS,Thm 2.15)**.** $M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_m = (0)$*, then* $l_R(M) = m$ *if and only if* M_i/M_{i+1} *is simple for all* i*.*

Example. Let $R = K$ a field, $M = V$ a vector space. Let $\{e_1, \ldots, e_n\}$ be a basis of V. Then we have the chain $M_0 = V \supsetneq \langle e_1, \ldots, e_{n-1} \rangle \supsetneq \cdots \supsetneq \langle e_1 \rangle \supsetneq (0)$. Hence $l_K(V) = \dim_K(V)$.

Example. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}$, we show that $l_{\mathbb{Z}}(\mathbb{Z}) = \infty$ as $\mathbb{Z} \supsetneq 2\mathbb{Z} \supsetneq 2 \cdot 13\mathbb{Z} \supsetneq \ldots$

Example. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}/m\mathbb{Z}$ where $m = p_1 \cdot \cdots \cdot p_r$, then $l_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}) = r$.

Let X be a scheme of pure dimension n. Let $V \subseteq X$ be a subvariety. Let $f \in R(X)^*$. We want to define $\operatorname{ord}(f, V)$. To do this, let $f = a/b$ where $a, b \in \mathcal{O}_X$. Then

$$\operatorname{ord}(f, V) = l_{\mathcal{O}_{V,X}}(\mathcal{O}_{V,X}/(a)) - l_{\mathcal{O}_{V,X}}(\mathcal{O}_{V,Z}/(b))$$

Where by $O_{V,X}$ we mean the localisation of $\mathcal{O}_X$ at $I(V)$, i.e., let S be the complement of $I(V)$, then $\mathcal{O}_{V,X} = \mathcal{O}_X S^{-1}$.

Example. Consider $\mathbb{C}^2_{(x,y)}$, let $V = \{x = 0\}$, $f = (x)^2$. Then $\operatorname{ord}(f, V) = l_{\mathcal{O}_{V,X}}(\mathcal{O}_{V,X}/(x)^2) = 2$ as $\mathcal{O}_{V,X}/(x)^2 \supsetneq \mathcal{O}_{V,X}/(x) \supsetneq (0)$. (For this example, $\mathcal{O}_X = \mathbb{C}[x, y]$ and $S = \mathbb{C}[x, y] \backslash (x)$, so $\mathcal{O}_{V,X} = \mathbb{C}[x, y]S^{-1} \ni \{f/g | x \nmid g\}$)

2.2 Divisors

Let X be a variety over K of dimension n. Let $Z_{n-1} = \{\sum_{\text{finite}} a_i [V_i] | a_i \in \mathbb{Z}, V_i \, \text{subvariety of } X \text{ of dim} = n-1\}$. We call an element $D \in Z_{n-1}$ a *Weil Divisor* and an element $[V_i]$ a *prime divisor*.

Definition 2.2. Let $f \in R(X)$, we define a divisor associated to f as $\operatorname{div}(f) = \sum_V \operatorname{ord}(f, V)[V]$. We call then *principal divisors*.

Definition 2.3. The *class group* of X is $\mathrm{Cl}(C) = Z_{n-1}/\text{principal divisors}$.

Definition 2.4. Let $D \in Z_{n-1}$ is *effective* if $a_i \geq 0$ for all i.

Definition 2.5. A *Cartier Divisor* is a collection $\{(U_i, f_i)\}_{i \in I}$ such that:

- $\{U_i\}$ is an open cover of X
- $f_i \in R(U_i)$
- For all i, j, $f_i/f_j \in \mathcal{O}^*(U_i \cap U_j)$.

Let $D\{(U_i, f_i)\}$, we can associate to it a Weil divisor: $[D] = \sum \mathrm{ord}(D, V)[V]$ where $\mathrm{ord}(D, V) = \mathrm{ord}(f_i, V)$ for any i such that $U_i \cap V \neq \emptyset$.

We can also associate to it a line bundle: $\mathcal{O}(D)$ with transition data $\{(U_i, f_i)\}$ (so a section of it is a collection $r = \{r_i\}$ where $r_i \in \mathcal{O}(U_i)$ and $r_i = f_i/f_j r_j$)

$\mathrm{Pic}(X) := \text{Cartier Div}/\text{Principal Div} \cong \text{Line Bundle}/\text{Isom}$.

Let X be a scheme over K of dimension n. Define $Z_k = \{\sum a_i[V_i] | V_i \subseteq X \text{ subvariety of dimension } k\}$. We call $C \in Z_k$ a k-cycle.

Example. Let Y be a scheme of pure dimension m, $Y_1, \ldots, Y_l$ its irreducible components. We have $Z_m(Y) \ni [Y] = \sum m_i[Y_i]$ where $m_i = l_{\mathcal{O}_{Y_i,Y}}(\mathcal{O}_{Y_i,Y})$.

If $Y \subseteq X$ subscheme then $[Y] \in Z_m(X)$.

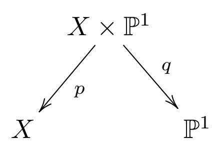
2.3 Rational equivalence

We want to consider $A_k := Z_k/\sim$ (where $\sim$ is to be determined), which we will call this the *Chow group*. There are two equivalent way to define the equivalence

1. Let $W \subseteq X$ be a subvariety of dimension $k+1$, $r \in R(W)$. Then $0 \sim \mathrm{div}(r) = \sum \mathrm{ord}(r, V)[V] \in Z_k(W)$ but we can also think of $\mathrm{div}(r) \in Z_k(X)$

Definition 2.6. $D_1, D_2 \in Z_k(X)$ are equivalent, $D_1 \sim D_2$, if $D_1 - D_2 = \sum \mathrm{div}(r_i)$ for some $r_i \in R(W_i)$.

2. Consider



Let $Y \subseteq X \times \mathbb{P}^1$ variety of dimension $k+1$, $f = q|_Y$ is dominant. $p_*[f^{-1}(0)] - p_*[f^{-1}(\infty)] \sim 0$

To see why they are equivalent see [Ful,Prop 1.6]

Remark.

1. $Z_k(X) \cong Z_k(X_{\mathrm{red}})$
2. If $k = m$, then $A_m(X) = Z_m(X)$

2.4 Pushforward

Let $f : X \to Y$ be a proper morphism. Let $V \subseteq X$ be a subvariety, this gives $f(V) = W$ a variety in Y. We can define $f_* : Z_k X \to Z_k Y$ by $[V] \mapsto \begin{cases} 0 & \dim W < \dim V \\ \deg(V, W) \cdot [W] & \text{otherwise} \end{cases}$ (where $\deg(V, W) := [R(V) : R(W)]$.

Theorem 2.7 (Ful, Thm 1.4)**.** *If $\alpha \sim 0$ then $f_*\alpha \sim 0$, hence we have well defined $f_* : A_k(X) \to A_k(Y)$.*

2.5 Pullback

Let $f : X \to Y$ be a flat morphism of relative dimension m. (Relative dimension means: if $V \subseteq Y$ a subvariety, then $f^{-1}(V)$ has every component of dimension $m + \dim(V)$)

We can to define $f^*[V] = [f^{-1}(V)]$. This extend by linearity to a map $f^* : Z_k(Y) \to Z_{k+m}(X)$.

Theorem 2.8 (Ful, Thm 1.7)**.** *If $\alpha \sim 0$ then $f^*\alpha \sim 0$, hence we have well defined $f^* : A_k(Y) \to A_{k+m}(X)$.*

Example.

1. Consider the open embedding $i : Y \hookrightarrow X$. Then i^* is just the restriction map, that is $[V] \mapsto [V \cap Y]$.
2. Let Z be a scheme of pure dimension m, consider $f : X \times Z \to X$. Then f^* is defined by $[V] \mapsto [V \times Z]$.
3. Consider $p : E \to X$ an affine (projective) bundle, then we still have p^*.

   Proposition 2.9. *If $p : E \to X$ is an affine bundle, $p^* : A_k X \to A_{k+m} E$ is surjective.*

2.6 Intersection with divisors

Consider $\alpha \in Z_k(X)$ and let D be a Cartier divisor on X. Then we want to define $D \cdot \alpha \in A_{k-1}(V)$. By linearity, we can assume $\alpha = [V]$. Two cases:

1. $V \nsubseteq \operatorname{supp}(D)$. Then D intersects with V, let $D = \sum a_i[W_i]$, then $D \cdot V = \sum a_i[W_i \cap V]$.
2. $V \subseteq \operatorname{supp}(D)$. We can not simply intersect. Let $i : V \hookrightarrow X$. From D consider the line bundle $\mathcal{O}(D)$. Consider the line bundle on V, $i^*\mathcal{O}(D)$. There is a Cartier divisors C on V such that $i^*\mathcal{O}(D) \cong \mathcal{O}(C)$. Then $[C] = V \cdot D \in A_{k-1}(V)$.

3 Chern Classes (Ian Vincent)

3.1 Motivation

(Following Eisenbud)

Let $\pi : E \to X$ of rank n be a vector bundle and there exists sections $s_1, \dots, s_n$ of π such that for every $p \in X$, $s_1(p), \dots, s_n(p)$ are linearly independent (in each fibre). Make some changes of coordinates so that $s_1(p), \dots, s_n(p)$ is a basis for each fibre.

Idea: If we have enough global sections finding their forced linear dependence measures the non-triviality (twisting) of $\pi : E \to X$.

3.2 Chern classes of line bundles

Let L be a line bundle over a scheme X. We define a function $c_1(L) \cap - : A_k(X) \to A_{k-1}(X)$ in the following way. If $[V] \in A_k(X)$ then choose a Cartier divisor C on V such that $L|_V \cong \mathcal{O}_V(C)$ then $c_1(L) \cap [V] := [C]$. We extend linearly to get a homomorphism $A_k(X) \to A_{k-1}(X)$

Remark. This is well defined. If $L = \mathcal{O}_X(D)$ then if $\alpha = [V]$ we have $c_1(L) \cap [V] = D \cdot \alpha$ as defined last time.

Properties (Fulton Prop 3.1)

1. Commutativity: Let L, L' be line bundles on X then $c_1(L) \cap (c_1(L') \cap \alpha) = c_1(L') \cap (c_1(L) \cap \alpha) \in A_{k-2}(X)$
2. Projection formula: Let $f : X' \to X$ be a proper morphism, L a line bundle on X and $\alpha \in A_k(X')$. Then $f_*(c_1(f^*L) \cap \alpha) = c_1(L) \cap f_*(\alpha)$
3. Pullback: Let $f : X' \to X$ be a flat morphism of relative dimension n, L a line bundle on X and $\alpha \in A_k(X)$. Then $c_1(f^*L) \cap f^*\alpha = f^*(c_1(L) \cap \alpha)$

4. Additivity: Let L, L' be line bundles on X, $\alpha \in A_n(X)$ then $c_1(L \otimes L') \cap \alpha = c_1(L) \cap \alpha + c_1(L') \cap \alpha$. In particular, $c_1(L^{-1}) \cap \alpha = -c_1(L) \cap \alpha$.

Example. Consider $X = \mathbb{P}^n$ and let L^k be a linear subspace of $\mathbb{P}^n$ with dimension k. Then $\mathcal{O}_{\mathbb{P}^n}(1) \leftrightarrow H$ hyperplane section of $\mathbb{P}^n$. Then $c_1(\mathcal{O}_{\mathbb{P}}(1)) \cap [L^k] = [L^{k-1}]$. More generally, if $X \subseteq \mathbb{P}^n$ is a subvariety, then $c_1(\mathcal{O}_{\mathbb{P}}(1)) \cap [X] = [X \cap H]$.

3.3 Segre classes

Let $\pi : E \to X$ be a vector bundle of rank $e+1$ on X. Let $P = P(E)$ (turn E into projective space), $\mathcal{O}_P(1)$ is the "canonical line bundle on P". Define homomorphism $s_i : A_k(X) \to A_{k-i}(X)$ by $s_i(E) \cap \alpha = \pi_*(c_1(\mathcal{O}_p(1))^{e+1} \cap \pi^* \alpha)$ where π^* is a flat pullback from $A_n(X) \to A_{k+e}(P)$. The product $c_1(\mathcal{O}_P(1))^{e+i}$ is just composition. This is called the ith Segre class.

Properties (Fulton 3.1)

1. Similarly we have commutativity
2. Projection
3. Pullback
4. For $\alpha \in A_k(X)$, $s_i(E) \cap \alpha$ if $i < 0$ and $s_0(E) \cap \alpha = \alpha$.

3.4 General Chern class

Let $\pi : E \to X$ be a vector bundle of rank $n = e+1$. We define $s_t(E) = 1 + s_1(E)t + s_2(E)t^2 + \ldots$. Then the Chern class $c_t(E)$ is the coefficient of the inverse power series, i.e., $c_t(E) = \sum c_i(E)t^i = s_t(E)^{-1}$.

Explicitly, $c_0(E) = 1$ (i.e., $c_0(E) \cap \alpha = \alpha$), $c_1(t) = -s_1(E)$. In general we have

$$c_i(E) = (-1)^i \det \begin{pmatrix} s_1(E) & 1 & 0 & \ldots & \ldots & 0 \\ s_2(E) & s_1(E) & 1 & 0 & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ \vdots & & & \ddots & \ddots & 0 \\ \vdots & & & & \ddots & 1 \\ s_i(t) & \ldots & \ldots & & s_2(E) & s_1(E) \end{pmatrix}$$

Remember $s_i(E)$ are endomorphism of $A_*(X)$ hence products here means compositions of functions.

Definition 3.1. The *total Chern class* is $c(E) = 1 + c_1(E) + \cdots + c_{e+1}(E)$

Properties (Fulton, Thm 3.2)

1. Commutativity
2. Projection
3. Pullback
4. Vanishing: $c_i(E) = 0$ for $i > \operatorname{rk} E$
5. Whitney sum: For any short exact sequences of Vector bundle on X: $0 \to E' \to E \to E'' \to 0$, then $c_t(E) = c_t(E')c_t(E'')$.

An important ingredients for this proof is the splitting construction: Let $\mathcal{S}$ be a finite collection of vector bundles on X. There is a scheme X and a flat morphism $f : X' \to X$ such that $f^* A_* X \to A_* X'$ is injective and furthermore for each vector bundle $E \in \mathcal{S}$, fE has a filtration of subbundles $E = E_r > \cdots > E_0 = 0$ such that $E_i/E_{i+1} = L_i$ a line bundle. Then $c_t(t) = \prod(1 + c_1(L_i)t)$.

3.5 Examples

- Consider $T_{\mathbb{P}^n}$ (the tangent bundles of $\mathbb{P}^n$), we have an exact sequence $0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(n+1)} \to T_{\mathbb{P}^n} \to 0$ (which is the dual of $0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n} \to 0$). If $H = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$, by the splitting principle then $c_t(\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)}) = (1+Ht)^{n+1}$. Now $\mathcal{O}_{\mathbb{P}^n}$ is a trivial bundle on $\mathbb{P}^n$ so by Whitney formula $c_t(T_{\mathbb{P}n}) = (1+\mathrm{id}\, t)^{n+1}$

- Let $X \subseteq \mathbb{P}^n$ be a smooth hypersurface in $\mathbb{P}^n$ of degree d. Let $i : X \hookrightarrow \mathbb{P}^n$ be a closed embedding. Then we have the sequence $0 \to T_X \to i^*T_{\mathbb{P}^n} \to \mathcal{N} \to 0$. We have $c_t(i^*T_{\mathbb{P}^n})$ is the restriction of $(1+Ht)^{n+1}$ to X. Now $c_1(\mathcal{N}) = c_1(i^*\mathcal{O}_{\mathbb{P}^n}(X)) = c_1(i^*\mathcal{O}_{\mathbb{P}}(d)) = dH$ by surjectivity of Chern classes of line bundles. So by Whitney formula, $c_t(T_X) = \frac{(1+Ht)^{n+1}}{(1+dHt)}$.

Theorem 3.2 (Fulton Thm 3.3)**.** *Let $\pi : E \to X$ be a vector bundle of rank r. The flat pullback $\pi^r : A_{k-r}(X) \to A_k(P(E))$ is an isomorphism for every $k \geq r$. In particular, each element $\beta \in A_k(P(E))$ is uniquely expressible in the form $\beta = \sum_{i=1}^{r} c_1(\mathcal{O}_{P(E)}(1))^i \cap \pi^r \alpha_i$ for some $\alpha_i \in A_{k-r+i}(X)$*

4 Segre Classes (Tom Ducat)

In the previous section we learned about Segre and Chern classes for line bundles.

Notation. $h_X = c_1(\mathcal{O}_X(1))$.

Brief recap of last section: Let $\pi : E \to X$ be a vector bundle over a scheme X of rank $e+1$, consider $\mathbb{P}(E)$, $\mathcal{O}_{\mathbb{P}(E)}(1)$ then the Segre class $s_i(E)$ is given by the formula: $A_kX \to A_{k-i}X$ defined by $\alpha \mapsto \alpha \cap s_i(E) = \pi_*(h_{\mathbb{P}(E)}^{i+e} \cap \pi^*\alpha)$. The Chern classes $c_i(E)$ are defined by $\sum_{i\geq 0} c_i(E)t^i = \left(\sum_{i\geq 0} s_i(E)t^i\right)^{-1}$

In this section, we want to generalised Chern classes to more general objects than vector bundles.

4.1 Cones

Consider $\mathcal{F}^\bullet = \oplus_{i\geq 0}\mathcal{F}^i$ to be a graded sheaf of $\mathcal{O}_X$-algebras over a scheme X. (Caveats: $\mathcal{O}_X \twoheadrightarrow \mathcal{F}_0$ surjective, $\mathcal{F}_1$ coherent and generate $\mathcal{F}^\bullet$). Then the *cone* of X is $C = \operatorname{Spec} \mathcal{F}^\bullet \xrightarrow{\pi} X$. There are two ways of getting a projective cone over X:

1. Projectivised cone $\mathbb{P}(C)$. $\mathbb{P}(C) = \mathrm{Proj}_X \mathcal{F}^\bullet$.
2. Projective closure $\overline{C}$. $\overline{C} = \mathrm{Proj}(\oplus_{0\leq i\leq d}\mathcal{F}^i z^{d-i}) \xrightarrow{\overline{\pi}} X$

Remark. $C \subseteq \overline{C}$ is a dense affine open subset and $\overline{C} \backslash C \cong \mathbb{P}(C)$.

The hyperplane section $h_{\overline{C}} \cap [\overline{C}] = [\mathbb{P}(C)]$.

For an arbitrary coherent sheaf $\mathcal{F}$ we can do this construction using $\mathrm{Sym}\mathcal{F} = \oplus_{i\geq 0}\mathcal{F}^{\otimes i}/\mathrm{sym\,perm}$.

Definition 4.1. The *Segre class* $s(C)$ is defined to be $s(C) = \overline{\pi}_*(\sum_{i\geq 0} h_{\overline{C}}^i \cap [\overline{C}]) \in A_*X$.

Proposition 4.2.

1. *If E is a vector bundle over X then $s(E) = c(E)^{-1} \cap [X]$ (where $c(E) = 1 + c_1(E) + \cdots + c_r(E)$ the total Chern class as defined in the previous section)*
2. *If C has irreducible components $c_1, \ldots, c_k$ with geometric multiplicities $m_1, \ldots, m_k$ then $s(C) = \sum_i m_i s(C_i)$.*

Proof.

1. The only issue that needs to be checked is $\overline{E} = \mathrm{ProjSym}(E \oplus \mathcal{O}_X)$. Now the short exact sequence $0 \to \mathcal{O}_X \to \overline{E} \to E \to 0$ gives rise to $c(\overline{E}) = c(E)c(\mathcal{O}_X) = c(E)$
2. This follows from $[\overline{C}] = \sum m_i [\overline{C}_i]$

□

Remark. If we have a short exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{E} \to 0$ where $\mathcal{E}$ is locally free, then $s(\mathcal{F}) = s(\mathcal{G}) \cap c(E)$.

4.2 Normal Cones

Take a closed subscheme $X \subset Y$ with ideal sheaf $\mathcal{I} = \mathcal{I}_{X/Y}$. The *normal cone* of X in Y is $C_X Y := \mathrm{Spec} \oplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$. The Segre class of X in Y is $s(X,Y) := s(C_X Y) \in A_* X$.

Recall: The blow-up of X in Y is $\mathrm{Bl}_X Y := \mathrm{Proj}_Y \oplus_{n\geq 0} \mathcal{I}^n \xrightarrow{\sigma} Y$ and $E = \sigma^{-1}(X)$ the exceptional divisor, has ideal sheaf $\mathcal{O}(1)$. $E = \mathrm{Proj}(\oplus \mathcal{I}^n \otimes_{\mathcal{O}_Y} \mathcal{O}_X) = \mathrm{Proj}(\oplus \mathcal{I}^n/\mathcal{I}^{n+1}) = \mathbb{P}(C_X Y)$.

Trick: $X \subset Y$, consider $\mathbb{A}^1 \times Y \supset \{0\} \times X$, define $M_X Y := \mathrm{Bl}_{X\times\{0\}} Y \times \mathbb{A}^1 \to Y$. The exceptional divisor is isomorphic to $\overline{C_X Y}$.

Example. $X \subset Y$ is embedded regularly, i.e., the normal cone is a vector bundle then $s(X,Y) = c(C_X Y)^{-1} \cap [X]$.

Lemma 4.3. *Let $X \subset Y$, Y pure dimensional with irreducible components $Y_1, \dots, Y_k$ and multiplicities $m_1, \dots, m_k$ then $s(X,Y) = \sum m_i s(X_i, Y_i)$ where $X_i = X \cap Y_i$.*

Proof. Consider $M_X Y$ has irreducible components $M_{X_i} Y_i$, $[M_X Y] = \sum m_i \, [M_{X_i} Y_i] \in A_* M_X Y$. So we get $\left[\overline{C_X Y}\right] = \sum m_i \left[\overline{C_{X_i} Y_i}\right]$. □

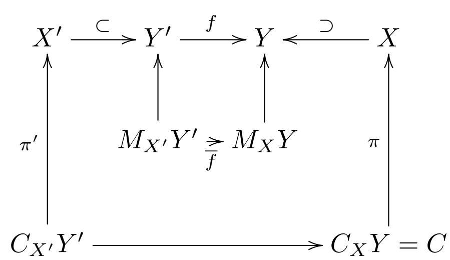
Proposition 4.4. *If $f : Y' \to Y$ is a morphism of pure dimensional schemes, $X' \subset Y'$, $X \subset Y$ are closed subschemes such that $X' = f^{-1}(X)$ is the scheme theoretic pull-back. Then*

1. *Push-forward: If f is proper, Y irreducible, each components of Y' maps onto Y then $f_* S(X', Y') = \deg(Y'/Y) s(X,Y) \in A_* X$.*
2. *Pull-back: If f is flat then $f^* s(X,Y) = s(X',Y') \in A_* X'$*

Note that $\deg(Y'/Y) = \sum m_i \deg(Y'_i/Y)$

Proof.

1. Reduce to Y' irreducible,



2.

$$\begin{aligned} f^* s(X,Y) &= f^* \pi_* (\sum_{i\geq 0} h^i_{\overline{C}} \cap [C]) \\ &= \pi'_* \overline{f}^* (\sum_{i\geq 0} h^i_{\overline{C}} \cap [\overline{C}]) \\ &= \pi'_* (\sum h^i_{\overline{C}} \cap [\overline{C}]) \\ &= s(X', Y') \end{aligned}$$

□

Corollary 4.5. *Consider $\sigma : \widetilde{Y} : \mathrm{Bl}_X Y \to Y$ with exceptional divisor E then $s(X,Y) = \sum_{i\geq 1} (-1)^{i-1} \sigma_*(E^i)$.*

Example. Let Y be a surface, and let A, B, D be effective Cartier Divisors. Let A, B intersect transversely at smooth points $p \in Y$. Let X be the scheme theoretic intersection $(A+D)\cap(B+D)$. Then $s(X,Y) = [D]-[D^2]+[p]$. To see this, let $\sigma : \widetilde{Y} = \mathrm{Bl}_p Y \to Y$, $\widetilde{X} = \sigma^* D + E$ (where E is the exceptional divisor). Then

$$\begin{aligned} S(X,Y) &= \sigma_* s(\widetilde{X}, \widetilde{Y}) \\ &= \sigma_*((1-\widetilde{X})[\widetilde{X}]) \\ &= \sigma_* \widetilde{X} - \sigma_*(\sigma^* D^2 + 2\sigma^* DE + E^2) \\ &= [D] - [D^2] + [p] \end{aligned}$$

5 The basic construction (Simon)

5.1 The basic construction

The basic set up is the following: A fibre square is

$$\begin{array}{ccc} W & \xrightarrow{j} & V \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \overset{i}{\hookrightarrow} & Y \end{array}$$

where

- $i : X \hookrightarrow Y$ is a regular embedding of dimension d
- V is purely k-dimensional, $f : V \to Y$ morphism
- $W = f^{-1}(X)$ is the inverse image scheme

Some preliminary definitions and facts:

- $N := g^* N_X Y$ a bundle of W (Recall $N_X Y = C_X Y = \mathrm{Spec}(\oplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ where $\mathcal{I}$ is the ideal sheaf of X in Y), with $\pi : N \to W$ the projection and $s : W \to N$ the zero section

Fact 5.1. *Recall that $\pi^* : A_{k-d}(W) \to A_k(N)$ is an isomorphism. We define $s^* = (\pi^*)^{-1} : A_k(N) \to A_{k-d}(W)$*

- $C = C_W V$ the normal cone

Fact 5.2. *If $\mathcal{I}$ is the ideal sheaf of X in Y, $\mathcal{J}$ the ideal sheaf of W in V, then there is a surjective morphism $\oplus_n f^*(\mathcal{I}^n/\mathcal{I}^{n+1}) \to \oplus_n \mathcal{J}^n/\mathcal{J}^{n+1}$. This gives a closed embedding* $\begin{array}{ccc} C & \hookrightarrow & N \\ & \searrow & \downarrow{\scriptstyle \pi} \\ & & W \end{array}$

So we now have

$$\begin{array}{ccc} C \hookrightarrow & N & \\ \searrow & \downarrow{\scriptstyle \pi} \uparrow{\scriptstyle s} & \\ & W & \xrightarrow{j} V \\ & {\scriptstyle g}\downarrow & \downarrow{\scriptstyle f} \\ & X & \overset{i}{\hookrightarrow} Y \end{array}$$

Fact 5.3. *C is purely k-dimensional, so $[X] \in A_k(N)$.* (This can be seen by: the blow up of $V \times \mathbb{A}^1$ in $W \times \{0\}$ is purely $(k+1)$-dimensional. So the exceptional divisor $P(C \oplus 1)$ is Cartier. Hence $P(C \oplus 1)$ is purely k-dimensional and contains Cas a dense open subset)

Definition 5.4. The *intersection product of V by X in Y* is $X \cdot V = X \cdot_Y C := s^*[C] \in A_{k-d}(W)$ (i.e., the unique class $[Z]$ such that $\pi^*[Z] = [C]$)

Proposition 5.5.

1. *$X \cdot V = \{c(N) \cap s(W,V)\}_{k-d}$ (where $s(W,V) = s(C) = q_*(\sum_{i \geq 0} c_1(O(1))^i \cap P(C \oplus 1))$ with $q : P(C \oplus 1) \to W$)*

2. *If $d = 1$ (so X is a Cartier divisor), V a variety and f a closed embedding, then $X \cdot V$ is the same as intersection with a divisor (as defined before)*

3. *If Y is pure dimensional, f a regular embedding, then $X \cdot V = V \cdot X = (V \times X) \cdot \Delta_Y$. ie., setup:*

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle \delta} \\ V \times X & \hookrightarrow & Y \times Y \end{array}$$

4. *If $W \hookrightarrow V$ is a regular embedding of codimension d' with normal bundle $N' = C_W V$. Then $X \cdot V = c_{d-d'}(W/N') \cap [w]$.*

5. *If $X \times \mathbb{P}^1 \hookrightarrow \mathcal{Y}$ is a family of regular embeddings, $\mathcal{V}$ a subvariety of $\mathcal{Y}$, $\mathcal{V}$ and $\mathcal{Y}$ are flat over $\mathbb{P}^1$. Then $X \cdot_{Y_t} \cdot V_t$ are equal for all t.*

5.2 Distinguished components and canonical decomposition

Assume $[C] = \sum m_i [C_i]$ with C_i the irreducible components of C. $W \geq Z_i := \prod(C_i)$ are the distinguished components of $X \cdot V$. For $N_i := N|_{Z_i}$, s_i its zero section, $\alpha_i := s_i^*[C_i]$. Then $X \cdot V = \sum m_i \alpha_i$ is the canonical decomposition of $X \cdot V$.

Example. Let $Y = \mathbb{P}^2_{[x,y,z]}$, $X_1 = \{xy = 0\}$, $X_2 = \{x = 0\}$ and $P = \{x = y = 0\}$. We have to possibilities to intersect X_1 and X_2.

1.

$$\begin{array}{ccc} W = X_2 & \rightarrow & V = X_2 \\ \downarrow & & \downarrow \\ X = X_1 & \longrightarrow & Y \end{array}$$

Hence $C = C_W V = X_2$. In particular, X_2 is the only distinguished components

2.

$$\begin{array}{ccc} W = X_2 & \hookrightarrow & V = X_1 \\ \downarrow & & \downarrow \\ X = X_2 & \longrightarrow & Y \end{array}$$

Then let $I = \langle \overline{x} \rangle$ in $k[x,y]/\langle xy \rangle$. We have $\oplus_{n \geq 0} I^n / I^{n+1} \cong k[x,y,T]/\langle x, yT \rangle$. So we can see that C has two components, namely $\{x = y = 0\}$ and $\{x = T = 0\}$. Now $N = \mathrm{Spec}(k[x,y,T]/\langle x \rangle)$. So the distinguished components are X_2 and P.

5.3 Refined intersection

Given our fibred square

$$\begin{array}{ccc} W & \xrightarrow{j} & V \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \overset{i}{\hookrightarrow} & Y \end{array}$$

we have a homomorphism $i^! : Z_k(V) \to A_{k-d}(W)$ defined by $\sum n_i[V_i] \mapsto \sum n_i(X \cdot V_i)$ (note that $X \cdot V_i$ are actually lies in $A_{k-d}(X \cap V_i)$).

Fact (Non-trivial)**.** *This passes to rational equivalence!*

We have *refined Gysin homomorphism* $i^! : A_k V \to A_{k-d} W$.

Notation. If $V = Y$ and $f = \mathrm{id}$ we write $i^! = i^* : A_k Y \to A_{k-d} X$. In this case the map is $[Z] \mapsto s_N^*[C_{Z\cap X} V]$.

Remark. For any purely k-dimensional cycle $[Z]$, $i^![Z] = X \cdot Z$.

Theorem 5.6. *Given the fibre diagram*

$$\begin{array}{ccc} X'' & \longrightarrow & Y'' \\ \downarrow{\scriptstyle q} & & {\scriptstyle p}\downarrow \\ X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{i} & Y \end{array}$$

where $i : X \to Y$ is a regular embedding of codimension d.

1. (Push-forward) *If p is proper, $a \in A_k Y''$, then $i^! p_*(\alpha) = q_*(i^!\alpha)$ (note that the first $i^!$ is with respect to* $\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$*, while the second $i^!$ is with respect to* $\begin{array}{ccc} X'' & \longrightarrow & Y'' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$

   Merit: e.g., we can compute $X \cdot Y'$ by calculating $X \cdot (\text{some blowup of } Y'')$. Therefore we see the advantage of allowing arbitrary *morphism* to Y.

2. (Pull-back) *If p is flat of regular dimension n, $\alpha \in A_k Y'$ then $i^! p^*(\alpha) = q^* i^! \alpha$*

   Merit: we can compute (part of) intersections products of locally by restricting to open subschemes

5.4 The intersection ring

Assumption: Y is smooth which implies $\delta : Y \to Y \times Y$ (defined by $y \mapsto (y, y)$) is a regular embedding

Setup: For $x \in A_k(Y)$, $y \in A_l(Y)$

$$\begin{array}{ccc} Y & \xrightarrow{\delta} & Y \times Y \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ Y & \xrightarrow[\delta]{} & Y \times Y \end{array}$$

we define $x \cdot y := \delta^*(x \times y) \in A_{k+l-n}(Y)$

Theorem 5.7. *This makes $A_*(Y)$ into a graded (by codimension), commutative ring with unit $pY]$.*

The assignment Y to $(A_*(Y),)$ is a contravariant function form smooth varieties to rings.

6 Schubert Calculus (Aurelio Carlucci)

6.1 Recap on $G(k, n)$

Let V be a complex vector space of dimension n, let $G(k, V) = \{k - subspace\, of\, V\}$, $G(k, n) = G(k, \mathbb{C}^n)$.

Let $\Lambda \in G(K, n)$

$$\begin{pmatrix} v_1 \\ \vdots \\ v_K \end{pmatrix} = \begin{pmatrix} v_{1,1} & \dots & v_{1k} \\ \vdots & & \\ v_{k1} & & v_{kk} \end{pmatrix}$$

where $\operatorname{rk} K = k$. Let $I = \{i_1, \dots, i_k\} \subset \{1, \dots k\}$, $V_I = \operatorname{Span}\{e_i | i \notin I\}$, $U_i = \{\Lambda : \Lambda \cap V_I = \emptyset\}$, I^{th} matrix non-singular

We have a map $\phi_I : U_I \to \mathbb{C}^{k(n-k)}$. We have that $\phi_I(U_I \cap U_{I'})$ is open. Let $\Lambda^I_{I'}$ be the I'-th minor of Λ^I, we have $\Lambda^I = (\Lambda^I_{I'})^{-1} \cdot \Lambda^{I'}$.

6.2 Cell decomposition

Let $\mathcal{V}$ be a *flag*, that is $\mathcal{V} = \{V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = \mathbb{C}^n\}$. Let $\mathbb{P}^n = G(1, n+1)$, we can consider $W_i \cong \mathbb{C}^{i-1} = \{l \subsetneq \mathbb{C}^{n+1} : l \subset V_i, l \nsubseteq V_{i-1}\}$, we have $\mathbb{P}^n = \mathbb{C}^0 \cup \cdots \cup \mathbb{C}^n$.

Let $\mathcal{V}$ be a generic flag. Let $\Lambda \in G(K, n)$, we have $\Lambda \cap V_i = \begin{cases} \text{zero dim} & i \leq n-k \\ (1+k-n) \dim & \text{otherwise} \end{cases}$. Let $(a_1, \dots, a_k) = a$ be a cycle, let $\sum_a(\mathcal{V}) = \{\Lambda \in G(K, n) | \dim(V_{n-k+i-a_i} \cap \Lambda) \geq i\}$

Remark. If $a_i > n-k$, then $\dim V_{n-k+1-a_i} < a_i$ and $\sum_a = \emptyset$.

Let $\sigma_a = [\Sigma_a]$, this construction is independent of the choice of flag. This is called a Schubert class.

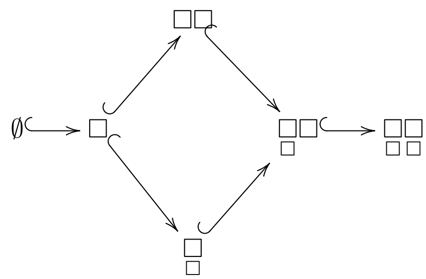
Remark. We have that $\sigma_a \subset \sigma_b$ if and only if $a \geq b$ (i.e., $a_i \geq b_i$ for all i)

Example. Consider $G(2, 4)$

- $(1, 0)$: $\sigma_{1,0} = \{\Lambda : \dim(\Lambda \cap V_{n-k+i-a_i}) \geq i\}$, i.e, $\dim(\Lambda \cap V_2) \geq 1$ and $\dim(V \cap V_4) \geq 2$ which is trivial. So $\sigma_{1,0} = \{\Lambda | \dim(\Lambda \cap V_2) \geq 1\}$.
- $(1, 1)$, $\sigma_{1,1}$: we need $\dim(\Lambda \cap V_2) \geq 1$ and $\dim(\Lambda \cap V_3) \geq 2$, but as the second implies the first, we have $\sigma_{1,1} = \{\Lambda : \Lambda \subset V_3\}$
- $(2, 0)$, $\sigma_{2,0}$: we need $\dim(\Lambda \cap V_1) \geq 1$, so $V_1 \subset \Lambda$ and $\dim(\Lambda \cap V_4) \geq 2$ which is trivial, so $\sigma_{2,0} = \{\Lambda : V_1 \subset \Lambda\}$
- $(2, 1)$, $\sigma_{2,1}$: we need $\dim(\Lambda \cap V_1) \geq 1$, so $V_1 \subset \Lambda$ and $\dim(\Lambda \cap V_3) \geq 2$ so $\Lambda \subset V_3$. Hence $\sigma_{2,1} = \{\Lambda : V_1 \subset \Lambda \subset V_3\}$.

So we have $V_1 \subset V_2 \subset V_3 \subset \mathbb{C}$, so take the flag $\{P\} \subset l_0 \subset H$ (a point, line and hyperplane). So translating we have

- $\sigma_{1,0} = \{l \cap l_0 \neq \emptyset\}$
- $\sigma_{1,1} = \{l \subset H\}$
- $\sigma_{2,0} = \{P \in l\}$
- $\sigma_{2,1} = \{p \in l \subset h\}$



Choose bases e_i of V and let $V_i = \text{span}\{e_1, \dots, e_i\}$. Let $\Lambda \subset \sum_{a_1 \dots a_k}$, then we can find v_1 with $\Lambda \cap V_{n-k+1-a_1} \supset \langle v_1 \rangle$, and we can normalise v_1 so that $\langle v_1, e_{n-k+1-a_1} \rangle = 1$. We can find v_2 with $\langle v_1, v_2 \rangle \subseteq \Lambda \cap V_{n-k+2-a_2}$ such that $\langle v_2, e_{n-k+1-a_1} \rangle = 0$ and $\langle v_2, e_{n-k+1-a_2} \rangle = 1$. We can continue this process to find more v_i. Basically, we are just apply Gaussian elimination. So we end up with $\sum_{j=1}^{n}(n-k+j-a_j-1) - \sum_{j=1}^{k}(k-j) = k(n-k) - \sum_j a_j$.

Fact. *The Schubert classes are a free basis for $A_*(G(K,n))$.*

6.3 Complementary codimension

Proposition 6.1. *Let $\mathcal{V}$ and $\mathcal{W}$ be general flags. Consider $\Sigma_a(V), \Sigma_b(W)$ with $|a| + |b| = k(n-k)$, then*

- *they intersect in a unique point if $a_i + b_{k-1-i} = n - k \forall i$*
- *They are disjoint otherwise.*

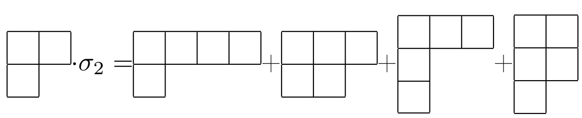
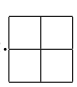
Proposition 6.2. $A_*(G(K,n)) \cong \mathbb{Z}^{\binom{n}{k}}$.

If $[\Gamma] \in A^m(G(k,n))$ with $[\Gamma] = \sum_{|a|=m} \gamma_a \sigma_a$ where $\gamma_a = \deg([\Gamma] \cdot \sigma_{a^*}) = \#(\Gamma \cap \Sigma_{a^*}(\mathcal{V}))$ where $\mathcal{V}$ is a generic flag.

We have the multiplication of Schubert classes: $\sigma_a \sigma_b = \sum_{|c|=|a|+|b|} \gamma_{a,b,c} \sigma_c$. There is a formula for Special Schubert classes, i.e., the one of the forms $\sigma_\alpha = \sigma_{\alpha,0,\dots,0}$

Proposition 6.3. *Let $\sigma_\alpha \in A(G(K,n))$, $\beta \in \mathbb{N}$. Then $\sigma_\beta \cdot \sigma_a = \sum_{|e|=|a|+\beta, a_i \le e_i \le e_{i-1}} \sigma_e$*

For example

- $\sigma_1 \cdot \sigma_e =$ sum of all Young diagram obtained from a.
- 
- $(\sigma_\square)^2 = 2\cdot$  (which in $2 \cdot \{\text{pt}\}$ in $G(2,4)$)

6.4 Giambelli's formula

Consider $\sigma_{a_1 \dots a_k}$. This is equal to

$$\det \begin{pmatrix} \sigma_{a_1} & \sigma_{a_1+1} & \sigma_{a_1+2} & \cdots & \sigma_{a_1+k-1} \\ \sigma_{a_2-1} & \sigma_{a_2} & & & \\ \sigma_{a_3-2} & & & & \\ \vdots & & & & \\ & & & & \\ \sigma_{a_k-k+1} & & & & \sigma_{a_k} \end{pmatrix}$$

Example. We have $\sigma_{2,1} = \begin{pmatrix} \sigma_2 & \sigma_3 \\ \sigma_0 & \sigma_1 \end{pmatrix} = \sigma_2\sigma_1 - \sigma_3$

- $\sigma_{11} = \sigma_1^2 - \sigma_2$, so $\sigma_1^2 = \sigma_2 + \sigma_{11}$ (which we had calculated above)
- $\sigma_1^2\sigma_2 = \sigma_2^2$
- $\sigma_1\sigma_{21} = \sigma_{22}$

So we find that $A_*(\mathbb{G}(1,\mathbb{P}^3)) = \mathbb{Z}[\sigma_1,\sigma_2]/(\sigma_1^3 - 2\sigma_1\sigma_2, \sigma_1^2\sigma_2 - \sigma_2^2)$

Suppose we have four lines in $\mathbb{P}^3$, l_1, l_2, l_3, l_4. We want to know how many general lines intersect l_i. We can use Schubert calculus. We calculate $\sigma_1(l_i)$ (i.e., choosing a flag consisting of one component l_i). Since $\sigma_1^4 = 2 \cdot \begin{smallmatrix}\square\square\\ \square\square\end{smallmatrix} = 2$

7 Riemann Roche (Miles Reid)

NB: This section needs some reworking, which will be done at a later stage

The statement of Riemann Roche is the following. Let X be smooth projective, we have $f : X \to Y$ defined by $\mathcal{F} \mapsto \sum(-1)^i R^i f_*\mathcal{F}$, gives rise to $f_! : K_0(X) \to K_0(Y)$. If Y is a point, $h^i(\mathcal{F}) \in K_0(\mathrm{pt})$ =dimension of finite dimensional vector space over k, so $\sum(-1)^i R^i f_*\mathcal{F}$ becomes $\chi(\mathcal{F})$.

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\mathrm{ch}} & A^*(X)\otimes\mathbb{Q} \\ \downarrow{\scriptstyle f_!} & & \downarrow{\scriptstyle f_*} \\ K_0(Y) & \xrightarrow{\mathrm{ch}} & A^*(Y)\otimes\mathbb{Q} \end{array}$$

This diagram only commutes after multiplying by Td_f. That is

$$\mathrm{ch}(f_!\mathcal{F}) = \mathrm{Td}_{X/Y} f_*(\mathrm{ch}(\mathcal{F}))$$

, where $\mathrm{Td}_{X/Y} = \mathrm{Td}_X \cdot (\mathrm{Td}_Y)^{-1}$. Let us define Td_X.

We have both K_0X and K^0X.

- K_0X is K_0(coherent sheave)
- K^0X is contravariant and is vector bundles over X divided by exact sequences.

If X is smooth then $K_0X = K^0X$. As we can take $\oplus$ and $\otimes$ we have that K_0 is a ring. We have $c(E\oplus F) = c(E)\cdot c(F)$. The *Chern character* of a line bundle by definition is $\mathrm{ch}(\mathcal{O}_X(D)) := 1 + D + \frac{D^2}{2} + \cdots = \exp(D)$. So we are turning addition to multiplication.

Let E be a general coherent sheaf, and write is as a sum of line bundles:$E = \sum \mathcal{O}_X(\alpha_i)$ (this is not true, but we can pretend that it is). Then by definition $\mathrm{ch}(E) := \sum \exp(\alpha_i)$.

Consider T_X, we are again going to pretend $T_X = \sum \mathcal{O}_X(x_i)$. We "have" $c(T_X) = \prod(1 + x_i)$. We define $\mathrm{Td}_X := \prod \frac{x_i}{1-e^{-x_i}}$. If we substitute $x_1 + x_2 = c_1, x_1x_2 = c_2$ etc, we find that:

$$\mathrm{Td}_X = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + c_1c_3 + 3c_2^2 - c_4) + \frac{1}{1440}\left(-c_1^5 + \dots\right)$$

So we get $\chi(\mathcal{F}) = [\mathrm{ch}(\mathcal{F}) \cdot \mathrm{Td}_X]_n$. From this we deduce $\chi(\mathcal{O}_X) = \mathrm{Td}_X[X]$.

Exercise. Let X be a smooth 3-fold, D a divisor on it and calculate $\mathrm{ch}(\mathcal{O}_X(D)) = \left(1 + D + \frac{D^2}{2} + \frac{D^3}{6}\right)\left(1 + \frac{1}{2}c_1 + \dots\right)$ evaluated at degree 3 terms. We should get $\chi(\mathcal{O}_X) + \frac{1}{12}Dc_2 + \frac{1}{12}D(D-K)(2D-K)$ (note $\chi(\mathcal{O}_X) = \frac{1}{24}c_1c_2$).

The advantage of this diagram is that it gives a stronger theorem for Riemann Roche while having a much simpler proof.

If $f = g \circ h$ ($g : Y \to Z, h : X \to Y$ and $f : X \to Z$), it is enough to prove this diagram commutes for g and h separately, i.e., show that $(gh)_! = g_! h_!$ and $\mathrm{Td}_g \cdot \mathrm{Td}_h = \mathrm{Td}_f$. Now for

$$\begin{array}{ccc} X & \xhookrightarrow{i} & Y \times \mathbb{P}^n \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle p} & \\ Y & & \end{array}$$

we can do i and p separately. Now p is just straightforward calculation. What about i? we do this as the inclusion of divisors followed by blowup. We can reduce the case to only looking at divisors.

Question: Why does $\frac{x}{1-e^{-x}}$ appear in Td_X. Think of $X \subset V$ a divisor, with the dimension of X and V being n and $n+1$ respectively.

$$0 \longrightarrow T_X \longrightarrow T_{V|X} \longrightarrow N_{V|X} \longrightarrow 0$$

Recall that $N_{V|X} = \left(\mathcal{I}_X/\mathcal{I}_X^2\right)^*$.

$$0 \longrightarrow \mathcal{O}_V(-X) \longrightarrow \mathcal{O}_V \longrightarrow \mathcal{O}_X \longrightarrow 0$$

$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_V) - \chi(\mathcal{O}_V(-X)) = \mathrm{Td}_V - \mathrm{Td}_V \cdot e^{-x}$.

8 Miss multiplicities (Diane)

Definition 8.1. A sequence $a_1, a_2, \ldots$ is *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$. I.e., $i \mapsto \log a_i$ is a concave function

This implies unimodal, i.e., one local maximal.

Question: Let X be a smooth projective variety of dimension d. Consider $Z \in A_k(X)$ for some k. Is $Z = [V]$ for some (reduced irreducible) $V \subseteq X$?

Example.

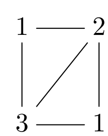
- $X = \mathbb{P}^d$, then $A_k(\mathbb{P}^d) = \mathbb{Z}$ (i.e., keeping track of degree). So the question is: is there an irreducible subvariety of $\mathbb{P}^d$ of dimension k and degree m? Here we know the answer is yes if $m > 0$
- $X = \mathbb{P}^2 \times \mathbb{P}^2$ $A_2(X) = \mathrm{span}([\mathbb{P}^2 \times \mathrm{pt}], [\mathbb{P}^1 \times \mathbb{P}^1], [\mathrm{pt} \times \mathbb{P}^2])$. Let $\zeta = a[\mathbb{P}^2 \times \mathrm{pt}] + b[\mathbb{P}^1 \times \mathbb{P}^1] + c[\mathrm{pt} \times \mathbb{P}^2]$ Is $\zeta = [v]$? The necessary conditions are $a, b, c \geq 0$ and $b^2 \geq ac$ (and they are sufficient for $\mathbb{P}^2 \times \mathbb{P}^2$). Note that (a, b, c) are log-concave.

Theorem 8.2 (Huh). *If $\zeta = \sum e_i[\mathbb{P}^i \times \mathbb{P}^{k-i}] \in A_k(\mathbb{P}^n \times \mathbb{P}^m)$, then there exists $l > 0$ with $l\zeta = [V]$ if and only if $(e_0, \ldots, e_p)$ is log-concave with no internal zeroes, or $\zeta = [\mathbb{P}^n \times \mathrm{pt}]$, $[\mathrm{pt} \times \mathbb{P}^m]$, $[\mathbb{P}^n \times \mathbb{P}^m]$ or $[\mathrm{pt} \times \mathrm{pt}]$.*

8.1 Chromatic polynomials

Let G be a finite graph. A colouring of G with q colours is a function $f : \mathrm{Vert}(G) \to \{1, \ldots, g\}$ for which $f^{-1}(i)$ is an independent set (i.e., no two vertices are adjacent)

Example 8.3.



Let $X_G(q)$ to be the number of ways to colour G with q colours. For example above $X_G(1) = X_G(2) = 0$ and $X_G(3) = 6$.

Theorem 8.4. *$X_G(q)$ is a polynomial in q with integer coefficients*

In our case $X_G(q) = q(q-1)(q-2)^2 = q^4 - 5q^3 + 8q^2 - 4q$.

Conjecture (1968)**.** *Write $X_G(q) = a_n q^n - a_{n-1}q^{n-1} + \cdots + (-1)^n a_0$, then $a_0, \ldots, a_n$ is log-concave*

This is now a theorem by Huh in 2012. The proof involves realising the a_i as intersection numbers.

8.2 Hodge index theorem

Theorem 8.5 (Hodge index theorem)**.** *Let X be a smooth projective surface and let H be an ample divisor on X, and suppose that D is a divisor with $D \cdot H = 0$, $D \not\equiv 0$ (there exists C such that $D \cdot C \neq 0$). Then $D^2 < 0$.*

This implies intersection pairing has signature $(1, -1, \ldots, -1)$.

Corollary 8.6. *If $D_1 = aD + bH$ and $D_2 = cD + dH$ where H is ample, $H \cdot D = 0$ and $D \not\equiv 0$ then $(D_1 \cdot D_2)^2 \geq (D_1^2)(D_2^2)$*

Proof. $D_1 \cdot D_2 = acD^2 + bdH^2$, $D_1^2 = a^2D^2 + b^2H^2$, $D_2^2 = c^2D^2 + d^2H^2$ so check $(D_1 \cdot D_2)^2 - D_1^2 D_2^2 = 2abcd(D^2)(H^2) - (a^2d^2 + b^2c^2)D^2H^2 = 2(D^2)(H^2)(abcd - \frac{a^2d^2+b^2c^2}{2}) \geq 0$ □

So we are going to refer to the corollary when we talk about Hodge index theorem. Let $\zeta = a[\mathbb{P}^2 \times \text{pt}] + b[\mathbb{P}^1 \times \mathbb{P}^1] + c[\text{pt} \times \mathbb{P}^2]$ and suppose that $\zeta = [V]$ where V is an irreducible surface in $\mathbb{P}^2 \times \mathbb{P}^2$. Let $D_1 = [\text{general line} \times \mathbb{P}^2]$ and $D_2 = [\mathbb{P}^2 \times \text{general line}]$ both in $\mathbb{P}^2 \times \mathbb{P}^2$. Then $D_1 \cdot D_2 = [\mathbb{P}^1 \times \mathbb{P}^1]$, $D_1^2 = [\text{pt} \times \mathbb{P}^2]$ and $D_2^2 = [\mathbb{P}^2 \times \text{pt}]$. So let $D_1' = i(D_1)$, $D_2' = i(D_2)$ as divisors in V. So $D_1' \cdot D_2' = \left[\mathbb{P}^1 \times \mathbb{P}^2\right] \cdot [V] = b$, $D_1'^2 = [\text{pt} \times \mathbb{P}^2] \cdot [V] = a$ and $D_2'^2 = [\text{pt} \times \mathbb{P}^2] \cdot [V] = c$. Therefore the Hodge index theorem implies that $b^2 \geq ac$.

8.3 Generalisations

Theorem 8.7. *Let X be an irreducible complete variety (scheme) of dimension n, and let $\delta_1, \ldots, \delta_n \in N^1(X)_{\mathbb{R}}$ (divisors up to numerical equivalence) be nef classes. Then $(\delta_1 \ldots \delta_n)^n \geq (\delta_1)^n \cdots (\delta_n)^n$.*

For a proof, see e.g., Lazarfeld "positivity bock" theorem 1.6.1.
A variant of this as follow:

Theorem. $(\alpha_1 \cdots \alpha_p \cdot \beta_1 \cdots \beta_{n-p})^p \geq (\alpha_1^p \beta_1 \cdots \beta_{n-p}) \cdots (\alpha_p^p \beta_1 \cdots \beta_{n-p})$.

Corollary 8.8 (Khovanskii, Teissier)**.** *Let X be an irreducible complete variety (scheme) of dimension n, let α, β be nef divisors. Set $s_i = \alpha^i \beta^{n-i}$. Then for $1 \leq i \leq n-1$, $s_i^2 \geq s_{i-1}s_{i+1}$.*

Proof. Apply the variant to the case $p = 2$, $\alpha_1 = \alpha$, $\alpha_2 = \beta$ and $\beta_1 \cdots \beta_{n-2} = \alpha^{1i-1}\beta^{n-i-1}$. □

Approach to chromatic polynomials: From the graph (say with $n+1$ edges and $r+1$ vertices), we get a $(n+1) \times (r+1)$ matrix of edges and vertices. Let $V^0 = \text{row}(A) \cap (K^\bullet)^n$ (the Torus $(K^\bullet)^{n+1}/K^\bullet$), let $\widetilde{V}$ be the closure of graph of the Cremona transformation restricted to V (recall that the Cremona transformation is $[x_0 : \cdots : x_n] \mapsto [\frac{1}{x_0} : \cdots : \frac{1}{x_n}]$) Note that $\widetilde{V} \subset \mathbb{P}^n \times \mathbb{P}^n$. Let $[\widetilde{V}] = \sum \mu^i [\mathbb{P}^{r-i} \times \mathbb{P}^i] \in A_r(\mathbb{P}^n \times \mathbb{P}^n)$. The claim is that the μ^i are the coefficients (up to sign) of $X_{\widetilde{G}}(q) := X_q(q)/(q-1)$ (Note $X_{\widetilde{G}}(q)$ is a polynomial since $X_q(1) = 0$). Easy exercise: μ^i is log-concave implies that the a_i are log-concave. Take $D_1 = [H \times \mathbb{P}^n]$, $D_2 = [\mathbb{P}^n \times H]$ then $\mu^i = D_1^i D_2^{r-i}[\widetilde{V}]$ (or maybe $\mu^i = D_1^{r-i} D_2^i [\widetilde{V}]$). Hence μ^i is log-concave.

9 Toric Intersection Theory (Magda)

Definition 9.1. A *Toric variety* is an irreducible variety X containing a torus $T_N(\mathbb{C}^*)^n$. This is a Zariski open subset such that the action of T_N on itself extends to an algebraic action of T_N on X.

Example 9.2. $X = \mathbb{C}^2$, $T_N = (\mathbb{C}^*)^2$. Then the action is $(s,t)(x,y) = (sx, ty)$.

$X = \mathbb{P}^2_{[x,y,z]}$, $T_N = (\mathbb{C}^*)^2$ consisting of points $xyz \neq 0$. The action is $[t_1 : t_2 : t_3][x : y : z] = [t_1x : t_2y : t_3z]$ where $t_1t_2t_3 \neq 0$. We look at orbits not, consider $[t_1 : t_2 : t_3][1 : 0 : 0] = [t_1 : 0 : 0] = [1 : 0 : 0]$, so $[1 : 0 : 0], [0 : 1 : 0]$ and $[0 : 0 : 1]$ are fixed points. The orbit of $[x : y : 0]$ is $[1 : a : 0]$, of $[x : 0 : y]$ it's $[1 : 0 : a]$ and for $[0 : x : y]$ it's $[0 : 1 : a]$ for $a \neq 0$. As for $[x : y : z]$ it is $[a : b : c]$ where $abc \neq 0$ (under the assumption that $xyz \neq 0$).

We have a correspondence between the orbit and the cones of a picture. Let $V(\sigma)$ denote the orbit corresponding to σ, $\Sigma(k)$ the set of k dimensional cone.

Let X be a n-dimensional variety. Recall that the Chow ring of X is $A^*(X) = \oplus_{k=0}^n A^k(X)$ where $A^k(X) = Z^k(X)/\sim$. Recall that for smooth variety we had a product $A^k(X) \times A^k(X) \to A^{k+1}(X)$ which agreed with intersection of transversal objects.

Let X_Σ be a complete smooth Toric variety.

Fact. *$[V(\sigma)]$ for σ of dimension k generates $A^k(X_\Sigma)$.*

Example. For $A^1(X_\Sigma) = \mathrm{Pic}(X_\Sigma) = \{T - \text{inv divisors}\}/\{T - \text{inv principal divisors}\} = \mathbb{Z}^{|\Sigma(1)|}/\langle \mathrm{div}(X^m); m \in \mathbb{Z}^n\rangle$, where $\mathrm{div}(X^m) = \sum_p \langle m, u_p\rangle D_p, m \in \mathbb{Z}^n$, where u_p is the "generator" of the rays in $\Sigma(1)$.

Example 9.3. $\mathrm{Pic}(\mathbb{P}^2) = \mathbb{Z}^3/\langle D_1 - D_3, D_{2"} - D_3\rangle \cong \mathbb{Z}$ (since $m_1 = (1,0)$, $m_2 = (0,1)$)

$\mathrm{Pic}(\mathrm{Bl}(\mathbb{P}^2)) = \mathbb{Z}^4/\langle D_1 - D_0 - D_3, D_2 - D_3\rangle \cong \mathbb{Z}^2$.

We know that:

- $D_{\sigma_1} \cdots D_{\sigma_k} = \begin{cases} V(\sigma) & \sigma = \langle \sigma_1, \dots, \sigma_k\rangle \\ \emptyset & \text{else} \end{cases}$.
- $\sum_p \langle m, u_p\rangle D_p = 0$

So being given a fan of a Toric variety, we can construct the following ring:

- With each $\rho_i \in \Sigma(1)$ associate a variable x_i and let $\mathbb{Z}[x_1, \dots, x_k]$ where $k = |\Sigma(1)|$ be a polynomial ring.
- Let $I \subset \mathbb{Z}[x_1, \dots, x_k]$ be the ideal generated by the monomials $x_{i_1}, \dots, x_{i_j}$ such that $\langle \rho_{i_1}, \dots, \rho_{i_j}\rangle \notin \Sigma$.
- Let $J \subset \mathbb{Z}[x_1, \dots, x_k]$ be generated by the linear forms $\sum_p \langle m, u_p\rangle D_p$, $m \in \mathbb{Z}^n$, $n = \dim V$.

Then $R(\Sigma) := \mathbb{Z}[x_1, \dots, x_k]/(I + J)$ is generated by the monomials $x_{\rho_1}, \dots, x_{\rho_j}$ where all ρ_i's are distinct.

Theorem 9.4. *If X_Σ is complete and smooth then $R(\Sigma) \cong A^*(X_\Sigma)$*

Proof. See Fulton, Introduction to Toric varieties □

By the construction and from the definition of rational equivalence, we can see that if $\langle \rho_1, \dots, \rho_l\rangle \in E(k)$, if we assign the monomial $x_{\rho_1}, \dots, x_{\rho_l}$ to the cycle $[V(\langle \rho_1, \dots, \rho_l\rangle)]$, then we have a surjection $R(\Sigma) \to A^*(X_\Sigma)$. $X_{\rho_i} \mapsto [D_{\rho_i}]$ this gives an isomorphism.

Example. $A^*(\mathrm{Bl}(\mathbb{P}^2)) \cong \mathbb{Z}[x_0, x_1, x_2, x_3]/(I + J)$, $I = \langle x_0x_1, x_2x_3\rangle$, $J\langle x_1 - x_0 - x_3, x_2 - x_3\rangle$. So $A^*(\mathrm{Bl}(\mathbb{P}^2)) \cong \mathbb{Z}[x_1, x_2]/\langle (x_1 - x_2)x_1, x_2^2\rangle$.

- $A^0(X) \cong \mathbb{Z}$ so $\mathrm{rk}(A^0(X)) = 1$
- $A^1(X) = \mathrm{Pic}(X) = \langle x_1, x_2\rangle$ so $\mathrm{rk}(A^2(X)) = 2$
- $A^2(X) = \langle x_1^2, x_1x_2, x_2^2\rangle = \langle x_1^2\rangle$ (from the relations) so $\mathrm{rk}(A^2(X)) = 1$

- $A^n(X) = 0$ for $n > 2$ since the relations cancel everything down.

From this we can get that $D_0^2 = -1$ as follows: Note that $D_1 \cdot D_2 = 1 \cdot V(\langle \rho_1, \rho_2 \rangle)$.

$$\begin{aligned}
x_0^2 &= (x_1 - x_2)^2 \\
&= x_1^2 - 2x_1x_2 + x_2^2 \\
&= x_1x_2 - 2x_1x_2 + 0 \\
&= -1 \cdot x_1x_2
\end{aligned}$$

We expect that $D_2^2 = D_3^2 = 0$, which we do since, $x_2^2 = 0 = 0 \cdot x_1x_2$, $x_3^2 = x_2^2 = 0$. So let us calculate D_1^2, we have that $x_1^2 = 1 \cdot x_1x_2$, so $D_1^2 = 1$.