Schottky groups and Mumford curves

References: Gerritzen - Van der Put, Fresnel - Vand der Put.
Silverman (for week 4), Bosch (for week 5-6, lecture notes on rigid geometry)

Contents

0 Introduction / Overview (Jeroen):

Start with $\mathbb{Q}$ and look at its completion:

- $\mathbb{R}$ and then its algebraic closure is $\mathbb{C}$
- $\mathbb{Q}_p$ (where we say $\left|\frac{a}{b}\right| = p^{-n}$ if $\frac{a}{b} = p^n \frac{a_0}{b_0}$ with $p \nmid a_0, b_0$. Its algebraic closure is $\overline{\mathbb{Q}_p}$ and the completion of this is $\mathbb{C}_p$

0.1 Uniformisations over $\mathbb{C}$

Simplest case: E a genus 1 curve over $\mathbb{C}$. Then $E \cong E_\Lambda = \mathbb{C}/\Lambda$ where $\Lambda \cong \mathbb{Z}^2$ a lattice inside $\mathbb{C}$.

Meromorphic functions on E correspond to elliptic functions on $\mathbb{C}$ (meromorphic, doubly periodic with respect to Λ)

Similar results holds for line bundles.

Given $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ where $\operatorname{im}(\tau) > 0$. Let $q = e^{2\pi i \tau}$. Then E_Λ is isomorphic to the algebraic curve $E : y^2 + xy = x^3 + a_4(q)x + a_6(q)$ where $a_4(q) = -5S_3(q)$, $a_6(q) = \frac{=5S_3(q)+7S_5(q)}{12}$ and $S_k(q) = \sum_{n \geq 1} \frac{n^k q^n}{1-q^n}$.

Moral of the story, we know exactly how to go from one to the other.

Remark.

- a_i have nice integrality properties
- Application: construction of CM curves ($\Lambda \subset \mathcal{O}_K$ ideal inside imaginary quadratic number field)
- The story changes for curves of higher genus: if C over $\mathbb{C}$ is a curve of genus > 1, then $C \cong \Gamma \backslash \mathcal{H}$ where $\mathcal{H}$ is the upper half plane, and $\Gamma < \mathrm{PSL}_2(\mathbb{R}) \circlearrowright \mathcal{H}$.

0.2 Uniformisations over $\mathbb{C}_p$

Let E be a genus 1 curve over $\mathbb{C}_p$. We can not expect $E \cong \mathbb{C}_p/\Lambda$, because additive subgroups of $\mathbb{C}_p$ have an accumulation point at 0 (consider elements $p^n\lambda$ for $\lambda \in \Lambda$).

Over $\mathbb{C}$ there is an isomorphism $\mathbb{C}/\Lambda \overset{z \mapsto \exp(2\pi i z)}{\to} \mathbb{C}^*/\langle q \rangle$, with $|q| < 1$ because $\operatorname{im}(\tau) > 0$. This also works over $\mathbb{C}_p$.

Now consider the quotient $E_q = \mathbb{C}_p^*/\langle q \rangle$ where $|q| < 1$.

Theorem 0.1 (Tate)**.** *The same series $a_4(q), a_6(q)$ converge and give an algebraic structure to the quotient E_q. Moreover if $q \in \mathbb{Q}_p$,, then E_q is defined over $\mathbb{Q}_p$ too.*

Let $L/\mathbb{Q}_p$ be algebraic, then the homomorphism $L^* \to E_q(L)$ is surjective, with kernel $\langle q \rangle$, and it is Galois equivariant for the action of $\mathrm{Gal}(L/\mathbb{Q}_p)$ on both sides.

Remark. Starting with $|q| < 1$, one obtains exactly those E over $\mathbb{C}_p$ for which $|j(E)| > 1$. Over $\mathbb{Q}_p$: one obtains the curves E with multiplicative reduction. The equations give the split multiplicative E over $\mathbb{Q}_p$.

0.3 Schottky groups

Mumford generalisation of Tate to higher genus.

From groups to curves

Let $\Gamma < \mathrm{PGL}_2(\mathbb{C}_p) \circlearrowright \mathbb{P}^1(\mathbb{C}_p)$.

Definition 0.2. $P \in \mathbb{P}^1(\mathbb{C}_p)$ is called *a limit point of* Γ if there is $q \in \mathbb{P}^1(\mathbb{C}_p)$ and distinct $\gamma_n \in \Gamma$ such that $P = \varinjlim_{\infty} \gamma_n(q)$.

Set of limit points of Γ: $L(\Gamma)$

Definition 0.3. Γ is called *Schottky* if:

- $L(\Gamma) \neq \mathbb{P}^1(\mathbb{C}_p)$
- Γ is finitely generated and torsion free

Theorem. *Let $\Omega_\Gamma = \mathbb{P}^1(\mathbb{C}_p) - L(T)$. Then the quotient $\Gamma\backslash\Omega_\Gamma$ has the structure of an algebraic curve over $\mathbb{C}$.*

Remark.

- Schottky groups have nice fundamental domains.
- Reduction of $\Gamma\backslash\Omega_\Gamma$ is totally split, dual graph is the quotient of the tree on Ω_Γ (subset of Bruhat-Tits tree)
- Modular forms for Γ are completely classified as products of Θ-functions; these can be used to fund the canonical embedding of $\Gamma\backslash\Omega_\Gamma$.

From curves to groups

Definition 0.4. X curve over $\mathbb{Q}_p$ is *totally split* if X has a (flat) model $\underline{X}$ over $\mathbb{Z}_p$ such that $X_0 = \underline{X} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is a union of rational curves intersecting transversely in $\mathbb{F}_p$-rational points.

Theorem 0.5 (Mumford)**.** *Every totally split curve over $\mathbb{Q}_p$ is a Mumford curve; they can be obtained as quotients $\Gamma\backslash\Omega_\Gamma$ of domains $\Omega_\Gamma \subset \mathbb{P}^1(\mathbb{C}_p)$ by Schottky groups Γ.*

0.4 p-adic geometry

To obtain Ω: glue affine patches to the universal cover of the reduction graph.

Fundamental problem

The usual topology is totally disconnected. Tate found a solution by using the theory of affinoid subdomains.
Idea: restrict the subsets and coverings that are used.

Goal:

To understand parts of the "groups to curves" and "curves to groups" sections.
Topics:

1. $\mathbb{P}^1$ as a topological space (Marc)
2. $\mathbb{P}^1$ as an analytic space (Samir)
3. Group action (Chris W)
4. The Tate curves (Heline)
5. Affinoid spaces, rigid spaces (part 1) (Chris B)
6. Affinoid spaces, rigid spaces (part 2) (Céline)
7. Reduction of curves (Angelos)
8. Modular functions and Mumford curves (Haluk)
9. Totally split curves as Mumford curves (Jeroen)

1 $\mathbb{P}^1$ as a topological space (Marc)

1.1 Trees

Reference: [Mumford] An analytical Construction of degenerating curves..., [Chris W] 4th year essay

Goal: To attach tree to a compact subset of $X \subset \mathbb{P}^1(K)$, where K is a local field.

Motivation:

Real case: $\mathrm{PGL}_2^+(\mathbb{R})$ acts on $\mathcal{H} = \{z \in \mathbb{C} : \mathrm{im}(z) > 0\}$ via $z \mapsto \gamma z = \frac{az+b}{cz+d}$ (where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$) by isometries, transitively. It has boundary $\partial\mathcal{H} = \mathbb{P}^1(\mathbb{R})$. $\Gamma \subset \mathrm{PSL}_2^+(\mathbb{R})$ is a discrete cocompact group with no elements of finite order, $\mathcal{H}/\Gamma$ is a Riemann surface of some genus g.

Theorem 1.1. *Any Riemann Surface of genus $g \geq 2$ is of this form.*

Complex case: $\mathrm{PSL}_2(\mathbb{C})$ acts isometrically and transitively on $\mathbb{H}$ =hyperbolic 3-space (Can think of as $\mathbb{C}\times\mathbb{R}_{>0}$). We have $\partial\mathbb{H} = \mathbb{P}^1(\mathbb{C})$. Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ Kleinian group and finitely generated. $\mathbb{H}/\Gamma \supset (\partial\mathbb{H}\backslash \text{limit points of } \Gamma)/\Gamma \cong$Riemann Surface of genus g(It is a theorem of Maskit that Γ is a $\mathbb{C}$-Schottky, free on g generators.)

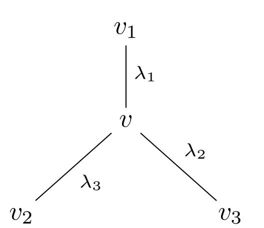
p-adic case: $\mathrm{PGL}_2(K)$ where K is a p-adic field acts on Δ (called Brahut - Tits tree), a tree. We have $\partial\Delta \cong \mathbb{P}^1(K)$. If $\Gamma \subset \mathrm{PGL}_2(K)$ is Schottky (to be defined) then we will obtain curves as $\partial\Delta/\Gamma$.

Notation. Let K be a local field: a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$. Let $|\ |$ be its valuation, $\mathbb{Z}_K$ be the value ring and $\mathbb{Z}_K \supseteq m_K = (\pi)$, $|\pi| < 1$, $k = \mathbb{Z}_K/\pi$. $\mathrm{PGL}_2(K)$ acts on $\mathbb{P}^1(K) = (K \times K \setminus \{0,0\})/\sim = K \cup \{\infty\}$ via $z \mapsto \frac{az+b}{cz+d}$. Consider lattices $M \subset K \times K$ (Rank 2 $\mathbb{Z}_K$-lattice), we say that $M \sim M'$ if $M' = \lambda M$, $\lambda \in K^*$ (M, M' are *homothetic*). Set $\Delta^{(0)}$ =set of classes $[M]$ (call them *vertices*)

Remark. $\mathrm{PGL}_2(K)$ acts transitively on $\Delta^{(0)}$, stabiliser of $[\mathbb{Z}_K + \mathbb{Z}_K] = \mathrm{PGL}_2(\mathbb{Z}_K)$, hence $\Delta^{(0)} \cong \mathrm{PGL}_2(K)/\mathrm{PGL}_2(\mathbb{Z}_K)$

Definition. Distance: Given v_1, v_2, we can find representative $v_1 = [M_1]$, $v_2 = [M_2]$ such that $M_1 = \langle a, b\rangle$ and $M_2 = \langle a, \alpha b\rangle$ (elementary divisor theorem). We define $\rho(v_1, v_2) := (\alpha)$ (the ideal generated by α). This is symmetrical, so defines a distance on $\Delta^{(0)}$.

"Triangle inequalities": Given 3 vertices $v_1, v_2, v_3 \in \Delta^{(0)}$, $\exists v \in \Delta^{(0)}$ such that $\rho(v_i, v_j) = (\lambda_i\lambda_j)$



Triples in $\mathbb{P}^1(K)$

Let x_1, x_2, x_3 pairwise distinct triple in $\mathbb{P}^1(K)$, defines a lattice $M(x_1, x_2, x_3)$ as follows: $x_i = [w_i]$, $w_i \in K^2 \setminus \{0,0\}$, $\lambda_1 w_1 + \lambda_2 w_2 + \lambda_3 w_3 = 0$ non-trivial relations, then $M(x_1, x_2, x_3) = \langle \lambda_1 w_1, \lambda_2 w_2\rangle$ (independent on ordering of the x_i)

Remark. $x_1 = 0 = [0,1]$, $x_2 = 1 = [1,1]$, $x_3 = \infty = [1,0]$, then $M = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = \mathbb{Z}_K + \mathbb{Z}_K \subset K + K$

Given any pairwise distinct triple $\underline{x}$, there exists a unique $\gamma \in \mathrm{PGL}_2(K)$ such that $\gamma(\underline{x}) = (0, 1, \infty)$. Hence all $v \in \Delta^{(0)}$ are classes of $M(\underline{x})$ for an appropriate $\underline{x}$.

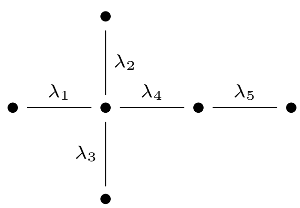
Adjacency: We say v_1, v_2 are adjacent if there exists representative $v_1 = [M_1], v_2 = [M_2]$ such that $M_1 \supsetneq M_2 \supsetneq \pi M_1$ (if and only if $\rho(v_1, v_2) = m_K$). This gives us the tree Δ called the Bruhut - Tits tree of $\mathbb{P}^1(K)$ of $\mathrm{PGL}_2(K)$.

Remark. Given $v = [M] \in \Delta^{(0)}$, there are as many adjacent vertices as there are $M \supsetneq M' \supsetneq \pi M$. The number of lines in $M/\pi M \cong k^2 = \#\mathbb{P}^1(k) = \#k + 1$.

$\Delta_X^{(0)} = \{[M(x_1, x_2, x_3) : x_i \in X\} \subset \Delta^{(0)}$.

Definition. A subset $\Delta_*^{(0)} \subset \Delta^{(0)}$ is *linked* if for all $v_1, v_2, v_3 \in \Delta_*^{(0)}$, the v in the triangle inequality if in $\Delta_*^{(0)}$.

Tree Theorem. *If $\Delta_*^{(0)}$ is linked, then it can be made to be the set of vertices of a connected tree with lengths such that $\rho(v, v') = \prod$ lenght of edges in path joining them (We get a tree Δ_X)*



Proposition. $\Delta_X^{(0)}$ *is a linked set of vertices*

Example. $X = \{p^n : n \in \mathbb{Z}\} \cup \{0, \infty\} \subset \mathbb{P}^1(\mathbb{Q}_p)$

Note: $\Gamma = \left\langle \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$ acts on Δ_X via "translation". Quotient $\Delta/\Gamma = \circlearrowleft$ (fundamental group is $\mathbb{Z}$)

Reduction point of view

Let $R : \mathbb{P}^1(K) \to \mathbb{P}^1(\mathbb{F}_K)$ be defined by $[x, y] \mapsto [\overline{x}, \overline{y}]$ if $x, y \in \mathbb{Z}_K$, $\max\{|x|, |y|\} = 1$. Given $\underline{a}$ pairwise distinct triple in $\mathbb{P}^1(K)$, there exists $\gamma_a \in \mathrm{PGL}_2(K)$ such that $\gamma_{\underline{a}}(\underline{a}) = (0, 1, \infty)$. We define $R_{\underline{a}} = R \circ \gamma_{\underline{a}}$.

If $X \subset \mathbb{P}^1(K)$ is compact, $\underline{a} \in X^3$, R_a determines a partition of $X = \sqcup_{p \in R_a(X)} R_a^{-1}(\{p\})$.

Definition. $\underline{a} \sim \underline{b}$ if $R_{\underline{a}}, R_{\underline{b}}$ gives identical partitions

$\underline{a}$ is adjacent to $\underline{b}$ if $\underline{a}$ gives a partition $X_1, \dots, X_s$ and $\underline{b}$ gives a partition $Y_1, \dots, Y_t$ and X_i is adjacent to Y_j for all i and j.

Turns out that the graph you get using these notions is Δ_X via $\underline{a} \mapsto [M(\underline{a})]$.

Boundary

Given a linked set $\Delta_*^{(0)}$, define $\mathrm{Ends}(\Delta_*) =$ equivalence classes of half-line (where equivalence is defined as differ at finitely many terms)

Define $\partial\Delta_* = \mathrm{Ends}(\Delta_*)$

Proposition.

1. *There is an injection $i : \partial\Delta_* \to \mathbb{P}^1(K)$ by intersecting nested lattices.*

2. *If $\Delta_*^{(0)} = \Delta_X^{(0)}$, then $i(\partial\Delta_X) = X$ (in particular, if $X = \mathrm{PGL}_2(K)$, then i is bijective)*

2 $\mathbb{P}^1$ as an Analytic Space (Samir)

Reference: Fresnel and Van der Put "Rigid Analytic Geometry and its Application" Chapter 2

The basic object of today is $\mathbb{P} = \mathbb{P}^1(\mathbb{C}_p)$. For this talk $K = \mathbb{C}_p$

Definition 2.1. (Disc). An *open disc* in $\mathbb{P}$ has the form $\{z \in \mathbb{C}_p : |z-a| < r\}$ for some $a \in \mathbb{C}_p$ and $r \in \mathbb{R}^+$, or $\{z \in \mathbb{C}_p : |z-a| > r\} \cup \{\infty\}$.

A *closed disc* in $\mathbb{P}$ has the form $\{z \in \mathbb{C}_p : |z-a| \leq r\}$ for some $a \in \mathbb{C}_p$ and $r \in \mathbb{R}^+$, or $\{z \in \mathbb{C}_p : |z-a| \geq r\} \cup \{\infty\}$.

A *connected affinoid subset* of $\mathbb{P}$ has the form $\mathbb{P} \setminus \cup D_i$ (finite non-empty union, and D_i are open disc). (Note that we can write this as $\mathbb{P} \setminus \coprod D_i'$ where D_i' are open disc, $h(X) =$"holes in X" $= \{D_i'\}$)

An *affinoid* of $\mathbb{P}$ is the finite union of connected affinoids.

Fact. *If F is an affinoid, then $F = \coprod_{i=1}^s F_i$ where F_i are connected affinoids. The F_i are the* connected components *of F. This decomposition is unique.*

Lemma 2.2. *Let $f \in \mathbb{C}_p(z) \setminus \{0\}$, $r > 0$. Consider $\{a \in \mathbb{P} : |f(a)| \leq r\}$, this is either an affinoid or empty.*

Example. $f(z) = z(z-1)$, $r = \frac{1}{p}$. Then

$$\begin{aligned}\left\{z : |f(z)| \leq \frac{1}{p}\right\} &= \left\{z : |z| \leq \frac{1}{p}\right\} \cup \left\{z : |z-1| \leq \frac{1}{p}\right\} \\ &= \mathbb{P} \setminus \left(\left\{z : |z| > \frac{1}{p}\right\} \cup \{\infty\}\right) \cup \mathbb{P} \setminus \left(\left\{z : |z-1| > \frac{1}{p}\right\} \cup \{\infty\}\right)\end{aligned}$$

2.1 Holomorphic Functions

Definition 2.3. Let F be an affinoid, $\mathrm{Rat}(F) := \{f \in \mathbb{C}_p(z) : \text{poles of } f \text{ are outside } F\}$.

Define $\|f\|_F = \sup_{a \in F} |f(a)| < \infty$.

The *holomorphic functions* on F, $\mathcal{O}(F) :=$completion of $\mathrm{Rat}(F)$ with respect to $\| \|$.

Fact.

1. *$F \mapsto \mathcal{O}(F)$ is a sheaf*
2. *$X \supseteq Y$ are connected affinoids then the image of $\mathcal{O}(X) \to \mathcal{O}(Y)$ is dense if and only if $h(X) \to h(Y)$ is surjective.*

Definition 2.4. $\mathcal{O}(F)^\circ := \{f \in \mathcal{O}(F) : \|f\| \leq 1\}$, this is an $\mathcal{O}_{\mathbb{C}_p}$-algebra.

$\mathcal{O}(F)^{\circ\circ} := \{f \in \mathcal{O}(F) : \|f\| < 1\}$.

$\overline{\mathcal{O}(F)} := \mathcal{O}(F)^\circ / \mathcal{O}(F)^{\circ\circ}$, this is an $\overline{\mathbb{F}_p}$-algbera

Example. Let $F = \{a \in \mathbb{P} : |a| \leq 1\} = \mathcal{O}_{\mathbb{C}_p}$.

1. $\mathcal{O}(F) = \{\sum_{n=0}^\infty c_n z^n : c_n \in \mathbb{C}_p, \lim c_n = 0\}$, $\|\sum c_n z^n\| = \max |c_n|$.
2. $\mathcal{O}(F)^\circ = \{\sum_{n=0}^\infty c_n z^n : c_n \in \mathcal{O}_{\mathbb{C}_p}, \lim c_n = 0\}$
3. $\mathcal{O}(F)^{\circ\circ} = \{\sum_{n=0}^\infty c_n z^n : c_n \in \mathfrak{m}_{\mathbb{C}_p}, \lim c_n = 0\}$.
4. $\overline{\mathcal{O}(F)} = \overline{\mathbb{F}_p}[z]$.

Lemma 2.5 (Division with Remainder)**.** *Let $F = \{a \in \mathbb{P} : |a| \leq 1\}$. Let $f \in \mathcal{O}(F)$ with $\|f\| = 1$, so $\overline{f} \in \overline{\mathbb{F}_p}[z]$ with degree $d \geq 0$. Then for any $g \in \mathcal{O}(F)$ there exists unique $q, r \in \mathcal{O}(F)$ such that*

1. *$g = qf + r$*
2. *$r \in \mathbb{C}_p[z]$ of degree less than d*

3. $\|g\| = \max(\|q\|, \|r\|)$.

Definition 2.6. Define $\mathcal{O}(F)^+ := \{f \in \mathcal{O}(F) : f(\infty) = 0\}$

Proposition 2.7 (Mittag - Leffler)**.** *Let F be a connected affinoid with $\infty \in F$, $h(F) = \{D_1, \ldots, D_S\}$, $D_i = \{z : |z - a_i| < |\pi_i|\}$, where $a_i \in \mathbb{C}_p$ and $\pi_i \in \mathbb{C}_p^*$. Let $F_i = \mathbb{P} \setminus D_i$, so $F = \cap F_i$. Then*

1. $\mathcal{O}(F)^+ = \oplus_{i=1}^s \mathcal{O}(F_i)^+$
2. $\mathcal{O}(F_i)^+ = \left\{\sum_{n>0} b_n \left(\frac{\pi_i}{z-a_i}\right)^n : b_n \in \mathbb{C}_p, \lim b_n = 0\right\}$.

If we let $f = \sum f_i$, then $\|f\| = \max \|f_i\|_{F_i}$. Also $\| \sum_{n>0} b_n \left(\frac{\pi_i}{z-a_i}\right)^n \|_{F_i} = \max |b_n|$.

Lemma 2.8. *Let $F = \coprod F_i$ be an affinoid. Then $\mathcal{O}(F) = \oplus \mathcal{O}(F_i)$.*

2.2 G-topology on $\mathbb{P}$

Definition 2.9. A G-topology is

1. A set X
2. *A* set $\mathcal{F} \subset \mathcal{P}(X)$ (power set of X). (The elements of $\mathcal{F}$ are called the *admissible*)
3. For each $U \in \mathcal{F}$ a set $\mathrm{Cov}(U)$ (a set of covering, called the admissible covering). $\mathrm{Cov}(U)$ are of the form $\{U_i\}_{i \in I}$ such that $U_i \in \mathcal{F}$ and $\cup U_i = U$.

satisfying

1. $\emptyset, X \in \mathcal{F}$
2. $U, V \in \mathcal{F}$ then $U \cap V \in \mathcal{F}$
3. $\{U\} \in \mathrm{Cov}(U)$
4. If $U \supseteq V$ are admissible and $\{U_i\}_{i \in I} \in \mathrm{Cov}(U)$, then $\{U_i \cap V\}_{i \in I} \in \mathrm{Cov}(V)$.
5. If $U \in \mathcal{F}$, $\{U_i\}_{i \in I} \in \mathrm{Cov}(U)$ and for $\mathcal{U}_i \in \mathrm{Cov}(U_i)$, then $\cup \mathcal{U}_i \in \mathrm{Cov}(U)$.

We can define presheafs, sheafs, sheafification and Cech Cohomology in the expected way, following this topology.

Definition 2.10. The *weak G-topology* on $\mathbb{P}$ is

1. $X = \mathbb{P}$
2. $\mathcal{F} = \{\emptyset, \mathbb{P}\} \cup \{\text{affinoid}\}$
3. $\mathrm{Cov}(U)$ are $\{U_i\}$, $U_i \subseteq U$ are affinoid and U is the union of finitely many U_i.

Theorem 2.11. *$\mathcal{O}$ is a sheaf. ($\mathcal{O}(U) \to \overset{\vee}{H}{}^0(\mathcal{U}, \mathcal{O})$ is an isomorphism)*

Furthermore $\overset{\vee}{H}{}^i(\mathcal{U}, \mathcal{O}) = 0$ for all $i > 0$.

3 Schottky groups and their actions (Chris Williams)

3.1 Discontinuous groups

Let K be any local field, $\Gamma \leq \mathrm{PGL}_2(K)$

Definition 3.1. $\alpha \in \mathbb{P}^1(K)$ is a *limit point* for Γ if there exists $(\gamma_n)_{n=1}^{\infty} \subset \Gamma$, $\beta \in \mathbb{P}^1(K)$ such that

1. $\gamma_m \neq \gamma_n$ for all $m \neq n$
2. $\alpha = \lim \gamma_n(\beta)$.

Write $L = L(\Gamma)$ for the set of limit points of Γ

Definition 3.2. $\Gamma \leq \mathrm{PGL}_2(K)$ is *discontinuous* if

1. $L(\Gamma) \neq \mathbb{P}^1(K)$
2. For any $\alpha \in \mathbb{P}^1(K)$, $\overline{\Gamma_\alpha}$ is compact

Remark. If K is a local field, then condition 2. is automatic.
Discontinuous implies Discrete. In particular, $\gamma_n \to \gamma$, then $\gamma_n\gamma^{-1} \to I$, implying $L(\Gamma) = \mathbb{P}^1(K)$.

3.1.1 Classification of elements of $\mathrm{PGL}_2(K)$

Definition 3.3. Let $\gamma \in \mathrm{PGL}_2(K)$ with eigenvalue λ, μ. Say γ is

1. hyperbolic if $|\lambda| \neq |\mu|$
2. Elliptic if $|\lambda| = |\mu|$ but $\lambda \neq \mu$
3. Parabolic if $\lambda = \mu$

Proposition 3.4. *Let* $\lambda \in \mathrm{PGL}_2(K)$

1. γ *is hyperbolic if and only if it is conjugate in* $\mathrm{PGL}_2(K)$ *to* $\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$ *with* $0 < |q| < 1$
2. γ *is elliptic/parabolic if and only if* γ^2 *is conjugate to an element of* $\mathrm{PGL}_2(\mathcal{O}_K)$.

Proposition 3.5.

1. *Let* $\gamma \in \mathrm{PGL}_2(K)$ *be hyperbolic. Then* $\langle\gamma\rangle$ *is discontinuous*
2. *IF* Γ *is discontinuous and* $\gamma \in \Gamma$ *is elliptic/parabolic, then* γ *has finite order.*

Proof.

1. $\langle\gamma\rangle$ has 2 limit points, corresponding to eigenvectors of γ
2. γ is conjugate to
   (a) $\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$, $|\lambda| = 1$ or
   (b) $\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$

In the case a) $\langle\gamma\rangle \cong \{\lambda^n\}$, discrete subgroup of $\mathcal{O}_K^*$ hence finite
In the case b) $\langle\gamma\rangle \cong \{n_\mu\}$, discrete subgroup of $\mathcal{O}_K^*$, hence $\mu = 0$.

□

3.1.2 Investigating limit points

Without loss of generality, $\infty \notin L$

Let $\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \subset \Gamma$ be an infinite sequence. Compactness of $\mathbb{P}^1(K)$ implies we can take subsequence such that a_n/c_n, b_n/c_n, d_n/c_n converges. Without loss of generality, none of these are ∞, so $\begin{pmatrix} a_n/c_n & b_n/c_n \\ 1 & d_n/c_n \end{pmatrix} \to \begin{pmatrix} a & b \\ 1 & d \end{pmatrix}$. As Γ is discrete, then this does not lie in $\mathrm{PGL}_2(K)$ which implies $ad = b$.

For any $\beta \in \mathbb{P}^1$, we have $\lim_{n\to\infty} \gamma_n(\beta) = \frac{a\beta+b}{\beta+d} = \frac{a\beta+ad}{\beta+d} = a$, <u>unless</u> $\beta = -d = \lim \gamma_n^{-1}(\infty)$

Proposition 3.6.

1. *Suppose $x \notin L$. Then if we define $L(x)$ to be $\{\alpha \in L : \exists\, (\gamma_n) \text{ with } \gamma_n(x) \to \alpha\}$. Then $L = L(x)$*
2. *If $A = \{x, y, z, \} \subset \mathbb{P}^1(K)$ distinct points, then there exists $w \in A$ such that $L(w) = L$*

Proof.

1. $x \notin L$, so "$x \neq -d$" in the above
2. Assume $x, y, z \in L$. As for any sequence γ_m, either "$x \neq -d$" or "$y \neq -d$", we have $L = L(x) \cup L(y)$. So without loss of generality $z \in L(y)$. Then $L(z) \subset L(x)$, so $L = L(x) \cup L(z) \subset L(x) \subset L$, so $L = L(x)$.

□

Proposition. *L is compact*

Proof. If $|L| \leq 2$, then this is clear.

If $|L| > 2$, choose $x \in L$ such that $L = L(x)$, then $L = \overline{\Gamma}_x = L(x)$ is compact. □

Definition 3.7. A *Schottky group* is a finitely generated discontinuous subgroup of $\mathrm{PGL}_2(K)$ with no elements of finite order. (So no elliptic or parabolic elements)

We now assume that Γ is a Schottky group

To L, we associate a tree $\mathcal{T}(L)$. Γ acts on $\mathcal{T}(L)$ in a natural way.

Lemma 3.8. *$\mathcal{T}(L)/\Gamma$ is finite.*

Proof. Notation: If $\alpha \in \mathcal{T}(L)$, then $\mathcal{T}(L) \setminus \{\alpha\} = \coprod T_i$ where T_i are tree. Say $\mathrm{fin}(\alpha) := \cup_{T_i \text{finite}} T_i$. Fix α. Pick $\mathcal{U}$ to be the minimal subtree such that for $\Gamma' \subset \Gamma$ to be a finite generated set (containing I inverses).

1. $\forall \gamma \in \Gamma'$, $\gamma(\alpha) \in \mathcal{U}$
2. $\forall \beta \in \mathcal{U}$, $\mathrm{fin}(\beta) \subset \mathcal{U}$.

Define $\mathcal{V} = \cup_{\gamma\in\Gamma} \gamma\mathcal{U}$. Then we claim $\mathcal{V} = \mathcal{T}(L)$. To see this take $\beta \in \mathcal{T}(L)$, without loss of generality, there is a halfline in $\mathcal{T}(L)$ starting at α through β. From Marc's talk, this halfline correspond to a limit point $z = \lim \gamma_n(z_0)$. So in particular, β lies in a path from $\gamma_n(z_0)$ to $\gamma_{n+1}(z_0)$ for some n. Therefore $\beta \in \mathcal{V}$, as $\gamma_n(z_0)$ and $\gamma_{n+1}(z_0) \in \mathcal{V}$ (the halfline starts at α) □

Corollary 3.9. *Any Schottky group is free*

Proof. $\mathcal{T}(L)$ is the universal cover of $\mathcal{T}(L)/\Gamma$, covering translations Γ. As $\mathcal{T}(L)/\Gamma$ is finite, Van Kampen implies the result. □

3.1.3 Fundamental Domain

Take $B_1, \dots, B_g, C_1, \dots, C_g$ disjoint open balls in $\mathbb{P}^1(\mathbb{C}_p)$ with centres in K

Suppose there exists $\gamma_1, \dots, \gamma_j \in \mathrm{PGL}_2(K)$ with $\gamma_i(\mathbb{P} \setminus B_i) = \overline{C_i}$ and $\gamma_i(\mathbb{P} \setminus \overline{B_i}) = C_i$.

Let $\Gamma := \langle \gamma_1, \dots, \gamma_g \rangle$. Then:

- Γ is non-abelian free,
- In particular, no elements of finite order

Define $F := \mathbb{P}^1(\mathbb{C}_p) \setminus (\cup B_i \cup C_i)$. Define $\Omega = \cup_{\gamma \in \Gamma} \gamma F \neq \mathbb{P}^1(\mathbb{C}_p)$.

Theorem 3.10.

1. *$\mathcal{L}(\Gamma) = \mathbb{P}^1(\mathbb{C}_p) \setminus \Omega$*
2. *Γ is Schottky*
3. *Moreover, every Schottky groups occurs in this way*

Ω/Γ is a curve of genus g.

4 The Tate curve (Heline)

4.1 Introduction

With an elliptic curve over $\mathbb{C}$, we get a parametrisation $\mathbb{C}/\Lambda$ where $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ is a lattice.

We want to do this over $\mathbb{Q}_p$. Note that if we have $0 \neq t \in \Lambda \subset \mathbb{Q}_p$, then $p^n t \in \Lambda \forall n$, and $\lim_{n\to\infty} p^n t = 0$, so 0 is an accumulation point, so this method will not work.

Note that an elliptic curve over $\mathbb{C}$, $\mathbb{C}/\Lambda \cong \mathbb{C}^*/q^{\mathbb{Z}}$ where $z \in \mathbb{C}$, $z \mapsto u = e^{2\pi i z}$. And we have that $q^{\mathbb{Z}} \subset \mathbb{Q}_p^*$, so we want to show that the elliptic curve over $\mathbb{Q}_p$ gives rise to $\mathbb{Q}_p^*/q^{\mathbb{Z}}$.

Convention: K is a finite extension of $\mathbb{Q}_p$ with characteristic $k \neq 2, 3$. $q \in \mathbb{Q}_p^*$ such that $|q| < 1$ (where $|\ |$ is the absolute value associated to K)

4.2 Tate curve

Definition 4.1. $s_k(q) = \sum_{n\geq 1} \frac{n^k q^n}{1-q^n}$, $a_4(q) = -s_3(q)$, $a_6(q) = -\frac{5s_3(q)+7s_5(q)}{12}$.

Fact. *If $q \in K^*$ with $|q| < 1$ then $a_4(q)$ and $a_6(q)$ converges in K.*

Definition 4.2. Let E_q be the curve defined by $y^2 + xy = x^3 + a_4(q)x + a_6(q)$. This is called the *Tate curve*

Fact. *E_q is an elliptic curve with discriminant $\Delta(E_q) = q\prod_{n\geq 1}(1-q^n)^{24}$ and j-invariant $j(E_q) = q^{-1} + 744 + 196884q + \dots$. Note that $|j(E_q)| = \left|q^{-1}\right| > 1$*

Definition 4.3. $X(u, q) = \sum_{n\in\mathbb{Z}} \frac{q^n u}{(1-q^n u)^2} - 2s_1(q)$

$Y(u, q) = \sum_{n\in\mathbb{Z}} \frac{(q^m u)^2}{(1-q^n u)^3} + s_1(q)$

Fact. *For all $u \in \overline{K}$, $u \notin q^{\mathbb{Z}}$, $X(u, q)$ and $Y(u, q)$ converges*

Theorem 4.4 (Tate)**.** *Let E_q be a Tate curve. There exists a surjective homomorphism $\phi : \overline{K}^* \to E_q(\overline{K})$ defined by $u \mapsto \begin{cases} (X(u,q), Y(u,q)) & \text{if } u \notin q^{\mathbb{Z}} \\ \infty & \text{if } u \in q^{\mathbb{Z}} \end{cases}$. The kernel is $q^{\mathbb{Z}}$.*

ϕ is compatible with the Galois action, $\mathrm{Gal}(\overline{K}/K)$*. That is $\phi(P^\sigma) = \phi(P)^\sigma$ for all $\sigma \in \mathrm{Gal}(\overline{K}/K)$, $P \in \overline{K}^*$.*

So we get $E_q(\overline{K}) \cong \overline{K}^*/q^{\mathbb{Z}}$.

Sketch of Proof. We show it is a homomorphism: $u_1 u_2 = u_3$, $\phi(u_i) = P_i$, $P_1 + P_2 = P_3$. Note that $\phi(qu) = \phi(u)$, so we can assume $|q| < u_1 \leq 1$, $1 \leq |u_2| < \left|q^{-1}\right|$,and hence $|q| < |u_3| < \left|q^{-1}\right|$. So u_1 will be in a domain of convergence X, Y, $\phi(1) = 0$, so $u_1 \neq 1 \neq u_2$, $P_1 + P_2 = 0$. $X(u_i, q) = x_i$

If $x_1 \neq x_2$, we need to check addition law, identities.

Lemma. *When we have a map ϕ from a multiplicative group to an additive group which takes infinitely many distinct values and $\phi(u_1 u_2) = \phi(u_1) + \phi(u_2)$ for all $u_1 \neq \pm u_2$, then ϕ is a homomorphism.*

Proof of Lemma. Pick u such that $\phi(u) \neq \pm\phi(u_1)$, $\phi(u) \neq \phi(u_1) \pm \phi(u_2)$, $\phi(u) \neq \phi(u_1 u_2)$. Then $\phi(uu_1) = \phi(u) + \phi(u_1) \neq \pm\phi(u_2)$, and $\phi(u) + \phi(u_1 u_2) = \phi(uu_1 u_2) = \phi(uu_1) + \phi(u_2) = \phi(u) + \phi(u_1) + \phi(u_2)$. □

To show that we satisfy the lemma, note that for $t \in K^*$, $|t| < 1$, $|X(t+1, q)| = |t|^{-2}$, so we get infinitely many distinct value.

We will not prove the surjectivity part, just read Silverman pg 429 to 438. □

4.3 Elliptic curves over p-adic fields

In the complex case, $E \cong \mathbb{C}^*/q^{\mathbb{Z}}$ for some q.

Question: Is this also true in the p-adic case? The answer is no. Consider $|j(E_q)| = |q^{-1}| > 1$, so elliptic curve with $|j(E)| < 1$ can not be isomorphic to a Tate curve. But we will show that $|j(E)| > 1$ is a sufficient condition for E to be isomorphic to a Tate curve E_q.

Lemma 4.5. *Let $\alpha \in \overline{\mathbb{Q}_p^*}$, $|\alpha| > 1$. Then there exists a unique $q \in \mathbb{Q}_p(\alpha)^*$ such that $j(E_q) = \alpha$.*

Proof. Let $f(q) = j(E_q)^{-1} = q - 744q^2 + 356652q^3 + \cdots \in \mathbb{Z}[[q]]$.

Uniqueness Suppose $q, q' \in \mathbb{Q}_p(\alpha)^*$ are such that $j(E_q) = j(E_{q'})$. Then $0 = |f(q) - f(q')| = |q - q'| \, |1 - 744(q + q') + \ldots| = |q - q'|$, hence $q = q'$.

Existence There exists $g(q) \in \mathbb{Z}[[q]]$ such that $g(f(q)) = q$, in fact $g(q) = q + \text{h.o.t.}$ Let $\beta \in \overline{\mathbb{Q}_p^*}$ with $|\beta| < 1$, $g(\beta)$ converges. Then $|g(\beta)| = |\beta|$. We know that $|\alpha| > 1$, so $|\alpha^{-1}| < 1$, so set $q = g(\alpha^{-1})$. Then $0 < |q| = |g(\alpha^{-1})| < 1$. Also note that $j(E_q)^{-1} = f(q) = f(g(\alpha^{-1})) = \alpha^{-1}$, hence $j(E_q) = \alpha$.

□

Definition 4.6. Let E/K be an elliptic curve in long Weierstrass equation, with $j(E) \neq 0, 1728$. Letc_4 and c_6 be the "usual quantities". Define the Hasse invariant (γ-invaraint) to be defined as $\gamma(E/K) := -\frac{c_4}{c_6} \in K^*/(K^*)^2$.

Lemma 4.7.

1. *$\gamma(E/K)$ is well defined and independent of choice of Weierstrass equations*
2. *If $j \neq 0, 1728$ then $E \cong_K E'$ if and only if $j(E) = j(E')$ and $\gamma(E/K) = \gamma(E'/K)$.*
3. *If $j(E) = j(E')$ and $\gamma(E/K) \neq \gamma(E'/K)$, let $t = \sqrt{\frac{\gamma(E/K)}{\gamma(E'/K)}}$ and $L = K(t)$ then $E \cong_L E'$.*

Proof. Assume $E : Y^2 = X^3 + AX + B$

1. Let $u \in K^*$, $u^4 c_4 = c_4'$ and $u^6 c_6 = c_6'$, hence independent of the Weierstrass equations
2. $j(E) = j(E')$ implies $\frac{A'^3}{B'^2} = \frac{A^3}{B^2}$,and $j(E/K) = j(E'/K)$ implies $t \in K^*$ such that $\frac{A}{B}t = \frac{A'}{B'}$. Hence we get $t^4 A = A'$ and $t^6 B = B'$
3. The isomorphism is defined by $(x, y) \mapsto (t^2 x, t^3 y)$.

□

Theorem 4.8 (Tate's p-adic uniformisation)**.** *Let E/K be an elliptic curve $|j(E)| > 1$*

1. *There exists a unique $q \in K^*$ such that $E \cong E_q$ over $\overline{K}$.*
2. *The Following Are Equivalent:*
   (a) *$E \cong E_q$ over K*
   (b) *$\gamma(E/K) = 1$*
   (c) *E has split multiplicative reduction.*

Proof.

1. This follows from the Lemma
2.

$a) \iff b)$ $E \cong E_q$ over K is the same as $j(E) = j(E_q)$ and $\gamma(E/K)$ and $\gamma(E_q/K)$. So we just need to show that $\gamma(E_q/K) = 1$ for all q. We use the following lemma

Lemma. *Let $\alpha \in K^*$, $|\alpha| < 1$, then $1 + 4\alpha$ is a square in K.*

$\gamma(E_q/K) = \frac{1+240s_3(q)}{1-504s_5(q)}$, so we can use the lemma to see that $j(E_q/K) = 1$.

$a) \Rightarrow c)$ To see this, note that $|a_4(q)| = |a_6(q)| = |q| < 1$. So $\widetilde{E}_q : Y^2 + XY = X^3$.

$c) \Rightarrow b)$ Read Chris' 4th year project.

□

4.4 Application

Theorem 4.9. *Let K be a number field, E/K an elliptic curve with $j(E) \notin \mathcal{O}_K$ then* $\mathrm{End}(E) = \mathbb{Z}$.

Proof. Uses Tate's curve. □

5 General Theory of Affinoids (Chris Birkbeck)

Let K be a field, complete with respect to a non-archimedean norm $||$. Let $\overline{K}$ be its algebraic closure. If $K \subseteq E \subseteq \overline{K}$, $[E:K] < \infty$, then E is complete with respect to $||$.

Let $B_n(\overline{K}) = \left\{(x_1, \dots, x_n) \in \overline{K}^n \,|\, |x_i| \leq 1\right\}$.

Fact. *A formal power series with coefficients in K, $f = \sum_{v \in \mathbb{N}^n} c_v X^v$ converges on $B_n(\overline{K})$ if and only if* $\lim_{\sum v_i \to \infty} |c_v| = 0$.

Definition 5.1. The *Tate Algebra*, $T_n = K\langle x_1, \dots, x_n\rangle$ is the K-algebra of power series converging on $B_n(\overline{K})$.

Think of $f \in T_n$ as a map $B_n(\overline{K}) \to \overline{K}$.

Define a norm on T_n called *the Gauss Norm* as follows: take $f \in T_n$, $f = \sum_v c_v X^v$. Let $|f| = \max_v |c_v|$.

Exercise. Prove its a norm.

Let:

- $K^\circ = \{a \in K | \, |a| \leq 1\}$
- $K^{\circ\circ} = \{a \in K | \, |a| < 1\}$
- $\widetilde{K} = K^\circ / K^{\circ\circ}$

There exists a unique epimorphism $K^\circ \to \widetilde{K}$ defined by $c \mapsto \widetilde{c}$. This extends to an epimorphism $K^\circ\langle X_1, \dots, X_n\rangle \to \widetilde{K}[X_1, \dots, X_n]$. ($f$ is an epimorphism, $f : X \to Y$ if for all $\forall g_1, g_2 : Y \to Z$ such that $g_1 \circ f = g_2 \circ g \Rightarrow g_1 = g_2$)

Fact.

- *T_n is complete with respect to the Gauss norm*
- *If $f \in T_n$, $|f| = 1$, then $f \in T_n^*$ if and only if $\widetilde{f} \in \widetilde{K}^*$. In general $|f - f(0)| < |f(0)|$ if and only if $\widetilde{f} \in \widetilde{K}^*$.*
- *(Maximum principle) if $f \in T_n$, then $|f(x)| \leq |f|$, and there exists $x \in B_n(\overline{K})$ such that $|f(x)| = |f|$.*

Definition 5.2. Let $g \in T_n$, $g = \sum_{v=0}^{\infty} g_v X_n^v$ for $g_v \in T_{n-1}$. We say g is *X_n-distinguished of order s* if:

1. $g_s \in T_{n-1}^*$
2. $|g| = |g_s|$ and $|g_s| > |g_v|$ for all $v > s$.

If $|g| = 1$ then g X_n-distinguished of order s implies $\widetilde{g} = \widetilde{g}_s X_n^s + \cdots + \widetilde{g}_0 X_n^0$ with $\widetilde{g}_s \in \widetilde{K}^*$.

Order 0 if and only if g is a unit.

Corollary 5.3 (Weierstrass preparation)**.** *If $g \in T_n$ is X_n distinguished of order s, then there exists a unique $w \in T_{n-1}[X_n]$ of degree s and there exist $e \in T_n^*$ such that $g = ew$. Such w is called* Weierstrass polynomial.

Corollary 5.4 (Noether Normalisation)**.** *For a proper ideal $a \subsetneq T_n$ there is a K-algebra homomorphism $T_d \to T_n$ ($d = \mathrm{krulldim} T_n/a$) such that $T_d \to T_n \to T_n/a$ is a finite monomorphism.*

Fact.

- *T_n is Noetherian*
- *Each ideal is complete (hence closed)*
- *$B_n(\overline{K}) \twoheadrightarrow \mathrm{Max}(T_n)$ by $x \mapsto m_x = \{f \in T_n | f(x) = 0\}$. Here $f(x)$ is image of $f \in T_n/m_x$. For every $g \in T_n$, $g(x)$ denotes the image of $g \in T_n/m_x$. This is well defined up to $\mathrm{Gal}(\overline{K}/K)$*
- *$m \subseteq T_n$ is a maximal ideal, then $[T_n/m : K] < \infty$.*

Definition 5.5. A K-algebra A is an *affinoid algebra* if there exists an epimorphism $\alpha : T_n \to A$ for some n.

We define the *suprenum norm* as follows: let $f \in A$, set $|f|_{\text{sup}} = \sup_{x \in \text{Max}(A)} |f(x)|$. This is a seminorm, as $|f|_{\text{sup}} = 0$ does not implies $f = 0$. We do have $|f|_{\text{sup}} = 0$ if and only if f is nilpotent.

We define *Affinoid spaces* as follows: Let A be an affinoid algebra. Let $\text{Sp}(A)$ be the set $\text{Max}(A)$ + the "functions". The morphism $\text{Sp}(A) \to \text{Sp}(B)$ is defined by $\sigma : B \to A$, $\sigma^* : \text{Max}(A) \to \text{Max}(B)$.

$a \subseteq A$ is an ideal, $V(a) = \{x \in \text{Sp}(A) | f(x) = 0 \forall f \subseteq a\}$. If $Y \subseteq \text{Sp}(A)$ we can define $I(Y) = \{f \in A | f(y) = 0 \forall y \in Y\} = \cap_{y \in Y} m_y$.

Canonical topology: Let $X = \text{Sp}(A)$, $f \in A$, $\epsilon \in \mathbb{R}$. Write $X(f, \epsilon) = \{x \in X | \, |f(x)| < \epsilon\}$.

$X\left(\frac{f_1}{f_0,}, \ldots, \frac{f_n}{f_0}, 1\right) := X\left(\frac{f_1}{f_0}, 1\right) \cap \cdots \cap X\left(\frac{f_n}{f_0}, 1\right)$ with f_i no common zero. They are called *rational domains.*

Affinoid subdomain U is a finite union of rational domains.

6 Affinoid Subdomain (Céline)

6.1 Motivation and plan:

Zariski topology is too coarse, so we want to define a topology: Canonical topology induced by topology on K

- Define open sets
- Define Affinoid Subdomain
- Define affinoid functions.

Let $X = \mathrm{Sp}(A)$ an Affinoid K-space. Set $X(f,\epsilon) = \{x \in X \mid |f(x)| \leq \epsilon\}$ with $f \in A$, $\epsilon \in \mathbb{R}_{\geq 0}$.

Definition 6.1. The *canonical topology* is generated by sets of the type $X(f,\epsilon)$ where $f \in A$, $\epsilon \in \mathbb{R}_{\geq 0}$.

This implies that $U \subset X$ is open (with respect to the canonical topology) if and only if it is the union of finite intersections of $X(f,\epsilon)$.

Notation. $X(f) = X(f,1)$, $X(f_1,\dots,f_r) = X(f_1) \cap \dots \cap X(f_r)$.

Proposition 6.2. *The canonical topology is generated by sets of type $X(f)$ for f varying in A.*

Proof. Let $f \in A$, then the function $|f| : \mathrm{Sp}(A) \to \mathbb{R}_{\geq 0}$ takes values in $\left|\overline{K}\right|$. Thus, if $\epsilon \in \mathbb{R}_{\geq 0}$, we can write

$$X(f,\epsilon) = \bigcup_{\epsilon' \in \left|\overline{K}^*\right|, \epsilon' \leq \epsilon} X(f,\epsilon')$$

. For $\epsilon' \in \left|\overline{K}^*\right|$ we can find $c \in K^*$ and $s \in \mathbb{Z}$ such that $\epsilon'^s = |c|$. Hence

$$X(f,\epsilon') = X(f^s, \epsilon'^s) = X(c^{-1} f^s)$$

□

Lemma 6.3. *Consider $f \in A$, $x \in \mathrm{Sp}(A)$ such that $|f(x)| = \epsilon > 0$. Then there exists $g \in A$ with $g(x) = 0$ such that $|f(y)| = \epsilon$ for all $y \in X(g)$. This implies that $X(g)$ is an open neighbourhood of x contained in $\{y \in X \mid f(y) = \epsilon\}$*

Proof. To each x, there correspond a maximal ideal $m_x \subset A$. Write $\overline{f}$ for the residue class of f in A/m_x. Let $P(\zeta) = \zeta^n + c_1\zeta^{n-1} + \dots + c_n \in K[\zeta]$ is the minimal polynomial for $\overline{f}$ and let $P(\zeta) = \prod_{i=1}^n (\zeta - \alpha_i)$ its product decomposition over $\overline{K}$. Choose $A/m_x \hookrightarrow \overline{K}$, then $\epsilon = |f(x)| = \left|\overline{f}\right| = |\alpha_i| \; \forall i$ by uniqueness of valuation in $\overline{K}$. Consider $g = P(f) \in A$, then $g(X) = P(f(x)) = 0$. We claim that for $y \in X$ with $|g(y)| < \epsilon^n$ then $|f(y)| = \epsilon$. To see this, choose $A/m_y \hookrightarrow \overline{K}$, $|f(y) - \alpha_i| = \max\{|f(y)|, |\alpha_i|\} \geq |\alpha_i| = \epsilon \forall i$. Hence $|g(y)| = |P(f(y))| = \prod_{i=1}^n |f(y) - \alpha_i| \geq \epsilon^n$ which is a contradiction to the choice of y. Hence if $c \in K^*$ satisfies $|c| < \epsilon^n$, then $|f(y)| = \epsilon \forall y \in X(c^{-1}g)$. □

Open Sets:

- $\left\{x \in \mathrm{Sp}A \middle| f(x) \neq 0\right\}$
- $\left\{x \in \mathrm{Sp}A \middle| f(x) \leq \epsilon\right\}$
- $\left\{x \in \mathrm{Sp}A \middle| f(x) \geq \epsilon\right\}$
- $\left\{x \in \mathrm{Sp}A \middle| f(x) = \epsilon\right\}$
- $\left\{x \in \mathrm{Sp}A \middle| f(x) < \epsilon\right\}$
- $\left\{x \in \mathrm{Sp}A \middle| f(x) > \epsilon\right\}$

Proposition 6.4. *Let $x \in X$, Sets $X(f_1, \dots f_r)$ forms a basis of neighborhood for x.*

Proposition 6.5. *Continuity: Let $\phi^* : A \to B$ be morphism of Affinoid K-algebra and $\phi : \mathrm{Sp}B \to \mathrm{Sp}A$ associated morphism of affinoid K-spaces. For $f_1, \dots, f_r \in A$ then $\phi^{-1}((\mathrm{Sp}A), (f_1, \dots, f_r)) = (\mathrm{Sp}B)(\phi^*(f_1), \dots, \phi^*(f_r))$. Hence ϕ is continuous with respect to the canonical topology.*

Proof. $y \in \mathrm{Sp}B$, we have the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\phi^*} & B \\ \downarrow & & \downarrow \\ A/m_{\phi(y)} & \longrightarrow & B/m_y \end{array}$$

$A/m_{\phi(y)} \to B/m_y \hookrightarrow \overline{K}$. Then $|f(\phi(y))| = |\phi^* f(y)| \forall f \in A$. This implies $\phi^{-1}((\mathrm{Sp}A)(f)) = \mathrm{Sp}B(\phi^*(f))$, so take intersections and we are done. □

Definition 6.6.

1. $X(f_1, \dots, f_r) = \{x \in X | \, |f_i(x)| \leq 1\}$ is called *Weierstrass domain* in X
2. $X(f_1, \dots, f_r, g_1^{-1}, \dots, g_s^{-1}) = \{x \in X | \, |f_i(x)| \leq 1, |g_j(x)| \geq 1\}$ called *Laurent domains* in X
3. $X(\frac{f_1}{f_0}, \dots \frac{f_r}{f_0}) = \{x \in X | \, |f_i(x)| \leq |f_0(x)|\}$ for $f_0, \dots, f_r$ without common zeros, it is called a *rational domain* in X.

Definition 6.7. A subset $U \subset X$ is an affinoid subdomain of X if there exists a morphism of affinoid K-spaces: $\iota : X' \to X$ such that $\iota(X') \subset U$.

The following universal property must hold: If $\phi : Y \to X$ such that $\phi(Y) \subset U$, then there exists a unique $\phi' : Y \to X'$ such that the following diagram commutes

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & X \\ & \searrow^{\exists! \phi'} & \downarrow \iota \\ & & X' \end{array}$$

Lemma 6.8. *Notation as above. $X = \mathrm{Sp}A$, $X' = \mathrm{Sp}A'$, let $\iota^* : A \to A'$ be the associated K-morphims. Then ι is injective and $\iota(X') = U$ and bijection of sets $X' \cong U$.*

This let us identify $U \subset X$ with X', which in turn gives a structure of affinoid K-space on any affinoid subdomains $U \subset X$.

Proposition 6.9. *Weierstrass, Laurent and rational domains are called* special affinoid subdomains.

Proposition 6.10. *$V \subset X$ an affinoid subdomain, $U \subset V$ is an affinoid subdomain, then $U \subset X$ is also an affinoid subdomain.*

Remark. If $V \subset X$ is a Weierstrass (respectively rational) subdomain, and $U \subset V$ is Weierstrass (or respectively rational) then $U \subset X$is also Weierstrass (respectively rational). But this is not true for Laurent domain.

Theorem 6.11 (Gerritzen - Grauert)**.** *Let X be an affinoid K-space, $U \subset X$ an affinoid subdomain, then U is a finite union of rational subdomains of X.*

6.2 Affinoid functions

Denote $\mathcal{O}_X(U)$ the affinoid K-algebra corresponding to $U \subset X$ an affinoid subdomain. If $U \subset V$ is an inclusion of affinoid subdomain, then we have a canonical map $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$ of K-algebra. This is a restrictions of functions on V to U. More precisely: $\mathcal{O}_X$ is a presheaf of affinoid K-algebra on the category of affinoid subdomain of X, called presheaf of affinoid functions on X. This can not be sheafified, hence more topology will need to be defined.

7 Tate's Acyclicity Theorem (Angelos)

Let X be an affinoid domain and T_X the category of affinoid subdomain of X, with inclusions as morphisms. We have seen that $\mathcal{O}_X$ is a presheaf $\mathcal{F}$, where $\mathcal{O}_X$ is the set of affinoid functions on X such that $\mathcal{F}(U) = \mathcal{O}_X(U)$. We have the following sequence

$$\mathcal{O}_X(U) \longrightarrow \prod_{i\in I} \mathcal{O}_X(U_i) \rightrightarrows \prod_{i,j} \mathcal{O}_X(U_i \cap U_j) \quad (*)$$

$$f \longmapsto (f|_{U_i})_{i\in I}, (f_i)_{i\in I} \longmapsto \begin{matrix} f_i|_{U_i\cap U_j} \\ f_j|_{U_i\cap U_j} \end{matrix}$$

where $U \in T_X$ and $\Delta = (U_i)_{i\in I}$ of U and $U_i, U_j \in T_X$

Definition 7.1. If $A \longrightarrow B \rightrightarrows C$, we say that the *sequence is exact* if A is mapped bijectively to the subset of B such that the elements have the same images under the map $B \rightrightarrows C$.

Definition 7.2. For a presheaf $\mathcal{F}$ on X and a covering $\Delta = (U_i)_{i\in I}$ of X, $U_i \in T_X$, we say that $\mathcal{F}$ is a Δ-*sheaf*, if for all $U \in T_X$ we have that the sequence $(*)$ applied to $\Delta|_U = (U \cap U_i)_{i\in I}$ is exact.

Theorem 7.3 (Tate)**.** *Let X, $\mathcal{O}_X$ be as above, then $\mathcal{O}_X$ is a Δ-sheaf for any finite covering of X be affinoid subdomainn*

Comments:

1. The main idea is to reduce the general case to "well-known" cases such that an easy calculations proves the theorem.

   We can define Cech cohomology with respect to a covering Δ (finite) and our presheaf $\mathcal{F}$

Theorem 7.4 (Tate)**.** *Let X be an affinoid k-space and Δ a finite covering of X, then$H^q(\Delta, \mathcal{O}_X) = 0$ for $q > 0$. We say that Δ is* acyclic.t

7.1 Grothendiecks Topology

Definition 7.5. For any affinoid k-space X, the *Weak Grothendieck Topology T* on X consists of

1. Cat$T$ the category of affinoid subdomains of X with inclusion as morphism.
2. Cov$T$ the set of all finite families $(U_i \to U)_{i\in I}$ of inclusions of affinoid subdomains in X such that $U = \cup_{i\in I} U_i$.

Definition 7.6. Let X be an affinoid k-space the *Strong Grothendieck Topology* on X is given as follows:

1. A subset $U \subset X$ is called *admissible open* if there is a (not necessarily finite) covering $U = \cup_{i\in I} U_i$ by affinoid subdomains $U_i \subseteq X$ such that for all morphisms of affinoid k-spaces $\phi : Z \to X$ satisfying $\phi(Z) \subseteq U$ the covering $\left(\phi^{-1}(U_i)\right)_{i\in I}$ of Z admits a subcovering, which is a finite covering of Z by affinoid subdomains.
2. A covering $V = \cup_{i\in I} V_i$ of some admissible open subset $V \subseteq X$ by means of admissible open set V_i is called admissible if for each morphism of affinoid k-spaces $\phi : Z \to X$ satisfying $\phi(Z) \subset V$, the covering $\left(\phi^{-1}(V_i)\right)_{i\in I}$ of Z admits a subcovering, which is a finite covering of Z by affinoid subdomains.

Proposition 7.7. *Let X be an affinoid k-space for $f \in \mathcal{O}_X(X)$ and we define*

- $U = \{x \in X | \, |f(x)| < 1\}$
- $U' = \{x \in X | \, |f(x)| > 1\}$
- $U'' = \{x \in X | \, |f(x)| > 0\}$

Any finite union of set of this types is admissable open. Any finite covering by finite unions of sets of this type is admissible.

Corollary 7.8. *Let X be an affinoid k-space. The strong Grothendieck topology on X is finer than the Zariski, i.e., each Zariski open subset $U \subseteq X$ is admissible open, and each Zariski covering is admissable.*

The presheaf $\mathcal{O}_X$ of analytic functions is not a sheaf under the weak Grothendieck topology or the canonical topology, but it is a sheaf under the strong Grothendieck topology.

8 Reductions of curves (Haluk)

8.1 Recap

Set-up: $K = \overline{K}$ a non-archimedean complete valued field, $\mathbb{P} = (K^2 \setminus \{0,0\})/\sim$ the projective line over K

Open disks: $\{z \in L : |z-a| < r\}$ or $\{z \in K : |z-a| > r\} \cup \{\infty\}$

Connected affinoid subset of $\mathbb{P}$: $\mathbb{P} \setminus \{$finite union of open disc$\}$

Affinoid subset of $\mathbb{P}$: finite union of affinoid subsets

Tate Algebra: $T_n := K\langle z_1, \dots, z_n \rangle =$ formal power series in $z_1, \dots, z_n$ convergent on the polydisc $\mathbb{D}_n$

Affinoid Algebra: A=T_n /I for some $n \geq 1$ and $I \lhd T_n$

Affinoid Space: $X = \mathrm{Sp}(A) = \mathrm{Max}(A)$ (the set of maximal ideals) for some affinoid algebra A

Notes:

- $\mathrm{Sp}(T_n) \cong \mathbb{D}_n$
- $\phi : T_n \twoheadrightarrow A$ with $\ker(\phi) = I$, $\phi^* : \mathrm{Sp}(A) \hookrightarrow \mathbb{D}_n$, can view $\mathrm{Sp}(A)$ as zero set of I inside $\mathbb{D}_n$

Affine subdomain: $U \subseteq X = \mathrm{Sp}(A)$ such that there exists $\phi : A \to B$ (B unique) with $\phi^*(\mathrm{Sp}(B)) = U$ and some universal condition

Weak G-Topology on X: Open sets are affine subdomains, covers are finite covers.

8.2 Rigid analytic space

Definition 8.1. A *Rigid Analytic Space* $(X, \mathcal{O}_X)$ where

- X is a space with a G-topoligy
- $\mathcal{O}_X$ a sheaf of K-algebra

such that there is an admissable covering $\{X_i\}$ such that $\{X_i, \mathcal{O}_X|_{X_i}\}$ is an affinoid space with $\forall U \subseteq X_i$ is affinoid subdomain $\mathcal{O}|_{X_i}(U) = B$.

In practice, we start with $\{X_i\}$ and glue them:

- $\{X_i\}_{i \in I}$ affinoid spaces such that
  - $\forall (i,j) \in I^2$, $i \neq j$: there exists affinoid subdomain $X_{i,,j} \subseteq X_i$ and there exists isomorphism $\phi_{j,i} : X_{i,j} \to X_{j,i}$
  - $\phi_{i,j}^{-1} = \phi_{j,i}$
  - $\forall i,j,k \in I$, $\phi_{j,i}(X_{i,j} \cap X_{i,k}) = X_{j,i} \cap X_{j,k}$ and $\phi_{k,i} = \phi_{k,j} \circ \phi_{j,i}$ on $X_{i,j} \cap X_{i,k}$

There exists a unique Rigid Analutic Space X with G-topology T_X such that $U \subseteq X$ is in T_X if and only if $\forall i$ $U \cap X_i$ is admissible opne

Example. Take $\mathbb{P}$, $X_0 = \mathrm{Sp}\left(K\langle T_o \rangle\right) \cong \mathbb{D}_1$ and $X_\infty = \mathrm{Sp}\left(K\langle T_\infty \rangle\right) \cong \mathbb{D}_1$.

Then $X_{0,\infty} = \mathrm{Sp}\left(K\left\langle T_0, T_0^{-1}\right\rangle\right) \cong \partial\mathbb{D}_1$, $X_{\infty,0} = \mathrm{Sp}\left(K\left\langle T_\infty, T_\infty^{-1}\right\rangle\right) \cong \partial\mathbb{D}_1$.

We define $\phi : K\left\langle T_0, T_0^{-1}\right\rangle \to K\left\langle T_\infty, T_\infty^{-1}\right\rangle$ by $T_0 \mapsto T_\infty^{-1}$. This gives $\phi^* : \partial\mathbb{D}_1 \to \partial\mathbb{D}_1$ defined by $z \mapsto 1/z$.

Analytification: X/K an algebraic variety, this gives $X = X(K)$: we can put a Rigid Analytic Space structure on this X^{an}

8.3 Analytic Reduction of Rigid Analytic Space

Let $(X, \mathcal{O}_X), \{U_i\}$ be "nice" cover by affinoid spaces. We construct an algebraic variety $\overline{X}/k$.

Step 1 Fix $U_i = U$. $U = \mathrm{Sp}(A)$ for some A affinoid algebra.

$A^\circ = \left\{f \in A \middle| \|f\| \leq 1\right\}$ is a $\mathbb{Z}_K$-algebra

$A^{\circ\circ} = \left\{f \in A \middle| \|f\| < 1\right\}$ is an ideal of A°

$\overline{A} := A^\circ / A^{\circ\circ}$ is a k-algebra of finite type

$\overline{U} = \mathrm{Spec}(\overline{A})$ an algebraic variety over k.

Tehre is a surjection of sets, $\{\phi : A \to K\} = \mathrm{Set}(A) \twoheadrightarrow \mathrm{Sp}(\overline{A}) = \left\{\phi : \overline{A} \to k\right\}$.

Maximal Modulus Principle: $\|f\| = \max_{x \in U} |f(x)|$. This implies $\phi(A^\circ) \subseteq \mathbb{Z}_K$ and $\phi(A^{\circ\circ}) \subseteq m_k$.

Start with $\phi : A \to K$, $\phi|_{A^0} : A^\circ \to \mathbb{Z}_k$. Mod out by $A^{\circ\circ}$ we get $\overline{\phi} : \overline{A} \to k$.

Step 2 Glue $\overline{U}_i$ to get $\overline{X}/k$. We need $\overline{U_i \cap U_j} \rightrightarrows \frac{\overline{U_i}}{\overline{U_j}}$ to be "open immersion"

Example.

First Example: $X = \mathrm{Sp}(K\langle T\rangle) \cong \mathbb{D}_1$. $A^\circ = \mathbb{Z}_K\langle T\rangle$, $A^{\circ\circ} = m_k\langle T\rangle$. Hence $\overline{A} = k[t]$, $\overline{X} = \mathbb{A}^1$ over k

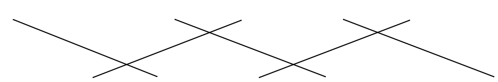
Second Example: $A = K\left\langle T, T^{-1}\right\rangle$, $X = \partial\mathbb{D}_1 = \mathrm{Sp}(A)$, $A^\circ = \mathbb{Z}_K\left\langle T, T^{-1}\right\rangle$, $A^{\circ\circ} = m_k\left\langle T, T^{-1}\right\rangle$. Hence $\overline{A} = k[T, T^{-1}]$, $\overline{X} = \mathbb{G}_m$ over k

Third Example: $\mathbb{P}$, $X_0 = \mathrm{S_p}\left(K\langle T_0\rangle\right) \to \overline{X}_0 = \mathbb{A}^1$ over k. $X_\infty = \mathrm{Sp}\left(K\langle T_\infty\rangle\right) \to \overline{X}_\infty = \mathbb{A}^1$ over k. Then $X_{0,\infty} = \mathrm{Sp}(K\left\langle T_0, T_0^{-1}\right\rangle) \to \overline{X_{0,\infty}} = \mathbb{G}_m$ over k. $X_{\infty,0} = \mathrm{Sp}\left(K\left\langle T_\infty, T_\infty^{-1}\right\rangle\right) \to \overline{X}_{\infty 0} = \mathbb{G}_m$ over k. Then we have the map $\overline{X}_{0,\infty} \to \overline{X}_{\infty,0}$ defined by $z \mapsto z^{-1}$. We have $\overline{X} = \mathbb{P}^1$ over k.

Fourth Example: Take $q \in K^*$ such that $0 < |q| < 1$. Let $\mathcal{L} = \{q^n | n \in \mathbb{Z}\} \cup \{0, \infty\}$ and $\mathcal{L}^* = \{0, \infty\}$. Consider $X = \mathbb{P} \setminus \mathcal{L}^*$. Consider the covering $\{X_n\}_{n \in \mathbb{Z}}$ where $X_n = \left\{z \in K^* : |q|^{\frac{n+1}{2}} \leq |z| \leq |q|^{\frac{n}{2}}\right\}$. This is an affinoid space $X_n = \mathrm{Sp}\left(K\left\langle q^{-\frac{n}{2}} z_n, q^{\frac{n+1}{2}} z_n^{-1}\right\rangle\right)$, $X_{n+1} = \mathrm{Sp}\left(K\left\langle q^{-\frac{n+1}{2}} z_{n+1}, q^{\frac{n+2}{2}} z_{n+1}^{-1}\right\rangle\right)$. We glue X_n with X_{n+1}, by sending $q^{\frac{n+1}{2}} z_n^{-1} \mapsto q^{-\frac{n+1}{2}} z_{n+1}$.

Now $\overline{X}_n$ is the union of two lines $l_{1,n}$ and $l_{2,n}$ meeting at P_n. Then we $\left\{|z| = |q|^{n/2}\right\} \to l_{1,n} \setminus \{P_n\}$ and $\left\{|z| = |q|^{-\frac{n+1}{2}}\right\} \to l_{2,n} \setminus \{P_n\}$ while the annulus $\left\{|q|^{-\frac{n+1}{2}} < |z| < |q|^{\frac{n}{2}}\right\} \to P_n$. We have $A = K\left\langle q^{-\frac{n}{2}} z, q^{\frac{n+1}{2}} z^{-1}\right\rangle$, $\overline{A} = k[u', z']/(uz)$.

To glue all of this together, note that we have the map $l_{2,n} \setminus \{P_n\} \to l_{1,n+1} \setminus \{P_{n+1}\}$ defined by $z \mapsto z^{-1}$. So we get that $\overline{X}$ is the union of copies of $\mathbb{P}^1$ over k each intersecting exactly two others.



Fact. *The intersection graph is a tree* .

$\left\{\begin{pmatrix} q^n & 0 \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}\right\}$ gives rise to Γ

Theorem 8.2. *Let $\mathcal{L}$ be an infinite compact subset of $\mathbb{P}$. Put $X = \mathbb{P} \setminus \mathcal{L}^*$. X has a certain Rigid Analytic Space structure and a certain cover $\{X_i\}$ which with respect to which the reduction $R : X \to \overline{X}$ has the following structure:*

1. $\overline{X}$ *is an algebraic variety over* k *(locally finite type schemes over* k*)*

2. *Each irreducible component of* $\overline{X}$ *is a* $\mathbb{P}^1$ *over* k.

3. *Intersections of irreducible components are either* $\emptyset$ *or an ordinary double points.*

4. *The intersection graph is a tree*

5. *Points in* $\mathcal{L} \setminus \mathcal{L}^*$ *are mapped down to non-singular points of* $\overline{X}$.

9 Schottky groups and Mumford curves (Jeroen)

Notation.

Rings:

- K: finite extension of $\mathbb{Q}_p$, with p odd
- R : Valuation ring of K
- k: Residue field of R

Curves:

- X: Curve over K
- X_R: model (flat, proper, regular) over R
- $\overline{X}_R = \overline{X}_{\mathcal{U}}$: reduction of X_R (special fiber), curve over k

Groups (as done by Chris W.)

- $\Gamma = \langle \gamma_1, \ldots, \gamma_g \rangle \subseteq \mathrm{PGL}_2(K)$ Schottky group
- $D = \mathbb{P}^1 \setminus \cup^{2g} B_i$ fundamental domain
- $\gamma_i(\beta_i) = \mathbb{P} \setminus \overline{B_{i+g}}$
- $\gamma_i(\overline{\beta_i}) = \mathbb{P} \setminus B_{i+g}$
- $\mathcal{L}_\Gamma$ set of limit points of Γ

Uniformisation

- $\Omega_\Gamma = \mathbb{P}^1 \setminus \mathcal{L}_\Gamma$
- $\rho : \Omega_\Gamma \to T_\Gamma$ (as in Haluk's talk)
- $T_\Gamma^* \subseteq \mathrm{BT}(K)$ dual graph

9.1 Stable models

Definition 9.1. X is said to admit a *semistable* (respectively *stable*, respectively *totally split*) *model* if it admits a model X_R such that

1. $\overline{X}_R$ is reduced with ordinary double points as singularity

(respectively in addition to 1. :

2. Each component of $\overline{X}_R$ that is isomorphism with $\mathbb{P}^1_k$ contains at least 3 ordinary double points

respectively in addition to 1. :

3. Each components of $\overline{X}_R$ has a normalisation isomorphic to $\mathbb{P}^1_k$ and all ordinary double points are rationals)

Example. An elliptic curve E over K has semistable reduction if and only if:

1. E has good reduction
2. E has multiplicative reduction

E has totally split reduction if E has split multiplicative reduction

Note. Any elliptic curve acquires semistable reduction over $K(E[12])$

Example. Let X be an hyperelliptic curve, $X \stackrel{2:1}{\to} \mathbb{P}^1$ ramified over $s_1, \ldots, s_n \in \mathbb{P}^1(K)$, so $n = 2g(X) + 2$. The reduction type of X then only depends on the reduction map $\rho_S : \mathbb{P}^!_K \to T_S$, where $S = \{s_1, \ldots, s_n\}$.

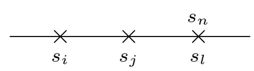
Construction 1: Let $M_!, \ldots, M_n$ be the lattices corresponding to the elements of $S^3 \setminus \Delta$. Then ρ_S is given by

$$\begin{array}{ccc} \rho_s : \mathbb{P}^1 & \xrightarrow{\prod_{i=1}^N \rho[M_i]} & \prod_{i=1}^N \mathbb{P}(M_i \otimes k) \\ & \searrow & \uparrow \\ & & \cup_{i=1}^N L_i = T_S \end{array}$$

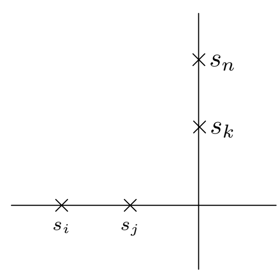
where $L_i = \prod_{j=1}^N U_j$, $U_j = \mathrm{Red}_{[M_j]}([M_i])$ and $U_i = \mathbb{P}(M_i \otimes k)$

Construction 2**:** iterative constructions. Suppose $\rho_{S'}$ for $S' = \{s_1, \ldots, s_n\}$ is constructed. To construct ρ_S:

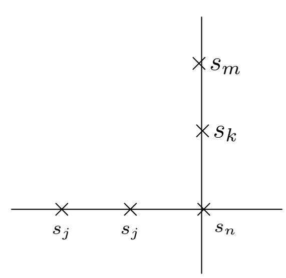
1. $\rho_S(s_n)$ is not a double point and not in $\rho_{S'}(S')$. Then put $\rho_S = \rho_{S'}$
2. $\rho_{S'}(s_n)$ is not double point but is in $\rho_{S'}(S')$.



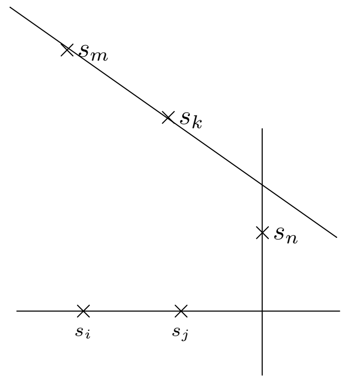
In the formula we have to add $M(s_i, s_j, s_n)$. This gives a blowup



3. $\rho_{S'}(s_n)$ is a double point:



Add the lattice $M(s_i, s_j, s_m)$ to get

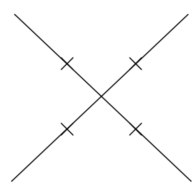


ρ_S gives rise to a cover of $\mathbb{P}^1$:

Generators are:

$$U(e^*) = \rho_S^{-1}(T_S \setminus \cup_{k \neq i,j} L_k)$$

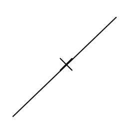
$$= \rho_S^{-1} \left(\quad \right)$$



Intersections are:

$$U(v^*) = \rho_S^{-1}(T_S \setminus \cup_{k \neq i} L_k)$$

$$= \rho_S^{-1} \left(\quad \right)$$



This is a special case of the cover defined by Haluk

Fact. *X has totally split reduction if and only if for all $L \subset T_S$ the partition of S obtained by contracting onto L contains at most two sets of odd cord.*

9.2 From groups to curves

Γ gives rise to $\mathcal{L} = \mathcal{L}_\Gamma$, $\Omega = \Omega_\Gamma = \mathbb{P}^1 \setminus \mathcal{L}$

Reduction of Ω:

$\mathcal{L}$ gives rise to a reduction $\mathbb{P}^1 \dashrightarrow T_{\mathcal{L}} = T_\Gamma$ (the reduction is only defined on $\Omega \subset \mathbb{P}^1$, so we get $\rho_{\mathcal{L}} : \Omega \to T_{\mathcal{L}}$)

Theorem 9.2.

1. *$X = \Gamma \setminus \Omega_\Gamma$ is a rigid analytic space defined by algebraic equations in some $\mathbb{P}^N$*
2. *X admits a cover $\mathcal{U}$ such that $\overline{X}_{\mathcal{U}}$ is totally split*
3. *The intersection graph of $\overline{X}_{\mathcal{U}}$ is isomorphism with $\Gamma \setminus T_\Gamma$.*

Proof. Consider $\rho : \Omega \to T_\Gamma$. Cover Ω with $U(e^*)$, $U(v^*)$. $X = \Gamma \setminus \Omega$ is obtained by considering the action of Γ on T_Γ and gluing/identifying the $U(e^*)$ according to this action.

Algebraically: Use theta function for Γ to embed into $\mathbb{P}^N$ use GAGA □

9.3 From curves to groups

Theorem 9.3. *Let X be a curve over K admitting a totally split model X_R. Then X is of the form $\Gamma \setminus \Omega_\Gamma$ for some Schottky group Γ.*

Proof. X_R gives $\rho : X \to \overline{X}_R$. Construct corresponding sets $U(e^*)$, $U(v^*)$ for e^*, v^* in the intersection graph of $\overline{X}_R$. Construct Ω: G^* intersection graph of $\overline{X}_R$. Let $\pi : T^* \to G^*$ be the universal cover. Set $\Omega(e') = U(\pi(e'))$, $\Omega(v') = U(\pi(v'))$ where $e' \in T^*$ edge and $v' \in T^*$ vertex. Glue $\Omega(e')$ to $\Omega(e'')$ via $\Omega(v)$ if the edges e, e' meet in v. Now $X = \pi_1(G^*)\backslash\Omega$ by construction (so let $\Gamma = \pi_1(G^*)$).

We want to embed $\Omega \hookrightarrow \mathbb{P}^1$. To do this, $\Omega \to T^*$ defined by $p \mapsto q$ an ordinary double point on $v_o \in T^*$ say. Define a sheaf $\mathcal{F}$ on Ω via $\mathcal{F}|_{\Omega(e)} = \mathcal{O}_{\Omega(e)}$ if v_0 is not a vertex of e. $\mathcal{F}|_{\Omega(e)} = \frac{1}{f_e}\mathcal{O}_{\Omega(e)}$ if v_0 is a vertex of e, where $f_e \in \mathcal{O}(\Omega(e))$ such that f_e is single ordinary at p. We get a Cech complex

$$0 \to \prod_e \mathcal{F}|_{\Omega(e)} \to \prod_v F|_{\Omega(v)} \to 0$$

Nakayam can be used to show that $H^0(\Omega, \mathcal{F}) = K \oplus Kf$.

Fact: f defines $\Omega \hookrightarrow \mathbb{P}^1$.

Fact: Γ acting on Ω extend to an action on $\mathbb{P}^1$. Then Γ, being free in g generators is a Schottky group □