EXERCISE SHEET FOR SUMMER SCHOOL ON "GEOMETRIC MEASURE THEORY FOR THE EVOLUTION OF DISLOCATIONS" (JYVÄSKYLÄ, AUGUST 2026)

Exercise 1. Compute the mass, the boundary and the mass of the boundary of the following currents in $\mathbb{R}^2$. For each one, say whether it is integral or not.

(1) $\mathrm{e}_1\mathcal{H}^1 \llcorner ([0,1]\times\{0\})$;
(2) $f(x)\mathrm{e}_1\mathcal{H}^1 \llcorner ([0,1]\times\{0\})$ with $f \in \mathrm{C}^1(\mathbb{R}^2;\mathbb{R})$;
(3) $\mathrm{e}_2\mathcal{H}^1 \llcorner ([0,1]\times\{0\})$;
(4) $\mathrm{e}_1\mathcal{H}^2 \llcorner [0,1]^2$.

Exercise 2. Define the **flat norm** of $T \in \mathrm{N}_k(\mathbb{R}^d)$ as

$$\mathbf{F}(T) := \inf\big\{\, \mathbf{M}(T-\partial S) + \mathbf{M}(S) \;:\; S \in \mathrm{N}_{k+1}(\mathbb{R}^d) \,\big\},$$

where we adopt the convention that $\inf\emptyset := \infty$. Prove the following:

(i) $\mathbf{F}(\delta_x - \delta_y) \le \min\{2, |x-y|\}$ where $x, y \in \mathbb{R}^d$;
(ii) let

$$T := \sum_{n=0}^{\infty}\big(\delta_{2^{-2n}} - \delta_{2^{-2n-1}}\big).$$

Show that the mass of T is *not* finite, while the flat norm of T is finite.

Exercise 3. Let Ω be star-shaped with vertex $p \in \Omega$, and let $T \in \mathrm{I}_k(\overline{\Omega})$. Define

$$H(t,x) := (1-t)p + tx, \qquad \overline{H}(t,x) := (t, H(t,x)),$$

and the cone

$$p \lhd T := \overline{H}_*(\llbracket (0,1) \rrbracket \times T).$$

(i) Prove that $p \lhd T \in \mathrm{I}_{1+k}([0,1]\times\overline{\Omega})$ and give an expression for $t \mapsto \mathrm{Var}(p\lhd T;[0,t))$.
(ii) Prove that

$$\partial(p \lhd T) = \delta_1 \times T - p \lhd \partial T.$$

Exercise 4. For every $N \in \mathbb{N}$, construct a space–time current $S_N \in \mathrm{I}_2([0,1]\times\overline{\Omega}_N)$ such that

$$\mathrm{Var}(S_N;[0,1]) = 2, \qquad \mathrm{Var}(\partial S_N;[0,1]) = 0, \qquad \|S_N\|_{\mathrm{L}^\infty([0,1])} \ge N.$$

This shows in particulat that slice masses cannot be bounded only in terms of $\mathrm{Var}(S;[0,1])$ and $\mathrm{Var}(\partial S;[0,1])$.

For those taking the course for credit, please submit a solution to the following:

Exercise 5. Let $t \mapsto T_t \in \mathrm{I}_k(\mathbb{R}^d)$, $t \in [0,1]$, with $\partial T_t = 0$ for every $t \in [0,1]$, have uniformly compact support and be absolutely continuous with respect to the **homogeneous flat norm**, defined for a $T \in \mathrm{N}_k(\mathbb{R}^d)$ as follows:

$$\mathbf{F}^\circ(T) := \inf\bigl\{ \mathbf{M}(S) \;:\; S \in \mathrm{N}_{k+1}(\mathbb{R}^d),\ \partial S = T \bigr\}.$$

That is, assume that there is $g \in \mathrm{L}^1([0,1])$ such that

$$\mathbf{F}^\circ(T_s - T_t) \le \int_s^t g(r)\,\mathrm{d}r$$

for all $s < t$. Prove the following statements, in order:

(i) There are $R_{t,h} \in \mathrm{N}_{k+1}(\mathbb{R}^d)$ such that

$$\partial R_{t,h} = T_{t+h} - T_t, \qquad \mathbf{M}(R_{t,h}) = \mathbf{F}^\circ(T_{t+h} - T_t).$$

(ii) The family $h^{-1}R_{t,h}$ is equibounded in mass for a.e. t, and there is an infinitesimal sequence $h_j \to 0$ such that

$$h_j^{-1} R_{t,h_j} \overset{*}{\rightharpoonup} R_t$$

for some $(k+1)$-current R_t on $\mathbb{R}^d$ with finite mass.

(iii) For every test k-form $\omega \in \mathcal{D}^k(\mathbb{R}^d)$, the scalar map

$$t \mapsto \langle T_t, \omega \rangle$$

is absolutely continuous.

(iv) Conclude that there exists a family of finite-mass $(k+1)$-currents R_t, $t \in [0,1]$, solving

$$\frac{\mathrm{d}}{\mathrm{d}t} T_t - \partial R_t = 0.$$

for $\mathcal{L}^1$-a.e. t, where

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t} T_t, \omega \right\rangle := \lim_{h \to 0} \left\langle \frac{T_{t+h} - T_t}{h}, \omega \right\rangle$$

for all $\omega \in \mathcal{D}^k(\mathbb{R}^d)$.

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