Geometric Measure Theory for the Evolution of Dislocations

F. Rindler

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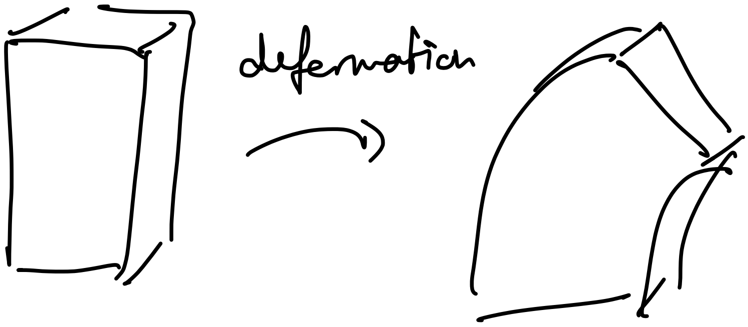
(joint work with Bonicatto, Del Nin, Hudson, Turnbull, ...)

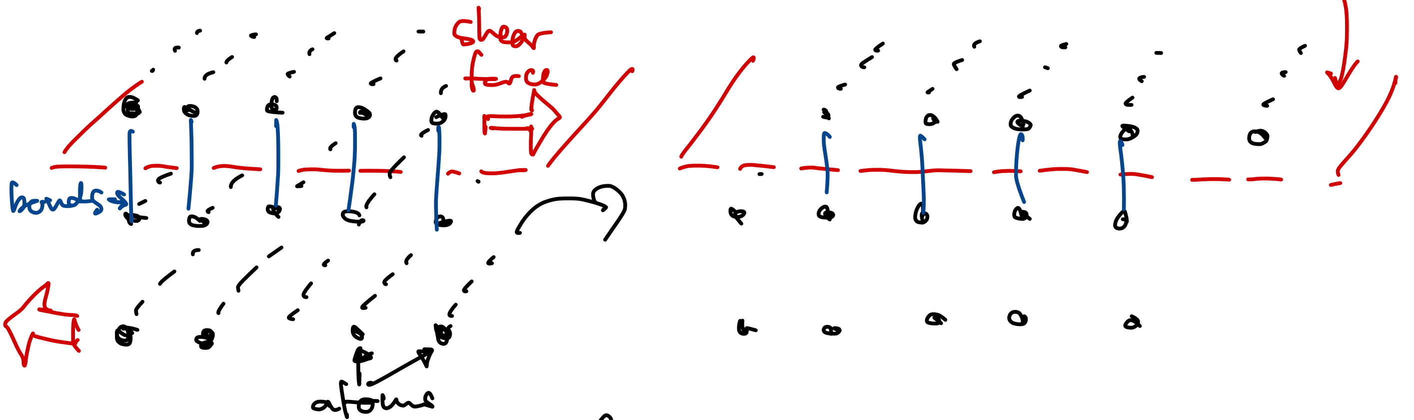
1. INTRODUCTION

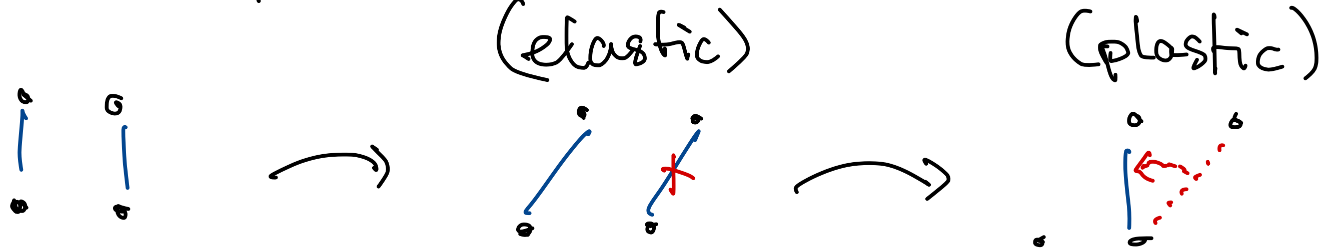
L1

Motivation: Elasto-plasticity in crystalline solids (e.g. metals):



Slip:

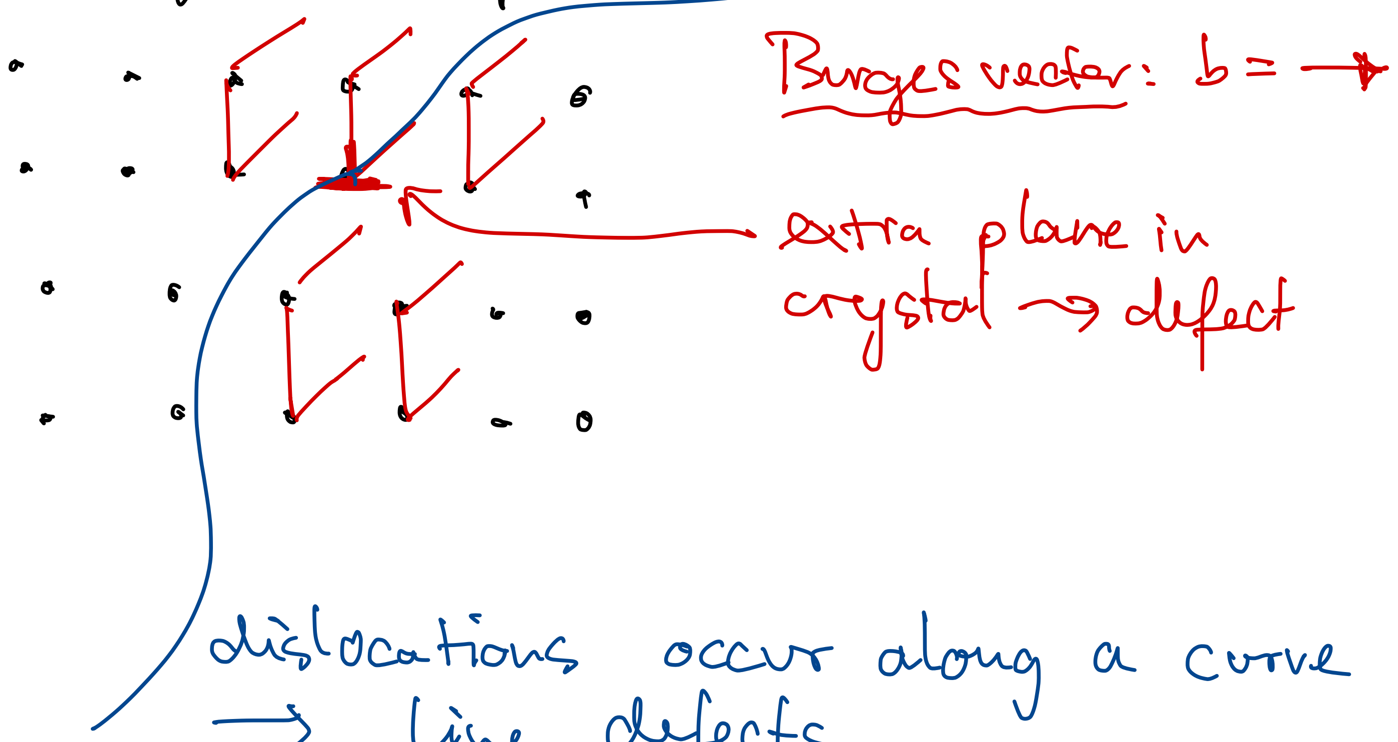




Q: But break all bonds over the slip plane at the same time?

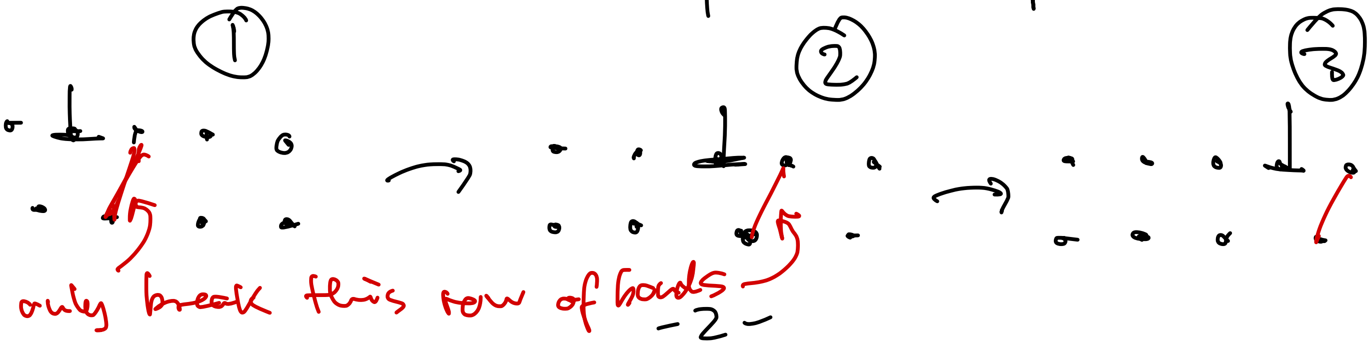
- No, gives wildly off prediction for the strength of metals !!
- Volterra/Taylor/Orowan, 100 years ago: Dislocation motion causes plasticity

$\underbrace{\text{Dislocation motion}}$ → crystalline line defects

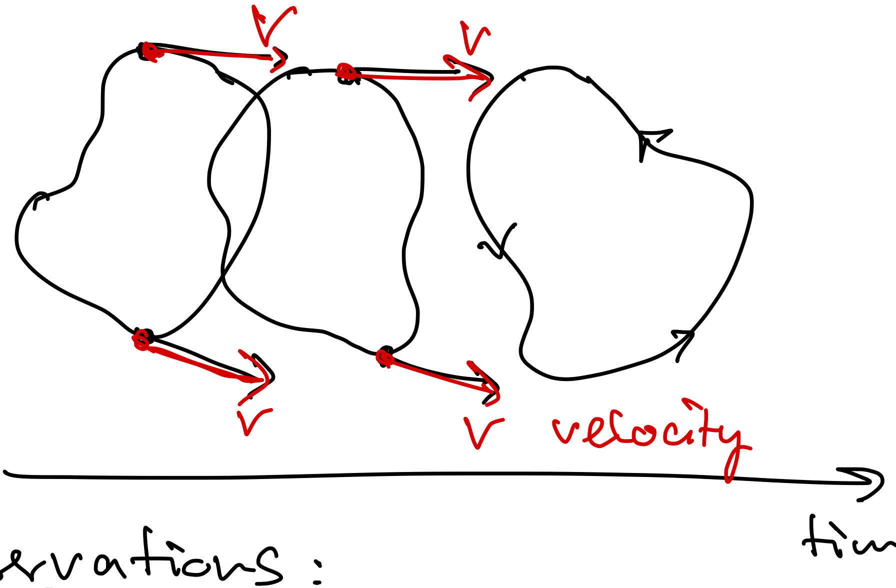


dislocations occur along a curve
→ line defects

Sequential mode of plastic slip:



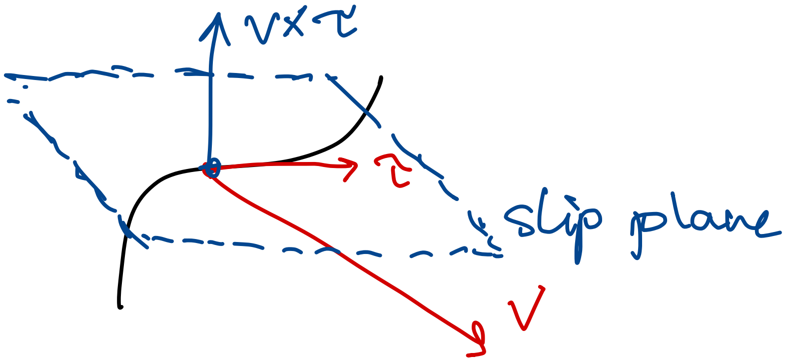
Abstract picture: "moving loops"



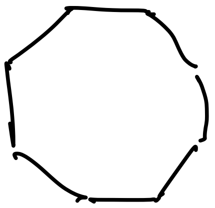
Observations:

- Loops for atomistic reasons

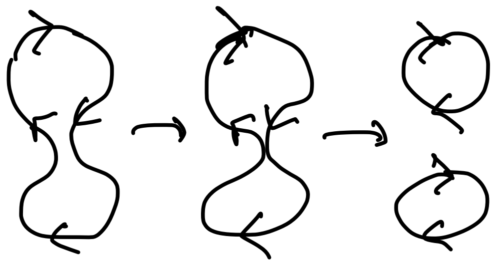
- Oriented loops to reflect crystal deformation direction



- Low regularity (corners) due to crystal anisotropy (only few allowed directions)



- Topological changes:



This leads to ....

... the theory of normal and integral currents, a form of "measure-theoretic geometry".
(Federer, Fleming, Almgren, Conti, Ortiz, Garroni, ...)

- integral 1-current:

$$T = \textcolor{red}{\vec{T}(x)}\ \textcolor{blue}{m(x)}\ \mathcal{H}^1 \llcorner \gamma$$

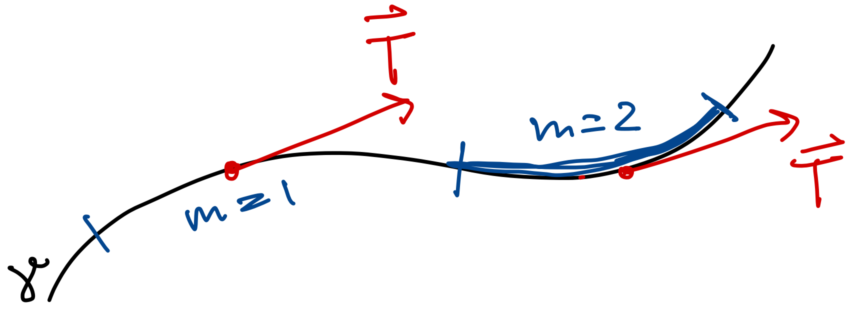
curve

orienting (tangent) vector $\in \mathbb{R}^d$ $(d = 2,3)$ with $|T(x)| = 1$

multiplicity $\in \mathbb{N}$

1D Hausdorff measure

$\mathcal{H}^1 \llcorner \gamma\,(A)$:= length of γ in A



- normal 1-current:

$$T = \vec{T}(x)\ \|T\|(x)$$

general measure, e.g. Lebesgue (volume), ...

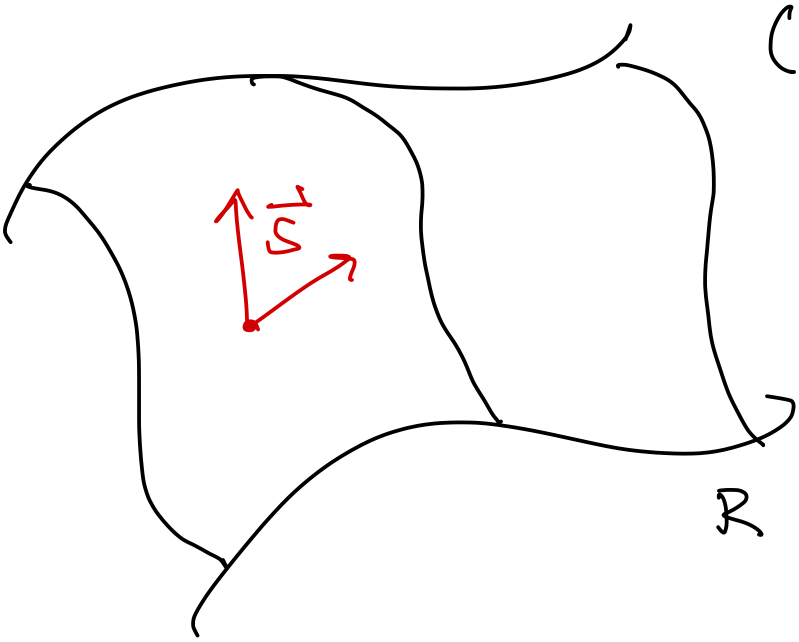
with the "boundary" also a measure
(↪ later)

- integral 2-current:

surface

$$S = \underbrace{\vec{S}(x)}_{\text{2-vector } (\to \text{later})} \, m(x) \, \mathcal{H}^2 \llcorner R$$

2D Hausdorff meas. ("area in $\mathbb{R}^n$")



- normal 2-current ...

- integral / normal k-currents ...

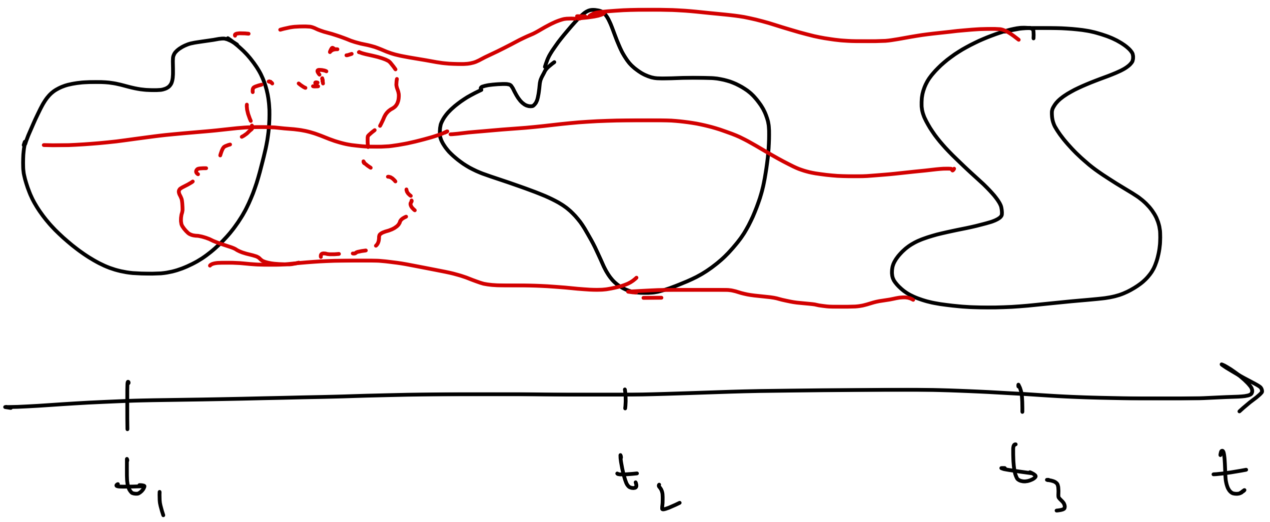
... all these are vector measures on $\mathbb{R}^d$ (→ later)

Q: Regularity of γ, R?

Q: What are k-vectors?

Q: Boundary?

The Space-Time Framework (Hudson-R. '22, R. '23, '25)

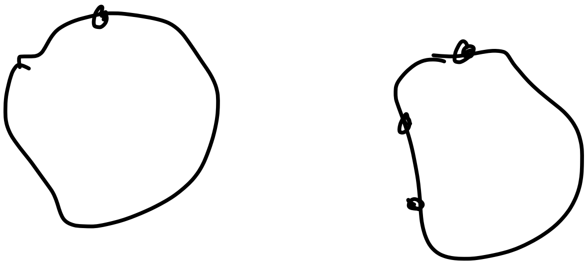


Say

$$[0,1] \ni t \longmapsto T_t \,, \quad \overbrace{\partial}^{\text{boundary of } T_t} T_t = 0$$

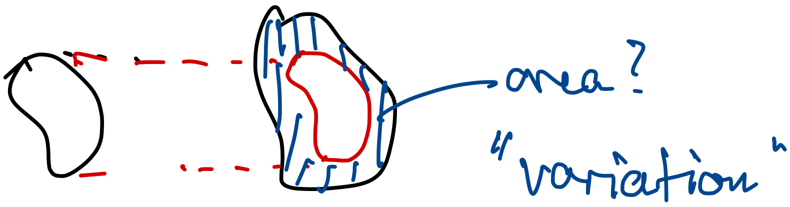
is an evolution of dislocation loops.

Q: How to define a "velocity" (in normal direction):

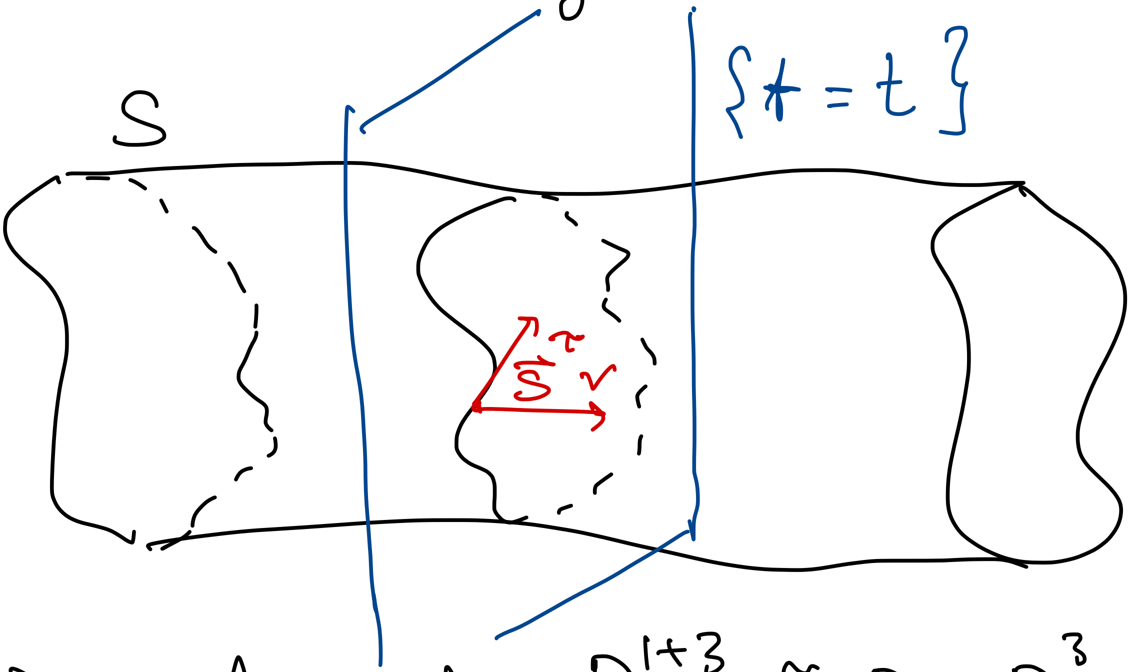


- how to compare pts?
- difference quotients?

Q: What area is traversed ($\approx$ dissipation)?



Idea: Instead of the path $t \mapsto T_t$ consider the space-time trajectory S "traced out" by the T_t:



→ 2D surface in $\mathbb{R}^{1+3} \cong \underset{\substack{\uparrow \\ \text{time}}}{\mathbb{R}} \times \underset{\substack{\uparrow \\ \text{space}}}{\mathbb{R}^3}$

→ slices $S|_t$ "=" $S \cap \{\boldsymbol{t} = t\}$ with respect to the time projection

$$\boldsymbol{t}(t,x) := t$$

→ $S(t) := \boldsymbol{p}_*(S|_t)$ pushforward w.r.t. the spatial projection $\boldsymbol{p}(t,x) := x$

Issues:

- Is this "equivalent" to the path?
- Define velocity?
- Define "variation"?
- Convergence of such surfaces?
- Satisfies a "transport equation"?

All these can be satisfactorily resolved in the Space-Time Framework

2. MEASURES & RECTIFIABILITY

Basic terminology:

- $\mathcal{B} = \mathcal{B}(\mathbb{R}^d)$ the Borel σ-algebra on $\mathbb{R}^d$
- Let V be a normed, finite-dim. vector space.

  A set function

  $$\nu : \mathcal{B}(\mathbb{R}^d) \longrightarrow V$$

  that is σ-additive, i.e.

  (i) $\nu(\emptyset) = 0$,

  (ii) $\nu\left(\bigcup_n E_n\right) = \sum_n \nu(E_n)$

  for any countable (or finite) family of disjoint Borel sets $E_n \in \mathcal{B}(\mathbb{R}^d)$

  is called a (vector) measure (on $\mathbb{R}^d$), written $\nu \in \mathcal{M}(\mathbb{R}^d; V)$.
- If only $\nu : \mathcal{B}(K) \longrightarrow V$ is a measure for all $K \subset \mathbb{R}^d$ compact, the $\nu \in \mathcal{M}_{loc}(\mathbb{R}^d; V)$, a Radon (vector) measure.
- If instead $\mu : \mathcal{B}(\mathbb{R}^d) \to [0, \infty]$ with (i), (ii) and such that $\mu(K) < \infty$ for $K \subset \mathbb{R}^d$ are called positive measures, $\mu \in \mathcal{M}^+(\mathbb{R}^d)$.

- The terms of null set ($\nu(N) = 0$), μ-completion $\mathcal{B}_\mu$ of $\mathcal{B}$, containing the μ-measurable sets ($A = B \cup N$, $B \in \mathcal{B}(\mathbb{R}^d)$, $\nu(N) = 0$) are used as usual.

- For $\nu \in \mathcal{M}(\mathbb{R}^d; V)$ and $f : \mathbb{R}^d \to \mathbb{R}$ ν-measurable, i.e.

$$f^{-1}(A) \in \mathcal{B}_\nu(\mathbb{R}^d) \quad \forall A \subset \mathbb{R} \text{ Borel},$$

the integral

$$\int f \, d\nu \in V$$

is defined as usual, first for step functions $f = \mathbb{1}_A$, $A \in \mathcal{B}_\nu(\mathbb{R}^d)$, then by approximation for general f.

- For $\nu \in \mathcal{M}(\mathbb{R}^d; V)$ def. the total variation measure $\|\nu\| \in \mathcal{M}^+(\mathbb{R}^d)$ for $A \in \mathcal{B}_\nu(\mathbb{R}^d)$ as

$$\|\nu\|(A) := \sup\left\{ \sum_{n \in \mathbb{N}} \|\nu(A_n)\| : A = \bigcup_{n \in \mathbb{N}} A_n : A_n \in \mathcal{B}_\nu(\mathbb{R}^d), \; A_m \cap A_n = \emptyset \text{ for } m \neq n \right\}.$$

Then, $\|\nu\| \in \mathcal{M}^+(\mathbb{R}^d)$.

Thm [1.13] (Besicovitch differentiation)

Let $\mu \in \mathcal{M}^+(\mathbb{R}^d)$, $\nu \in \mathcal{M}(\mathbb{R}^d;V)$. For μ-a.e. x, the "blow-up quotient" limit

open r-ball around x

$$\frac{d\nu}{d\mu}(x) := \lim_{r\downarrow 0} \frac{\nu(B_r(x))}{\mu(B_r(x))}$$

exists in V and the Radon-Nikodym decomposition

$$\nu = \frac{d\nu}{d\mu}\mu + \nu^s , \quad \nu^s := \nu \llcorner E$$

singular part

holds, where

$$E := (\mathbb{R}^d \setminus \operatorname{supp}\mu) \cup \left\{ x \in \operatorname{supp}\mu : \lim_{r\downarrow 0} \frac{\|\nu\|(B_r(x))}{\mu(B_r(x))} = \infty \right\}.$$

Thm [1.14] (Radon-Nikodym)

If additionally $\nu \ll \mu$, that is

$$\mu(A) = 0 \text{ for } A \in \mathcal{B}(\mathbb{R}^d) \Rightarrow \|\nu\|(A) = 0,$$

then $\nu = \frac{d\nu}{d\mu}\mu$ $\left(\text{i.e., } \nu(A) = \int_A \frac{d\nu}{d\mu}\, d\mu\right)$.

Lem [1.15] (Polar)

Let $\nu \in \mathcal{M}(\mathbb{R}^d;V)$. Then the polar

$$\frac{d\nu}{d\|\nu\|} \in L^1(\|\nu\|;V) = \left\{ f : \mathbb{R}^d \to V \ \|\nu\|\text{-measurable} : \int |f| \, d\|\nu\| < \infty \right\}$$

satisfies $\left\| \frac{d\nu}{d\|\nu\|} \right\| = 1$ $\|\nu\|$-a.e.

Weak* convergence

We say that $(\mu_j)_j \subset \mathcal{M}(\mathbb{R}^d; V)$ converges weakly* to $\mu \in \mathcal{M}(\mathbb{R}^d; V)$, written $\mu_j \overset{*}{\rightharpoonup} \mu$ if

$$\langle \psi, \nu_j \rangle := \int \left\langle \frac{d\nu_j}{d\|\nu_j\|}, \psi \right\rangle d\|\nu_j\| \longrightarrow \langle \psi, \nu \rangle$$

for all $\psi \in C_0(\mathbb{R}^d; V^*) = \overline{C_c(\mathbb{R}^d; V^*)}^{\|\cdot\|_\infty}$.

Here:

Thm [1.4] (Riesz)
Let $\Omega \subset \mathbb{R}^d$ be open. Then,

$$\left(C_0(\Omega; V^*)\right)^* = \mathcal{M}(\Omega; V) \quad .$$

Thm [1.19] (Banach-Alaoglu)
Let $(\nu_j)_j \subset \mathcal{M}(\mathbb{R}^d; V)$ with $\sup_j \|\nu_j\|(\mathbb{R}^d) < \infty$. Then there is a weakly* converging subsequence.

Hausdorff measure

Let ω_k be the volume of the unit ball in $\mathbb{R}^k$.

Set for $A \subset \mathbb{R}^d$:

Italian convention!

$$\mathcal{H}^k_\delta(A) := \frac{\omega_k}{2^k} \cdot \inf \left\{ \sum_{n\in\mathbb{N}} \operatorname{diam}(A_n)^k : A \subset \bigcup_n A_n, \ \operatorname{diam} A_n < \delta \right\}$$

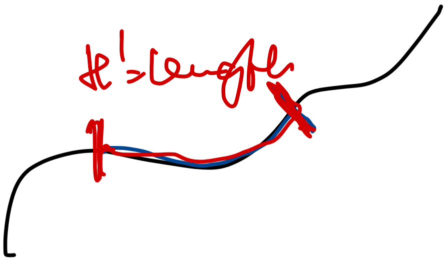
$$\mathcal{H}^k(A) := \lim_{\delta \downarrow 0} \mathcal{H}^k_\delta(A) \in [0,\infty].$$

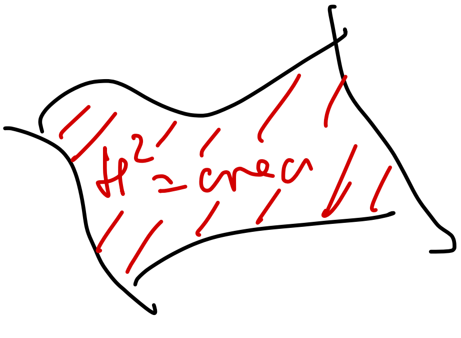
Rem: One could equivalently require the A_n to be open, closed or convex, but restricting their shape (e.g. to balls) yields different measures.

Intuition:

- $\mathcal{H}^1$ = curve length
- $\mathcal{H}^2$ = surface area
- $\mathcal{H}^3$ = volume

etc.





but without parametrization!

Prop [1.27] : Restricting $\mathcal{H}^k$ to the $\mathcal{H}^k$-measurable sets (recall: A is $\mathcal{H}^k$-measurable if $A = B \cup N$ with B Borel and $\mathcal{H}^k(N) = 0$) yields a positive measure on $\mathbb{R}^d$.

Lem [1.27] : Let $0 < k < k' < \infty$. Then

(i) $\mathcal{H}^{k'}(A) > 0 \Rightarrow \mathcal{H}^k(A) = \infty$.

(ii) $\mathcal{H}^k(A) < \infty \Rightarrow \mathcal{H}^{k'}(A) = 0$.

Lem [1.30] : Let $f: \mathbb{R}^d \to \mathbb{R}^m$ be Lipschitz with Lipschitz-constant L. Then, for $A \subset \mathbb{R}^d$,

$$\mathcal{H}^k(f(A)) \leq L^k \mathcal{H}^k(A).$$

Prop [1.31] : For any $A \in \mathcal{B}(\mathbb{R}^d)$,

$$\mathcal{L}^d(A) = \mathcal{H}^d(A).$$

Rectifiable sets

L3

A set $A \subset \mathbb{R}^d$ is called countably $\mathcal{H}^k$-rectifiable, $k \in \{0, \dots, d\}$ if

$$A \subset N \cup \bigcup_n h_n(\mathbb{R}^k)$$

where $\mathcal{H}^k(N) = 0$ and $h_n : \mathbb{R}^k \to \mathbb{R}^d$ is Lipschitz. If additionally $\mathcal{H}^k(A) < \infty$, then we call R $\mathcal{H}^k$-rectifiable.

Intuition: A "measure-theoretic" Lipschitz-regular k-manifold.

Ex: Let Γ be a Lipschitz (or C^1) k-graph, i.e. there is a projection π onto a k-plane π and $\varphi : \pi \to \pi^\perp$ Lip/C^1 s.t.

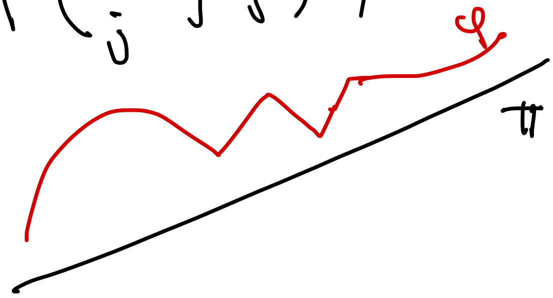
$$\Gamma = \{x \in \mathbb{R}^d : \varphi(\pi x) = \pi^\perp x\}.$$

Then Γ is k-rectifiable.

Pf: Let $v_1, \dots, v_k$ an OND of π. Then set

$$h(z_1, \dots, z_k) := \sum_j z_j v_j + \varphi\Big(\sum_j z_j v_j\Big),$$

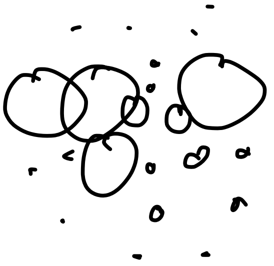
for which $\Gamma = h(\mathbb{R}^k)$.



Ex: Let $\{q_n\}_n$ be a countable dense set in $B_1 \subset \mathbb{R}^2$. Then,

$$S_\infty := \bigcup_n S_n \quad , \quad S_n := \partial B_{2^{-n}}(q_n)$$

is countably $\mathcal{H}^1$-rectifiable.



$\rightarrow$ rectifiable sets can be quite irregular!

Prop [2.18]: A $\mathcal{H}^k$-measurable set $R \subset \mathbb{R}^d$ is $\mathcal{H}^k$-rectifiable if and only if there exists a countable collection $\{\Gamma_n\}_n$ of Lipschitz (or C^1) graphs s.t.

$$\mathcal{H}^k\Big(R \setminus \bigcup_n \Gamma_n\Big) = 0.$$

Rem: One can even choose the Γ_n to be embedded C^1-manifolds.

Let $A \subset \mathbb{R}^d$ be $\mathcal{H}^k$-measurable. A k-dimensional subspace $\Pi \subset \mathbb{R}^d$ is called the <u>approximate tangent space</u> $T_{x_0}A$ at $x_0 \in A$ if

$$\lim_{r \downarrow 0} \int_{r^{-1}(A-x_0)} \varphi(y)\, d\mathcal{H}^k(y) = \int_{\Pi} \varphi(y)\, d\mathcal{H}^k(y)$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d)$.

<u>Prop</u> [2.20]: Let R be $\mathcal{H}^k$-rectifiable. Then, $T_{x_0}R$ exists for $\mathcal{H}^k$-a.e. $x_0 \in R$.

<u>Rem</u>: In fact, also the converse holds.

<u>Pf</u>: Assume that $R = h(K)$ for $K \in \mathbb{R}^k$ compact, h injective and Lipschitz. The general case follows by exhaustion.

Let $x_0 := h(z_0)$, $z_0 \in K$ and assume that $\Pi = \nabla h(z_0)[\mathbb{R}^k]$ is a k-dimensional subspace, which holds for $\mathcal{L}^k$-a.e. $z_0 \in K$ by the injectivity and Rademacher's theorem.

Set

$$h_r(y) := \frac{h(x_0+ry) - h(x_0)}{r},$$

$$K_r := r^{-1}(K - x_0)$$

and compute as follows:

$$\lim_{r\downarrow 0} \int_{r^{-1}(R-x_0)} \varphi(x)\, d\mathcal{H}^k(x) \underset{\Big\updownarrow}{=} \lim_{r\downarrow 0} \int_{K_r} \varphi(h_r(y))\, J_k h_r(y)\, dy$$

(a.e. x_0)

Thm [2.2] (area formula): Let $f:\mathbb{R}^k\to\mathbb{R}^m$ be Lipschitz, $k\le m$. Then, for any $\mathcal{L}^k$-measurable $g:\mathbb{R}^k\to\mathbb{R}$,

$$\int_{\mathbb{R}^k} g(x)\, J_k f(x)\, dx = \int_{\mathbb{R}^m} \sum_{x\in f^{-1}(y)} g(x)\, d\mathcal{H}^k(y),$$

where $J_k f := \sqrt{\det(\nabla f^T \nabla f)}$ "k-Jacobian".

In particular,

$$\mathcal{H}^k(f(A)) = \int J_k f\, dx.$$

__Mnemonic:__ $z = f(y) \Rightarrow d\mathcal{H}^k(z) = J_k f\, dy$

$$\ldots \underset{\uparrow}{=} \lim_{r\downarrow 0} r^{-k} \int_K \varphi\left(\frac{h(z)-h(z_0)}{r}\right) \cdot J_k h(z)\, dz$$

$z = x_0 + ry$
$dz = r^k dy$

$$\underset{\uparrow}{=} J_k h(z_0) \cdot \lim_{r\downarrow 0} \int_{K_r} \varphi\left(\frac{h(x_0+ry)-h(x_0)}{r}\right) dy$$

Lebesgue diff. thm

$$= \int_{\mathbb{R}^k} \varphi(\nabla h(x_0) y)\, J_k h(z_0)\, dy \overset{\text{area}}{=} \int_{\Pi} \varphi\, d\mathcal{H}^k. \qquad \square$$

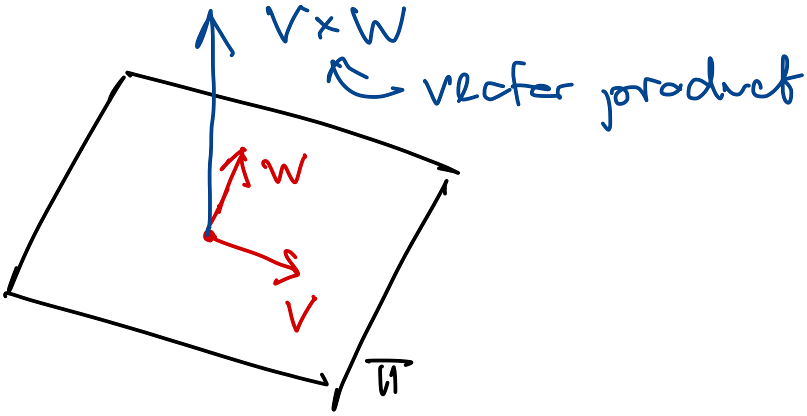
3. DIFFERENTIAL FORMS & CURRENTS

L4

Multi-linear algebra

Problem: Need an efficient way to specify a k-dimensional (tangent) space π in $\mathbb{R}^d$, together with an orientation.

An attempt: Use a normal: in $\mathbb{R}^3$



BUT:
- This only works in 2 or 3 ambient dimensions! (a 2-plane in $\mathbb{R}^4$ has $\underline{\underline{2}}$ normals)
- Even in 3D: What is the plane normal we get as the image $L\pi$ under a linear map L?

$$(Lv)\times(Lw) = (\det L)\, L^{-T}(v\times w)$$

BETTER: Keep information of v, w "modulo natural identifications" ⇝ "$\wedge$"

Prop [3.3]: For any finite-dimensional vector space V and $k = 0, \dots, \dim V$, there is a vector space

$$\Lambda_k V \quad (\neq \{0\})$$

together with the wedge product

$$\wedge : \underbrace{V \times \dots \times V}_{k \text{ times}} \longrightarrow \Lambda_k V$$

that is linear in every entry and s.t. for $v_1, \dots, v_k \in V$, $v_1' \in V$, $\lambda \in \mathbb{R}$ the following Gaussian rules hold: (like rows in determinant)

(a) $v_1 \wedge \dots \wedge (\lambda v_i) \wedge \dots \wedge v_j \wedge \dots \wedge v_k$

$$= v_1 \wedge \dots \wedge v_i \wedge \dots \wedge (\lambda v_j) \wedge \dots \wedge v_k$$

(b) $v_1 \wedge \dots \wedge v_i \wedge \dots \wedge v_j \wedge \dots \wedge v_k$

$$= v_1 \wedge \dots \wedge (v_i + \lambda v_j) \wedge \dots \wedge v_j \wedge \dots \wedge v_k$$

(c) $v_1 \wedge \dots \wedge v_i \wedge \dots \wedge v_j \wedge \dots \wedge v_k$

$$= v_1 \wedge \dots \wedge (-v_j) \wedge \dots \wedge v_i \wedge \dots \wedge v_k$$

(d) $\lambda (v_1 \wedge \dots \wedge v_k) = (\lambda v_1) \wedge \dots \wedge v_k$

(e) $v_1 \wedge \dots \wedge v_k + v_1' \wedge v_2 \wedge \dots \wedge v_k$

$$= (v_1 + v_1') \wedge v_2 \wedge \dots \wedge v_k$$

Pf: doesn't matter!

Lem [3.4]: Let $\xi \in \Lambda_k V$, $\eta, \zeta \in \Lambda_\ell V$, $\lambda \in \mathbb{R}$. Then:

(i) **Linearity**: $\xi \wedge (\eta + \zeta) = \xi \wedge \eta + \xi \wedge \zeta$

$$(\lambda \xi) \wedge \eta = \lambda (\xi \wedge \eta)$$

(ii) **Skew-commutativity**: $\xi \wedge \eta = (-1)^{k\ell} \eta \wedge \xi$

(in particular, $\xi \wedge \xi = 0$ if k is odd)

(iii) **Nilpotence**: $\xi \wedge \xi = 0$ if ξ is *simple*, i.e.

$\xi = v_1 \wedge \dots \wedge v_k$ for some $v_1, \dots, v_k \in V$.

(in particular, $v \wedge v = 0$ for all $v \in V$).

Ex: In $\mathbb{R}^4$ the 2-vector $e_1 \wedge e_2 + e_3 \wedge e_4$ is not simple:

$$(e_1 \wedge e_2 + e_3 \wedge e_4) \wedge (e_1 \wedge e_2 + e_3 \wedge e_4)$$

$$\overset{(i)}{=} e_1 \wedge e_2 \wedge e_1 \wedge e_2 + e_1 \wedge e_2 \wedge e_3 \wedge e_4$$

$$+ e_3 \wedge e_4 \wedge e_1 \wedge e_2 + e_3 \wedge e_4 \wedge e_3 \wedge e_4$$

$$\overset{(b),(d)}{=} 0 + 2 e_1 \wedge e_2 \wedge e_3 \wedge e_4 + 0 \neq 0,$$

so by (iii) this 4-vector cannot be simple.

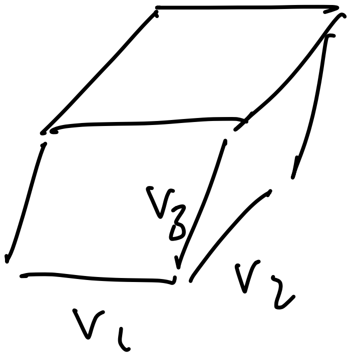
Intuition: The simple k-vector $v_1 \wedge \cdots \wedge v_k$ represents the (ordered) vector space

$$\mathrm{span}(v_1 \wedge \cdots \wedge v_k) := \mathrm{span}\{v_1, \ldots, v_k\}$$

Lem [3.5]: For $\xi, \eta \in \Lambda_k V$ simple

$$\mathrm{span}\,\xi = \mathrm{span}\,\eta \iff \eta = \lambda \xi .$$

Lem [3.6]: For $\xi = v_1 \wedge \cdots \wedge v_k \in \Lambda_k \mathbb{R}^d$, let $\xi = \lambda \xi_1$ with a unit ξ_1, that is, $\xi_1 = w_1 \wedge \cdots \wedge w_k$ with $\{w_i\}_i$ an orthonormal basis of span ξ_1, and $\lambda > 0$. Then λ is the volume of the parallelotope spanned by the $\{v_j\}_j$.



Pf: Reduce to cube by Gaussian rules (which leave the volume invariant). $\square$

Linear transformations: If $L: V \to W$ is linear, then

$$L(v_1 \wedge \cdots \wedge v_k) := (Lv_1) \wedge \cdots \wedge (Lv_k)$$

and extended by multilinearity to $\Lambda_k V$ (this is unique!).

Scalar product and norm: For $\xi, \eta \in \Lambda_k V$ with $\xi = v_1 \wedge \cdots \wedge v_k$, $\eta = w_1 \wedge \cdots \wedge w_k$:

$$(\xi, \eta) := \det [v_i \cdot w_j]^i_j$$

and extend by multilinearity. Also,

$$|\xi| := (\xi, \xi)^{1/2}.$$

Covectors: Elements of

here

$$\Lambda^k V := \Lambda_k V^*$$

are called *k-covectors*. We have

$$\Lambda^k V = (\Lambda_k V)^*$$

via

$$\langle u_1 \wedge \cdots \wedge u_k, \alpha^1 \wedge \cdots \wedge \alpha^k \rangle := \det [\langle u_i, \alpha^j \rangle]^i_j$$

and extended by multilinearity.

L5

Differential forms: Let $\Omega \subset \mathbb{R}^d$ be open.

An element of

$$\mathcal{D}^k(\Omega) := C_c^\infty(\Omega; \Lambda^k \mathbb{R}^d)$$

is called a (smooth, compactly supported) **differential k-form**.

The **exterior differential** $d: \mathcal{D}^k(\Omega) \to \mathcal{D}^{k+1}(\Omega)$ is given as:

(i) For $f \in \mathcal{D}^0(\Omega)$:

$$df := \sum \frac{\partial f}{\partial x_i} \underbrace{dx^i}_{(x_1, \dots, x_d) \mapsto x_i}$$

(ii) For $\omega = f \, dx^{i_1} \wedge \dots \wedge dx^{i_k} \in \mathcal{D}^k(\Omega)$,

$$d\omega := df \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

(iii) For general $\omega \in \mathcal{D}^k(\Omega)$ extend (ii) by linearity.

Lem $\boxed{3.10}$ (Leibniz rule): For $\omega \in \mathcal{D}^k(\Omega)$, $\eta \in \mathcal{D}^\ell(\Omega)$:

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$

Pf: By classical Leibniz rule. $\square$

Prop [3.11]: For $\omega \in \mathcal{D}^k(\Omega)$, $d\omega = d(d\omega) = 0$.

Pf: Assume w.l.o.g. $\omega = f\, dx^{i_1} \wedge \dots \wedge dx^{i_k}$, where $f \in C_c^\infty(\Omega)$. Then,

$$d^2\omega = \sum_{i,j} \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) dx^j \wedge dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$$= \sum_{i<j} \left[\frac{\partial}{\partial x^i}\frac{\partial f}{\partial x^j} - \frac{\partial}{\partial x^j}\frac{\partial f}{\partial x^i}\right] dx^i \wedge dx^j \wedge \dots$$

$$= 0. \qquad \square$$

"every exact form is closed"

$\downarrow$ $\omega = d\psi$ for some $\psi \in \mathcal{D}^{k-1}(\Omega)$

$\downarrow$ $d\omega = 0$.

Integration of differential forms

A k-form $\omega \in \mathcal{D}^k(\mathbb{R}^d)$ can be integrated over a k-box $R = [a_1, b_1] \times \dots \times [a_d, b_d]$ with $a_i \leq b_i$ for $i = 1, \dots, d$ s.t. $a_i < b_i$ for precisely k indices $i_1 < \dots < i_k$. Then

$$\int_R \omega := \int \langle e_{i_1} \wedge \dots \wedge e_{i_k}, \omega(x) \rangle \, dx$$

$\uparrow$ canonical orientation of R

This can be extended to "chains", i.e. formal sums of boxes, e.g. for $\lambda R_1 \oplus R_2$:

$$\int_{\lambda R_1 \oplus R_2} \omega := \lambda \int_{R_1} \omega + \int_{R_2} \omega$$

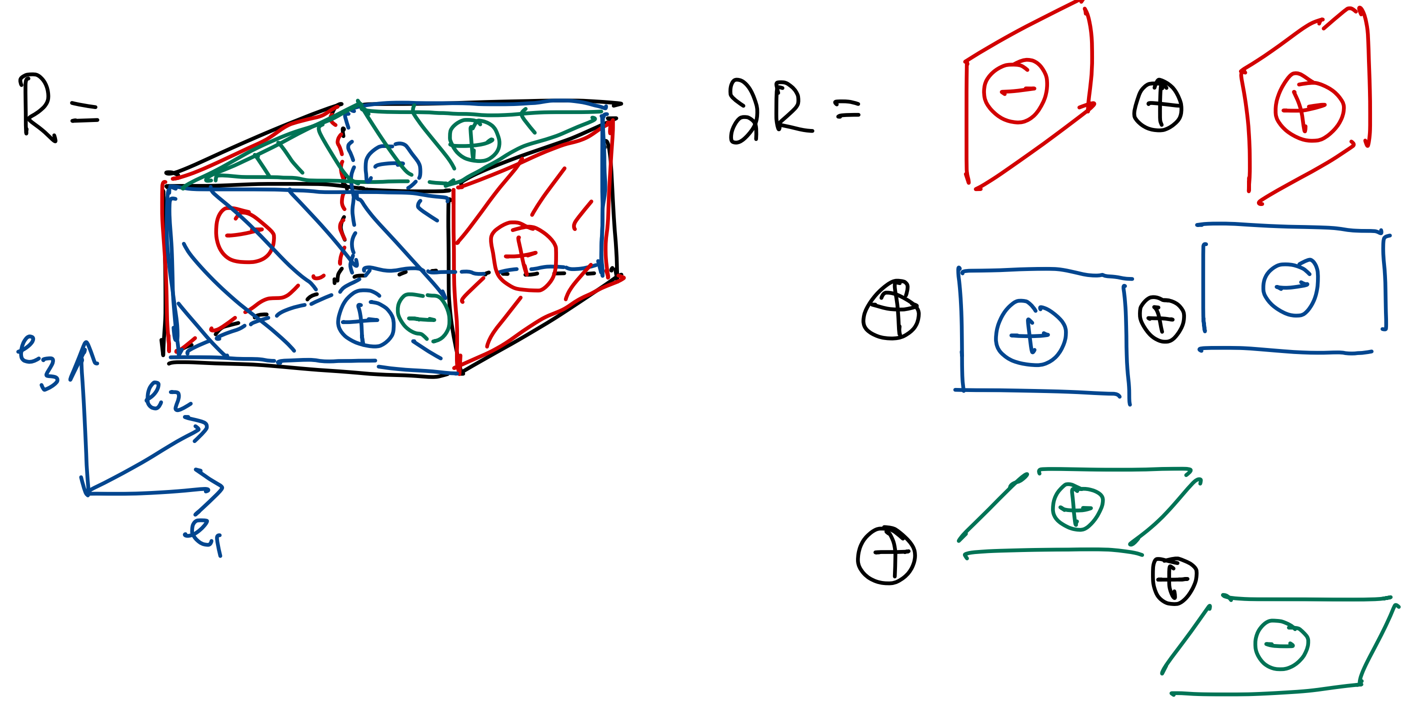
This is independent of the (oriented) parametrization!

Stokes Theorem:

There exists a $(k-1)$-chain, the boundary ∂R, for every k-box (which is a particular orientation of the topological boundary of R), such that

$$\int_{\partial R} \omega = \int_R d\omega$$

(this can also be extended to k-manifolds)



Currents

An integer-multiplicity rectifiable (IMR) k-current is a linear and continuous functional T on $\mathcal{D}^k(\mathbb{R}^d)$ [a current, $T \in \mathcal{D}_k(\mathbb{R}^d) := \mathcal{D}^k(\mathbb{R}^d)^*$] given as

$$T = \vec{T} \; m \; \mathcal{H}^k \llcorner R$$

where

- R is a countably $\mathcal{H}^k$-rectifiable carrier set
- $\vec{T} : R \to \Lambda_k \mathbb{R}^d$ is a $\mathcal{H}^k$-measurable orienting map: $\vec{T}(x)$ simple, unit, and $\operatorname{span} \vec{T}(x) = T_x R$ for $\mathcal{H}^k$-a.e. x,
- $m \in L^1_{loc}(\mathcal{H}^k \llcorner R; \mathbb{N})$ is an integer multiplicity,

which acts on $\omega \in \mathcal{D}^k(\mathbb{R}^d)$ via

$$\langle T, \omega \rangle := \int_R \langle \vec{T}(x), \omega(x) \rangle \, m(x) \, d\mathcal{H}^k .$$

(think: T is the "integral" and ω the "integrand")

Ex: For a k-box R, let $[\![R]\!]$ be the associated IMR k-current, that is,

$$\langle [\![R]\!], \omega \rangle := \int_R \omega .$$

For the oppositely-oriented box, we

$$[\![\ominus R]\!] := -[\![R]\!], \text{ i.e. } \langle [\![\ominus R]\!], \omega \rangle = -\int_R \omega .$$

The boundary ∂T of an (IMR) k-current is the linear functional on $\mathcal{D}^{k-1}(\mathbb{R}^d)$ given by

$$\langle \partial T, \omega \rangle := \langle T, d\omega \rangle , \quad \omega \in \mathcal{D}^{k-1}(\mathbb{R}^d).$$

Ex: For the boundary of a k-box R, the boundary ∂R induces an IMR $(k-1)$-current $[\![\partial R]\!]$ and for $\omega \in \mathcal{D}^{k-1}(\mathbb{R}^d)$ it holds that

$$\langle \partial [\![R]\!], \omega \rangle \overset{\text{def.}}{\underset{\text{of } \partial}{=}} \langle [\![R]\!], d\omega \rangle$$

$$= \int_R d\omega$$

$$\overset{\text{Stokes}}{\underset{\text{Thm}}{=}} \int_{\partial R} \omega \quad \leftarrow \text{a } (k-1)\text{-chain}$$

$$= \langle [\![\partial R]\!], \omega \rangle$$

So the definition of "∂" is natural.

The mass of an IMR k-current T is

$$\mathbb{M}(T) := \int_R m(x) \, d\mathcal{H}^k(x).$$

If T is an IMR k-current, ∂T is an IMR $(k-1)$-current and

$$\mathbb{M}(T) + \mathbb{M}(\partial T) < \infty,$$

then T is called an <u>integral k-current</u>, all of which are collected in the vector space $\mathbf{I}_k(\mathbb{R}^d)$.

<u>Why currents?</u>

- Construct a "measure-theoretic" form of geometry and use the large measure theory toolbox.
- "Calculate" with geometric structures.
- Do analysis with geometric structures....

<u>Thm</u> [5,6] (Federer-Fleming): Let $(T_j)_j \subset \mathbf{I}_k(\mathbb{R}^d)$ with

$$\sup_j \left(\mathbb{M}(T_j) + \mathbb{M}(\partial T_j) \right) < \infty.$$

Then, up to selecting a subsequence,

$$T_j \xrightarrow{*} T \in \mathbf{I}_k(\mathbb{R}^d) \text{ with}$$

$$\mathbb{M}(T) \leq \liminf_{j\to\infty} \mathbb{M}(T_j) \text{ and also } \partial T_j \xrightarrow{*} \partial T.$$

<u>Pf</u>: All are quite involved: Banach-Alaoglu, but hard to show that T is integral. $\square$

4. SPACE-TIME FRAMEWORK

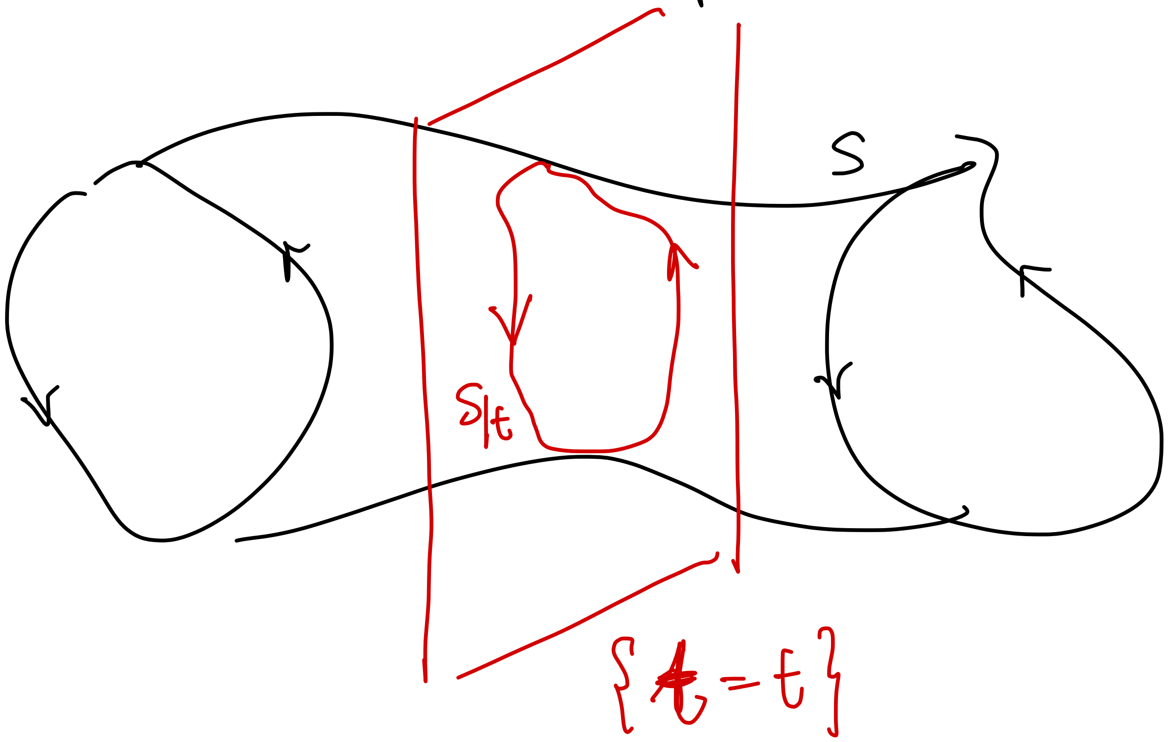
L6

A <u>space-time integral current</u>

$$S \in \mathbf{I}_{1+k}(\mathbb{R}^{1+d})$$

↖ space-time (Galilean)

describes the motion of its "slices".



Here, $\mathbf{t}(t,x) := t$, $\mathbf{p}(t,x) := x$.

To define the <u>slice</u> $S|_t$ (as an integral current!) at $t \in \mathbb{R}$ with

$$\mathbf{t}_\#(\|S\|+\|\partial S\|)\{t\} := (\|S\|+\|\partial S\|)(\mathbf{t}^{-1}(\{t\}))$$
$$= (\|S\|+\|\partial S\|)(\{t\}\times\mathbb{R}^d),$$

noting that the set of these t's is all of $\mathbb{R}$ without at most a null set, define $S|_t$ via the <u>cylinder formula</u>

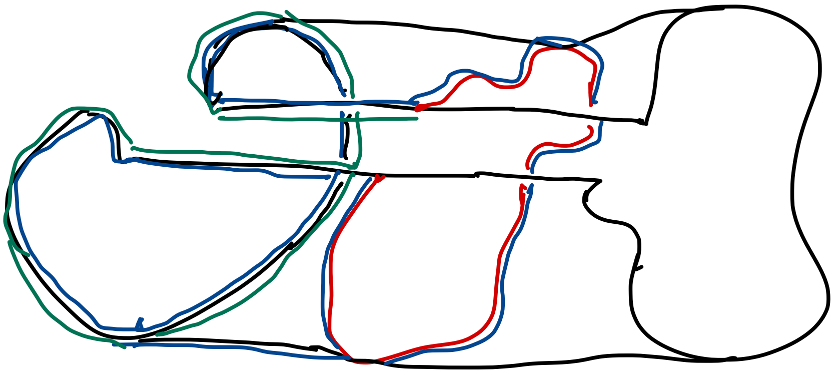
$$S|_t := \partial(S \llcorner \{\mathbf{t}<t\}) - (\partial S) \llcorner \{\mathbf{t}<t\},$$

meaning that for $\omega \in \mathcal{D}^k(\mathbb{R}^d)$,

$$\langle S|_t , \omega \rangle := \int_{\underbrace{\{\mathbf{t}<t\}}_{:= \{(s,x)\in\mathbb{R}^d \,:\, s<t\}}} \langle \vec{S}(\tau,x), d\omega(\tau,x) \rangle \, d\|S\|(\tau,x)$$

$$- \int_{\{\mathbf{t}<t\}} \langle \vec{\partial S}(\tau,x), \omega(\tau,x) \rangle \, d\|\partial S\|(\tau,x).$$

Intuition:



$$S|_t = \partial(S \llcorner \{\mathbf{t}<t\}) - (\partial S) \llcorner \{\mathbf{t}<t\}$$

Rem: There are various equivalent definitions for $S|_t$, but the above is the most geometric.

Let $S = \vec{S} \,m\, \mathcal{H}^{k+1} \llcorner R \in \mathcal{I}_{1+k}(\mathbb{R}^{1+d})$.

The <u>tangential differential</u> $D^R \mathbf{t} : T_{(t,x)} R \to \mathbb{R}$ at $(t,x) \in R$ is the restriction of the differential $D\mathbf{t} : \mathbb{R}^{1+d} \to \mathbb{R}$ $(D\mathbf{t}(t,x)[s,z] = s)$ to the approximate tangent space $T_{(t,x)} R$.
Then, the <u>tangential gradient</u> $\nabla^R \mathbf{t}$ is the $1 \times (1+d)$ matrix

$$\nabla^R \mathbf{t}(t,x) = \nabla \mathbf{t} \, P^T_{(t,x)} \qquad (\nabla \mathbf{t} = [1\, 0 \dots 0]),$$

where $P_{(t,x)}$ is the orthogonal projection onto $T_{(t,x)} R$ at (t,x). Set $\nabla^S \mathbf{t} := \nabla^R \mathbf{t}$.

<u>Note</u>: $\nabla^R \mathbf{t}(t,x) = 0$ if and only if $T_{(t,x)} R$ is "space-like", i.e. $T_{(t,x)} R$ is contained in $\{0\} \times \mathbb{R}^d$.

Define the <u>restriction</u> $\xi \llcorner \alpha \in \Lambda_{m-n} \mathbb{R}^N$ of $\xi \in \Lambda_m \mathbb{R}^N$ by $\alpha \in \Lambda^n \mathbb{R}^N$ as

$$\langle \xi \llcorner \alpha, \beta \rangle := \langle \xi, \alpha \wedge \beta \rangle, \quad \beta \in \Lambda^{m-n} \mathbb{R}^N.$$

Prop [6.5]:

For $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$:

(i) $R|_t := R \cap \{\mathbf{t} = t\}$ is countably $\mathcal{H}^k$-rectifiable,

(ii) For $\mathcal{H}^k$-a.e. $z \in R|_t$ with $\nabla^R \mathbf{t}(z) \neq 0$,

$$T_z R = T_z R|_t \oplus \operatorname{span}\{\xi(z)\})$$

where

$$\xi(z) := \frac{\nabla^R \mathbf{t}(z)}{|\nabla^R \mathbf{t}(z)|}$$

is a simple unit 1-vector.

(iii) Define

$$m_t(z) := \begin{cases} m(z) & \text{if } \nabla^R \mathbf{t}(z) \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\xi^* := \frac{D^R \mathbf{t}(z)}{|D^R \mathbf{t}(z)|} \quad (\text{wherever } D^R \mathbf{t}(z) \neq 0).$$

With

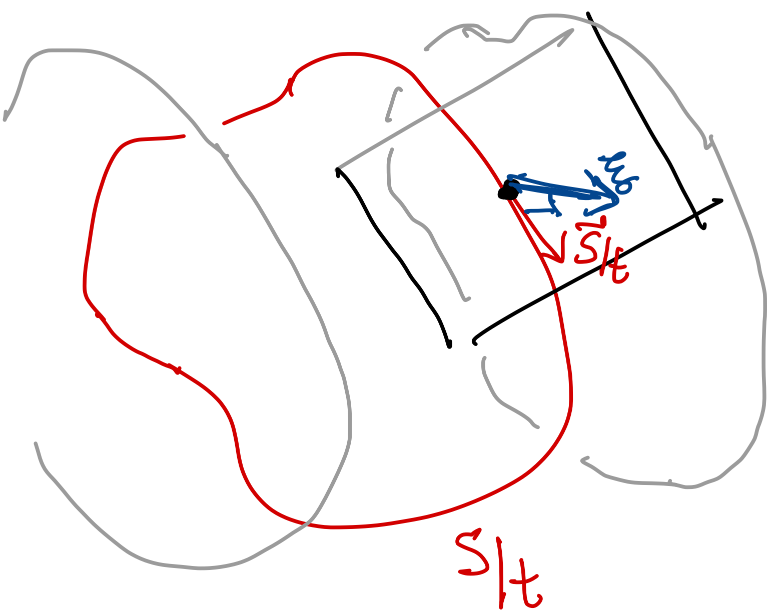
$$\vec{S}|_t(z) := \vec{S}(z) \llcorner \xi^*(z) \in \Lambda_k \operatorname{Tan}_z R|_t$$

it holds that

$$S|_t = \vec{S}|_t \, m_t \, \mathcal{H}^k \llcorner R|_t \in \mathcal{I}_k(\mathbb{R}^{1+d}).$$

Moreover, $\vec{S} = \xi \wedge \vec{S}|_t$.

Illustration:



$$\vec{S} = \xi \wedge \vec{S}|_t$$

$$\vec{S} \llcorner \xi^* = \vec{S}|_t$$

Note: In the space-time approach, ξ is given by the space-time current, but if we instead considered a path $t \mapsto S_t$, then we would not know ξ!

Prop [6-2] (Coarea formula for slices):
Let $S \in I_{1+k}(\mathbb{R}^{1+d})$ and $g : \mathbb{R} \to \mathbb{R}$ Borel. Then,

$$\int g(t)\, M(S|_t)\, dt = \int (g \circ t)\, |\vec{S} \llcorner dt|\, d\|S\|.$$

Regular and critical points

L7

Let $S \in I_{1+k}(\mathbb{R}^{1+d})$. Define the sets

$$\mathrm{Reg}(S) := \{ (t,x) \in \mathbb{R}^{1+d} : \overset{S}{\nabla} t(t,x) \neq 0 \}$$

$$\mathrm{Crit}(S) := \{ (t,x) \in \mathbb{R}^{1+d} : \nabla^S t(t,x) = 0 \}$$

of regular and critical points, respectively.

Lem [6.9]:

(i) $\|S_t\|(\mathrm{Crit}(S)) = 0$ for a.e. $t \in \mathbb{R}$.

(ii) $\nabla^S t \neq 0$ a.e. w.r.t. $dt \otimes S|_t$, where

$$(dt \otimes S|_t)(A) := \int S|_t(A)\, dt \quad , A \subset \mathbb{R}^{1+d} \text{ Borel.}$$

(iii) For a.e. $t \in \mathbb{R}$, $A \subset \mathbb{R}^{1+d}$ Borel with $\|S\|(A) < \infty$,

$$\int_A \frac{1}{|\nabla^S t|}\, d\|S|_t\| < \infty .$$

Pf: (i) By the coarea formula for slices with $g := \mathbb{1}_{\mathrm{Crit}(S)}$,

$$\int \|S|_t\|(\mathrm{Crit}(S))\, dt = \iint_{\{t\}\times\mathbb{R}^d} \mathbb{1}_{\mathrm{Crit}\,S}\, d\|S|_t\|\, dt = \int_{\mathrm{Crit}(S)} |\overset{S}{\nabla} t|\, d\|S\| = 0.$$

(ii): Follows from (i).

(iii): By the coarea formula for $g := |\nabla^S \mathbf{t}|^{-1} \mathbb{1}_{A \setminus \mathrm{Crit}(S)}$:

$$\int\int_A \frac{1}{|\nabla^S \mathbf{t}|} \, d\|S|_t\| \, dt = \int_{A \cap \mathrm{Reg}(S)} d\|S\| < \infty .$$

$\square$

Define (as before)

$$\xi(t,x) := \frac{\nabla^S \mathbf{t}(t,x)}{|\nabla^S \mathbf{t}(t,x)|} , \quad (t,x) \in \mathrm{Reg}(S).$$

By (ii), this is well-defined $(dt \otimes S|_t)$-a.e.
Moreover, $\mathbf{t}\xi \geq 0$.

Geometric derivative

Fix $(t,x) \in \mathrm{Reg}(S)$. Then,

$$|\nabla^S \mathbf{t}| = |(e_0 \cdot \xi)\xi| = |e_0 \cdot \xi| = |\mathbf{t}\xi| = \mathbf{t}\xi$$

(proj. of e_0 onto ξ)

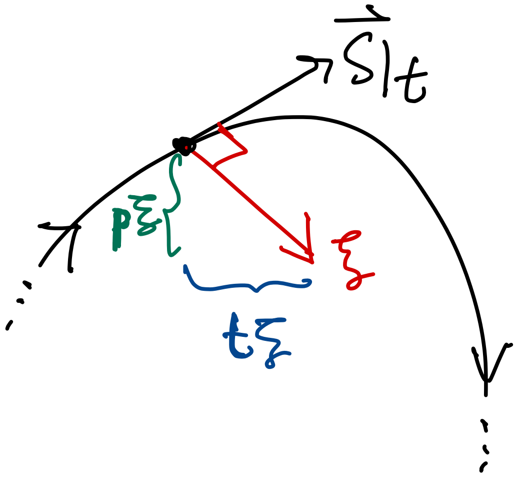
Also, since $S|_t = \vec{S} \llcorner \xi^* = |D^S \mathbf{t}|^{-1} \vec{S} \llcorner dt$,

$$\ldots = |D^S \mathbf{t}| = |\vec{S} \llcorner dt| .$$

Then define the <u>geometric derivative</u> as

$$\frac{D}{Dt} S(t,x) := \frac{\mathbf{p}\xi(t,x)}{\mathbf{t}\xi(t,x)}, \quad (t,x) \in \mathrm{Reg}(S).$$

Intuition:



$$\frac{D}{Dt} S = \frac{\mathbf{p}\xi}{\mathbf{t}\xi} \quad \left(\frac{\text{displacement}}{\text{time}}\right)$$

$\rightsquigarrow$ $\frac{D}{Dt}$ is the (normal) velocity of travel!

Ex: This corresponds to the dislocation velocity.

Prop [6.11]: $\frac{D}{Dt} S \in L^1(dt \otimes \|S|_t\|; \mathbb{R}^d)$ and

$$\left|\frac{D}{Dt} S\right| \leq \frac{1}{|\nabla^S t|} \quad (dt \otimes \|S|_t\|)\text{-a.e.}$$

Pf: Well-definedness follows from (iii) in Lemma. The estimate follows from the definition and by the coarea formula,

$$\int\int \left|\frac{D}{Dt} S\right| \, d\|S|_t\| \, dt \leq \int\int \frac{1}{|\nabla^S t|} \, d\|S|_t\| \, dt$$

$$= \mathbf{M}(S) < \infty. \qquad \square$$

Space-time variation

For $S \in \mathcal{I}_{1+k}([0,T]\times\mathbb{R}^d)$ and $I \subset [0,T]$ $\mathcal{L}^1$-measurable, define the (space-time) variation of S in I as (recall $\mathbf{p}(t,x) := x$)

norm

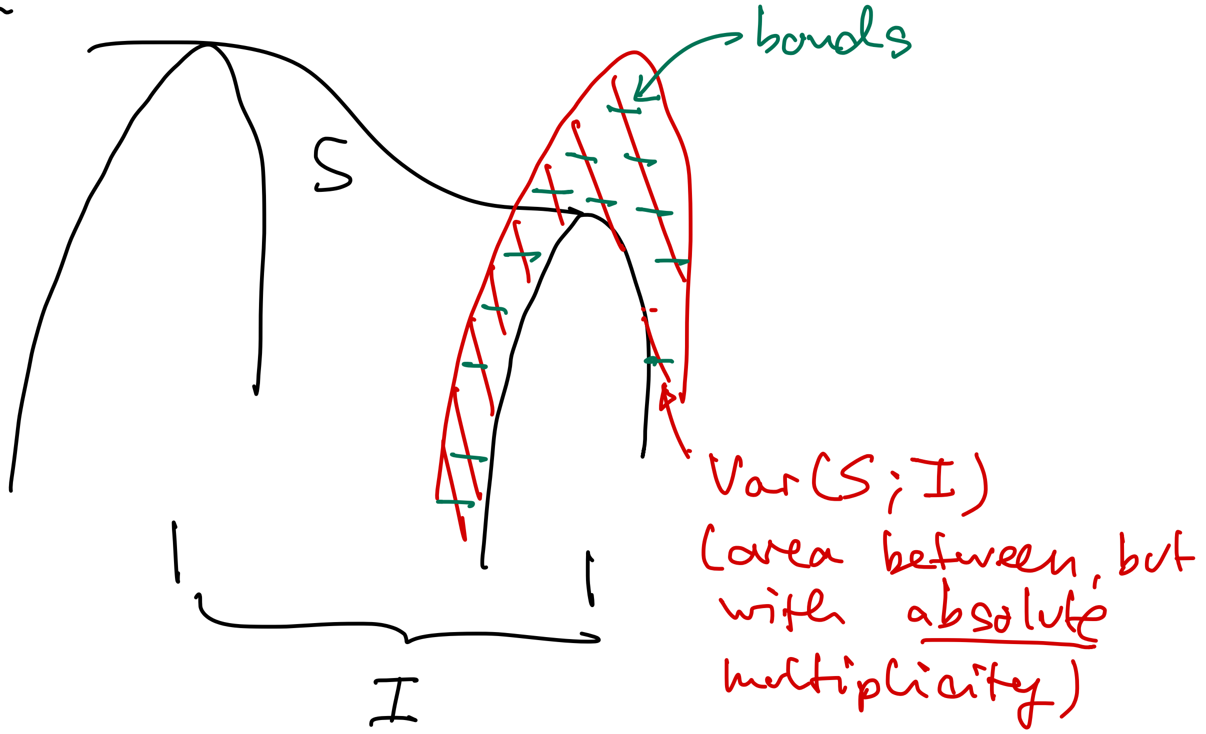
$$\mathrm{Var}(S;I) := \int_{I\times\mathbb{R}^d} |\mathbf{p}\vec{S}| \, d\|S\|$$

$\mathbf{p}$ extended to $\Lambda_{1+k}\mathbb{R}^{1+d}$

and the (space-time) L^∞-norm of S on I as

$$\|S\|_{L^\infty(I)} := \sup_{t\in I} \mathbf{M}(S|t) .$$

Intuition:



$\mathrm{Var}(S;I)$ (area between, but with absolute multiplicity)

Ex: In dislocation theory, this is (an approximation to) the dissipation ($\approx$ number of broken bonds).

Ex (BV-functions): Let $u \in BV(0,1)$, so that the distributional derivative $\dot{u}$ of u is a measure, i.e., there is $\mu \in \mathcal{M}((0,1);\mathbb{R})$ s.t.

$$-\int u\dot{\varphi}\,dt = \int \varphi\,d\mu \quad , \varphi \in C_c^\infty(0,1)$$

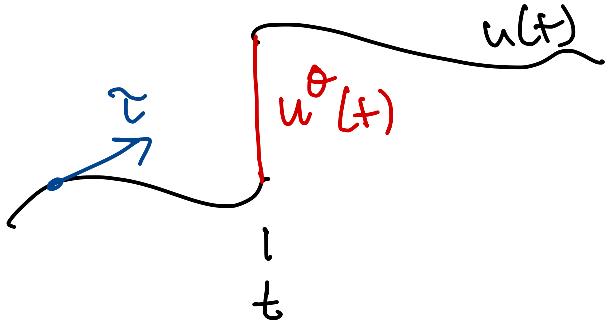
Then $\dot{u} := \mu$

(e.g. $u \in C^1(0,1)$ with $\int |\dot{u}|\,dt < \infty$).

Let

$$\text{graph}(u) := \{(t, u^\theta(t)) : t \in (0,1), \theta \in [0,1]\}$$

where $\theta \mapsto u^\theta(t)$ is the jump interpolant (constant in θ if t is not a jump point).



Set

$$S_u := \tau\,\mathcal{H}^1 \llcorner \text{graph}(u) \quad , \tau \text{ the tangent.}$$

For a.e. t, $\dot{u}(t)$ exists ($\dot{u} = \dot{u}(t)\,dt + \text{sing.}$) and

$$\tau = \frac{1}{\sqrt{1+|\dot{u}|^2}} \begin{bmatrix} 1 \\ \dot{u} \end{bmatrix}$$

Then:

1) $\frac{D}{Dt} S_u = \dot{u}$ at a.e. t

Pf: The slices are 0-dimensional and $\xi = \tau$. Thus, where τ is given as above,

$$\frac{D}{Dt} S_u = \frac{\mathbf{p}\tau}{\mathbf{t}\tau} = \dot{u}. \qquad \square$$

2) $\mathrm{Var}(u;I) = \mathrm{Var}(S_u;I)$, $I \subset [0,1]$ interval

Pf: By approximation, w.l.o.g. $u \in C^1$. Then, for I an interval (say),

$$\mathrm{Var}(u;I) \overset{\text{well}}{\underset{\text{known}}{=}} \int_0^1 |\dot{u}|\, dt$$

$$\overset{\text{area}}{\underset{\text{formula}}{=}} \int_{\mathrm{graph}(u) \cap (I \times \mathbb{R})} \frac{|\dot{u}|}{\sqrt{1+|\dot{u}|^2}}\, d\mathcal{H}^1$$

$$= \int_{\mathrm{graph}(u) \cap (I \times \mathbb{R})} |\mathbf{p}\tau|\, d\mathcal{H}^1$$

$$= \mathrm{Var}(S_u;I). \qquad \square$$

So: The space-time notions generalize the classical ones.

Estimates & convergence

L8

Clearly,

$$\mathrm{Var}(S;I) \le \mathbb{M}(S \llcorner I\times\mathbb{R}^d)$$

Also a reverse estimate is true:

Lem [6.13] ("Pythagoras"):

$$|t\xi|^2 + |\mathbf{p}\vec{S}|^2 = 1 \qquad \|S\|\text{-a.e.}$$

and for $I\subset\mathbb{R}$,

$$\|S\|(I\times\mathbb{R}^d) \le \mathcal{L}^1(I)\cdot\|S\|_{L^\infty} + \mathrm{Var}(S;I)$$

Pf: Every $\eta\in\Lambda_{1+k}\mathbb{R}^{1+d}$ can be written as

with e_0 without e_0

$$\eta = \sum_{J\in\Lambda(k,d)}{}' \eta_{0,J}\, e_0\wedge e_J + \sum_{J\in\Lambda(1+k,d)}{}' \eta_J\, e_J$$

where $\Lambda(k,d)$ are the ordered k-tuples in $\{1,\dots,d\}$ and for $J=(j_1,\dots,j_k)$, $e_J = e_{j_1}\wedge\dots\wedge e_{j_k}$.

Thus, $\eta = \eta\llcorner dt + \mathbf{p}\eta$ and this is an orthogonal decomposition. Hence,

$$\vec{S} = e_0\wedge(\vec{S}\llcorner dt) + \mathbf{p}\vec{S}$$

and

$$1 = |\vec{S}|^2 = |\vec{S}\llcorner dt|^2 + |\mathbf{p}\vec{S}|^2 = |t\xi|^2 + |\mathbf{p}\vec{S}|^2.$$

So,

$$\mathbf{M}(S \llcorner I \times \mathbb{R}^d) = \int_{I \times \mathbb{R}^d} \sqrt{\underbrace{|\vec{S} \llcorner dt|^2 + |\mathbf{p}\vec{S}|^2}_{=1}} \, d\|S\|$$

$\sqrt{a^2+b^2} \leq |a|+|b|$ and $|\vec{S} \llcorner dt| = |\nabla^S t|$

$$\leq \int_{I \times \mathbb{R}^d} |\nabla^S t| + |\mathbf{p}\vec{S}| \, d\|S\|$$

$$\overset{\text{coarea}}{=} \int \mathbf{M}(S|_t)\, dt + \mathrm{Var}(S; I)$$

$$\leq \|S\|_{L^\infty} + \mathrm{Var}(S; I).$$

$\square$

In the following we often assume for

$$S \in \mathrm{I}_{1+k}([0,T] \times \mathbb{R}^d)$$

that

$$\partial S \llcorner (0,T) = 0.$$

Then (by the cylinder formula),

$$\partial S|_t = 0 \quad \text{for a.e. } t \in (0,T).$$

Ex: In dislocation theory, the dislocations are closed loops for a.e. t.

Thm [6.25] (compactness): Let $(S_j)_j \subset I_{1+k}([0,\tau]\times\overline{\Omega})$ with $(\partial S_j) \llcorner (0,\tau)\times\mathbb{R}^d = 0$ for all j and

$$\sup_j \left(\|S_j\|_{L^\infty} + \mathrm{Var}(S_j;[0,\tau]) \right) < \infty.$$

Then, there exists $S \in I_{1+k}([0,\tau]\times\overline{\Omega})$ with $\partial S_j \llcorner (0,\tau)\times\mathbb{R}^d = 0$ and, up to a subsequence,

$$S_j \xrightarrow{*} S \quad \text{in } BV,$$

that is,

$$\begin{cases} S_j \xrightarrow{*} S & \text{in } I_{1+k}, \\ S_j|_t \xrightarrow{*} S|_t & \text{in } I_k \text{ for a.e. } t. \end{cases}$$

Moreover,

$$\mathrm{Var}(S;[0,\tau]) \le \liminf_{j\to\infty} \mathrm{Var}(S_j;[0,\tau]).$$

Pf: By the Pythagoras lemma,

$$\sup_j \mathbb{M}(S_j) < \infty.$$

By the Federer-Fleming compactness theorem, the first convergence $S_j \xrightarrow{*} S$ in I_{1+k} follows after selecting a subsequence.

Since (cylinder formula)

$$S|_t = \partial(S \llcorner \{\mathbf{t}<t\}) - (\partial S)\llcorner\{\mathbf{t}<t\},$$

If $\|S_j\| + \|\partial S_j\| \xrightarrow{*} \nu$ in $\mathcal{M}^+$ (up to a subsequence), then for t s.t. $\nu(\{t\}\times\mathbb{R}^d) = 0$, then

$$\langle S_j, \mathbb{1}_{\{\mathbf{t}<t\}} \wedge d\omega \rangle \to \langle S, \mathbb{1}_{\{\mathbf{t}<t\}} \wedge d\omega \rangle$$

$$\langle \partial S_j, \mathbb{1}_{\{\mathbf{t}<t\}} \wedge \omega \rangle \to \langle \partial S, \mathbb{1}_{\{\mathbf{t}<t\}} \wedge \omega \rangle$$

for all $\omega \in \mathcal{D}^k(\mathbb{R}^d)$. This is just measure theory because if $\mu_j \xrightarrow{*} \mu$ for vecter measures μ_j, μ and

$$\|\mu_j\| \xrightarrow{*} \lambda,$$

then

$$\int u \, d\mu_j \longrightarrow \int u \, d\mu$$

for any u whose discontinuity points are λ-negligible (elementary ....). Thus, for these t's, $S_j|_t \xrightarrow{*} S|_t$ in I_k. The lower semicontinuity follows from Reshetnyak's lower semicontinuity theorem or more general lower semicontinuity results (see my Calculus of Variations book). $\square$

Deformation

In the following, $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain.

Thm [6.27] (space-time deformation):

Let $T \in I_k(\overline{\Omega})$ with $\partial T = 0$. Then, for all $\varepsilon > 0$ there is $S \in I_{1+k}([0,1] \times \overline{\Omega}')$, where $\Omega' := \Omega + B_{c\varepsilon}$ $(c = c(d))$ s.t.

$$\partial S = \delta_1 \times P - \delta_0 \times T,$$

$$P = \sum_{F \in \mathcal{F}_k(\varepsilon)} p_F [\![F]\!], \quad \partial P = 0 \quad \text{(a "polyhedral chain")}$$

where $\mathcal{F}_k(\varepsilon)$ contains all k-faces of the cubes $\varepsilon z + (0,\varepsilon)^d$, $z \in \mathbb{Z}$, and $p_F \in \mathbb{Z}$.

Moreover,

$$\mathbf{M}(P) \leq C\,\mathbf{M}(T),$$

$$\mathrm{Var}(S) \leq C\varepsilon\,\mathbf{M}(T) \quad (\mathrm{Var}(S) := \mathrm{Var}(S;[0,1])),$$

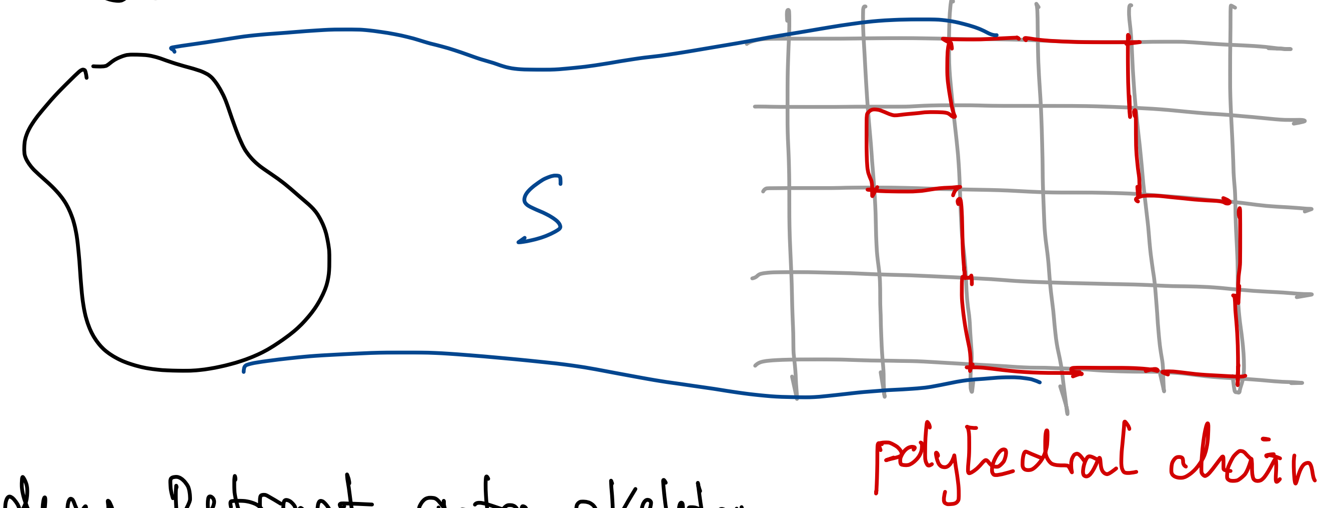
$$\|S\|_{L^\infty} \leq C\,\mathbf{M}(T),$$

and S can be chosen s.t. $t \mapsto \mathrm{Var}(S;[0,t])$ is a Lipschitz function.

Here: $\delta_0 \times T$ is the product current (on $\{0\} \times \mathbb{R}$) etc.

Pf: Similar to the classical deformation theorem (retract to skeleton). $\square$

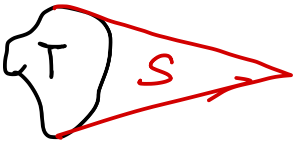
Illustration:



Pf idea: Retract onto skeleton.

Interlude: OMITTED

Thm [6.28] (isoperimetric inequality):
Let $T \in I_k(\bar{\Omega})$, $k \geq 1$, with $\partial T = 0$. Then there is $S \in I_{1+k}([0,1] \times \bar{\Omega}')$ (Lipschitz) with $\Omega' := \Omega + B_{C\mathbb{M}(T)^{1/k}}$ s.t.



$$\partial S = -\delta_0 \times T$$

and, for $C = C(d,k)$,

$$\mathrm{Vol}(S) \leq C\mathbb{M}(T)^{\frac{k+1}{k}}, \quad \|S\|_{L^\infty} \leq C\mathbb{M}(T).$$

Pf: Let P, S as in the deformation theorem for $\rho := (2C_{def}\mathbb{M}(T))^{1/k}$. By scaling,

$$\mathbb{M}(P) = N_\rho \rho^k \quad \text{for} \quad N_\rho \in \mathbb{N}.$$

Then, $\mathbb{M}(P) \leq C_{def}\mathbb{M}(T)$. Thus,

$$N_\rho \cdot 2C_{def}\mathbb{M}(T) = \mathbb{M}(P) \leq C_{def}\mathbb{M}(T).$$

Thus, $2N_\rho \leq 1$, whereby $N_\rho = 0$ and $P = 0$. □

Thm [G.28] (space-time connectivity): L9

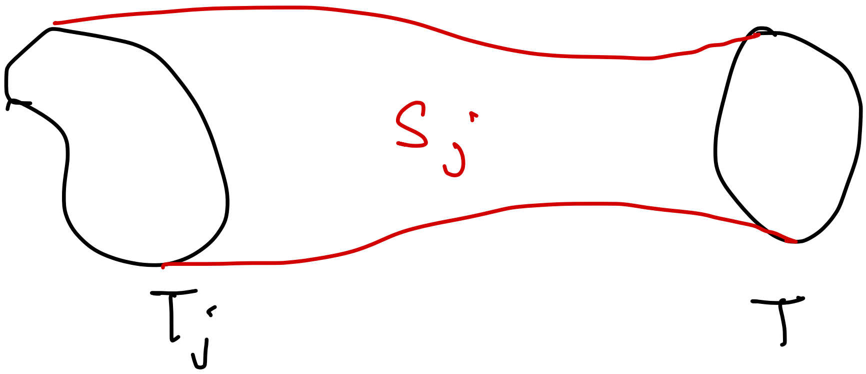
Let $T_j \xrightarrow{*} T$ in $\mathcal{I}_k(\overline{\Omega})$, $\partial T_j = 0$, $\sup_j \mathbf{M}(T_j) < \infty$.
Then there are $S_j \in \mathcal{I}_{1+k}([0,1]\times\overline{\Omega})$
s.t.

$$\partial S_j = \delta_1 \times T - \delta_0 \times T_j,$$

$$\mathrm{Var}(S_j) \to 0,$$

$$\limsup_{j\to\infty} \|S_j\|_{L^\infty} \leq C \cdot \limsup_{j\to\infty} \mathbf{M}(T_j) < \infty.$$

Illustration:



"S_j connects T_j to T"

Rem: • Also the other direction holds
$(\mathrm{Var}(S_j) \to 0 \implies T_j \xrightarrow{*} T)$.

- can also do "Lipschitz-in-time" regularity.
- this result is very important in applications!

Pf: Set $M := \sup_j \mathbf{M}(T_j)$. Define
for $T_0, T_1 \in \mathcal{I}_k(\overline{\Omega})$ with $\partial T_0 = \partial T_1 = 0$

$$\mathrm{dist}_{\overline{\Omega}}(T_0, T_1) := \inf\left\{ \mathrm{Var}(S) : \begin{array}{l} S \in \mathcal{I}_{1+k}([0,1]\times\overline{\Omega}), \\ \partial S = \delta_1 \times T_1 - \delta_0 \times T_0 \end{array} \right\}.$$

Claim: For $N \in \mathbb{N}$, there is a finite collection $\mathbb{P}_N \subset I_k(\overline{\Omega})$ s.t. for any $\hat{T} \in I_k(\overline{\Omega})$ with $\partial \hat{T} = 0$, $\mathbb{M}(\hat{T}) \leq M$ there is $P \in \mathbb{P}_N$ and

$$\mathrm{dist}_{\overline{\Omega}}(P, \hat{T}) < C 2^{-N} \qquad (C = C(d, k, \Omega)).$$

Pf of Claim:

1) First, we allow $S \in I_{1+k}([0,1] \times \overline{\Omega}')$ with $\Omega' := \Omega + B_{C\delta} \supset \Omega$ as in the deformation theorem with

$$\delta := \frac{2^{-N}}{C_{def} M} \qquad (C_{def} \text{ constant from there})$$

For this $\mathbb{P}_N$ we take the set of all polyhedral chains that satisfy the conclusions of the deformation theorem for any $\hat{T}$. This is a finite set (using the mass bound!).

2) Retract Ω' to Ω (which is possible since Ω has a Lipschitz boundary). ◇

Now, for every N find $P \in \mathbb{P}_N$ s.t. for infinitely many j's it holds that

$$\mathrm{dist}_{\overline{\Omega}}(T_j, P) < C 2^{-N}.$$

Applying this argument repeatedly, selecting a subsequence at every step, and using a diagonal procedure, we may find a subsequence, still denoted by $(T_j)_j$, such that

$$\mathrm{dist}_{\overline{\Omega}}(T_\ell, P_j) < 2^{-(j+1)}$$

for all $\ell \geq j$ and a $P_j \in I_k(\overline{\Omega})$.

By the triangle inequality,

$$\mathrm{dist}_{\overline{\Omega}}(T_j, T_{j+1}) < 2^{-j}.$$

So, there is $R_j \in I_{1+k}([0,1] \times \overline{\Omega})$ with

$$\partial R_j = \delta_1 \times T_{j+1} - \delta_0 \times T_j$$

$$\mathrm{Var}(R_j) < 2^{-j}$$

$$\|R_j\|_{L^\infty} \leq C \cdot \max\{M(T_j), M(T_{j+1})\} \leq CM.$$

Concatenate the R_j's for $\ell = j, \dots, j+m-1$, i.e.

$$S_j^m := R_{j+m-1} \circ R_{j+m-2} \circ \dots \circ R_j,$$

rescaled in time to $[0,1]$, so that

$$S_j^m \in I_{1+k}([0,1]\times\overline{\Omega}),$$

$$\delta S_j^m = \delta_1 \times T_{j+m} - \delta_0 \times T_j,$$

$$Var(S_j^m) = \sum_{\ell=0}^{m-1} Var(R_{j+\ell}) \leq 2^{-j+1}$$

$$\|S_j^m\| \leq C \cdot \sup_{\ell\geq j} M(T_j).$$

Passing to $m\to\infty$ by the compactness thm, we get

$$S_j \in I_{1+k}([0,1]\times\overline{\Omega}),$$

$$\delta S_j = \delta_1 \times T - \delta_0 \times T_j,$$

$$Var(S_j) \leq 2^{-j+1}$$

$$\|S_j^m\| \leq C \cdot \sup_{\ell\geq j} M(T_j).$$

So,

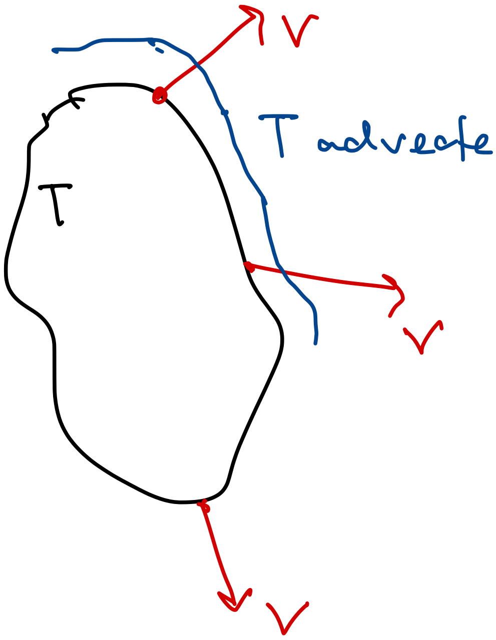
$$dist_{\overline{\Omega}}(T_j, T) \leq Var(S_j) \longrightarrow 0 \text{ as } j\to\infty.$$

Since this holds for all subsequences of the original sequence $(T_j)_j$, the claims follow. $\square$

5. THE GEOMETRIC TRANSPORT EQUATION

L10

A PDE-type approach to the transport of currents needs a directional derivative...



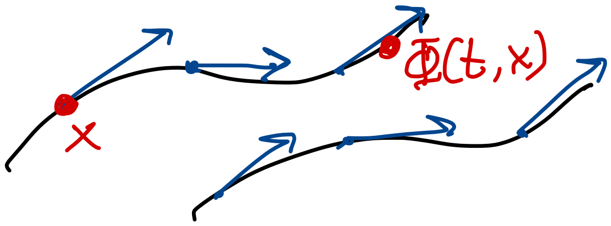
v: velocity field

How to go about this?

If $v: \mathbb{R}^d \to \mathbb{R}^d$ is smooth (or Lipschitz) and bounded, define its <u>flow</u> as the map

$$\Phi: \mathbb{R}^{1+d} \to \mathbb{R}^d,$$

$$\begin{cases} \frac{\partial}{\partial t}\Phi(t,x) = v(\Phi(t,x)) & , \quad (t,x) \in \mathbb{R}^{1+d} \\ \Phi(0,x) = x & , \quad x \in \mathbb{R}^d . \end{cases}$$



Prop. [3.16]:

Let $v: \mathbb{R}^d \to \mathbb{R}^d$ be Lipschitz and bounded.

(i) $\Phi^t(x) := \Phi(t,x)$ is Lipschitz for all $t \in \mathbb{R}$.

(ii) $(\Phi^t)^{-1} = \Phi^{-t}$

(iii) $(\Phi^t)_{t\in\mathbb{R}}$ is a group:

$$\Phi^t \circ \Phi^s = \Phi^{t+s} = \Phi^s \circ \Phi^t \quad , s,t \in \mathbb{R}.$$

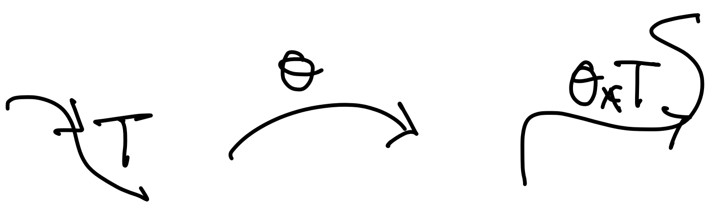
Let $T \in I_k(\Omega)$, $\theta: \Omega \to \Omega'$, where $\Omega \subset \mathbb{R}^d$, $\Omega' \subset \mathbb{R}^m$, be smooth and proper, i.e., preimages of compact sets are compact. Then, the pushforward $\theta_* T$ is the k-current on Ω' given by

$$\langle \theta_* T, \omega \rangle := \langle T, \theta^* \omega \rangle \quad , \omega \in \mathcal{D}^k(\Omega').$$

Here, $\theta^* \in \mathcal{D}^k(\Omega)$ is the pullback of ω,

$$\langle \xi, (\theta^*\omega)(x) \rangle := \langle D\theta(x)[\xi], \omega(\theta(x)) \rangle$$

for $\xi \in \Lambda_k \mathbb{R}^m$.



Lem [4.27]: If $\theta : \Omega \to \Omega'$ is smooth and proper and $T \in I_k(\Omega)$, then $\theta_* \in I_k(\Omega')$ and for $\omega \in \mathcal{D}^k(\Omega')$,

$$\langle \theta_* T, \omega \rangle := \int \langle D\theta(x)[\vec{T}(x)], \omega(\theta(x)) \rangle \, d\|T\|(x).$$

Thm [4.31]: The same holds for θ only Lipschitz (and T only "normal").

(for normal currents: Alberti - Marchese '16!)

For $v : \mathbb{R}^d \to \mathbb{R}^d$ Lipschitz (or smooth) and bounded, and $T_0 \in I_k(\mathbb{R}^d)$ consider

$$t \mapsto T_t := \Phi^t_* T_0 .$$

Then, for $\omega \in \mathcal{D}^k(\mathbb{R}^d)$,

$$\begin{aligned}\frac{d}{dt} \langle T_t, \omega \rangle &= \frac{d}{dt} \langle T_0, (\Phi^t)^* \omega \rangle \\ &= \lim_{h \to 0} \frac{1}{h} \langle T_0, (\Phi^{t+h})^* \omega - (\Phi^t)^* \omega \rangle \\ &= \langle \underbrace{\Phi^t_* T_0}_{= T_t}, \lim_{h \to 0} \frac{(\Phi^h)^* \omega - \omega}{h} \rangle .\end{aligned}$$

The quantity (if defined)

$$\mathcal{L}_v \omega := \lim_{h \to 0} \frac{(\Phi^h)^* \omega - \omega}{h}$$

is called the *Lie derivative* of ω in direction v. So, if we define also for $T \in I_k(\mathbb{R}^d)$ its *Lie derivative* as

$$\langle \mathcal{L}_v T, \omega \rangle := \langle T, \mathcal{L}_v \omega \rangle \quad , \omega \in \mathcal{D}^k(\mathbb{R}^d),$$

then we have shown

$$\frac{d}{dt} \langle T_t, \omega \rangle = \langle \mathcal{L}_v T_t, \omega \rangle,$$

that is, the *Geometric Transport Equation*

$$\boxed{\frac{d}{dt} T_t - \mathcal{L}_v T_t = 0 \qquad (\text{GTE})}$$

Prop [8.1]: $\mathcal{L}_v T = v \wedge \partial T + \partial (v \wedge T)$.
(which makes sense if only $v \in L^1(\|T\| + \|\partial T\|; \mathbb{R}^d)$)

Pf: Dualization of "Cartan's magic formula". □

Thm [8.14] (existence): For v as before, there is a solution to (GTE).

Pf: s.a. □

Thm 8.16 [uniqueness]:
This solution is unique (even in the class of "normal" currents.

Rem: Can also treat time-dependent. $v = v(t, x)$.

Rem: In coordinates, the GTE is:

- $k=0$: The continuity equation

$$\frac{d}{dt}\rho_t + \operatorname{div}(v_t \rho_t) = 0$$

- $k=1$ (dislocations!), boundaryless:

$$\frac{d}{dt}\tau_t - \operatorname{curl}(v_t \times \tau_t) = 0$$

(well-known in the field!)

- $k=2$ in $\mathbb{R}^3$: membrane transport
- $k=d$: The transport equation

$$\frac{d}{dt}u_t + v_t \cdot \nabla u_t = 0$$

Thm [8.18] (advection):

Let $S \in I_{1+k}([0,1] \times \mathbb{R}^d)$ be NC, that is,

$$\|S\| \llcorner \mathrm{Crit}(S) = 0$$

and $(\partial S) \llcorner (0,1) \times \mathbb{R}^d = 0$. Set

$$S(t) := \mathbf{p}_* S|_t .$$

Then

$$\frac{d}{dt} S(t) - \mathcal{L}_{V_t} S(t) = 0$$

for

$$V := \frac{D}{Dt} S.$$

Rem: One can also conversely start with a path $t \mapsto T_t$ and glue the T_t together if some (Lipschitz) continuity assumptions are met.

Conclusion: The space-time appraoch and the GTE are (more or less) equivalent!

END