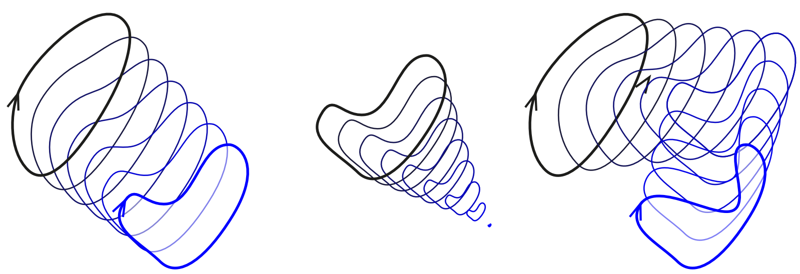
Transport of Geometric Structures in Space-Time

Lecture Notes for the Jyväskylä Summer School 2026

Filip Rindler

August 2026



Filip Rindler
Mathematics Institute
University of Warwick
Coventry CV4 7AL
United Kingdom
F.Rindler@warwick.ac.uk
http://www.warwick.ac.uk/filiprindler

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Introduction

The topic of these lecture notes is a recently developed theory describing the transport of “rough” geometric structures, such as Lipschitz curves or surfaces, via a space-time approach based on Geometric Measure Theory.

A particular motivation for this study comes from the theory of elasto-plasticity in crystalline materials, such as metals. While elasticity theory only considers *reversible* deformations of materials, plasticity is the study of *permanent* deformation. Both effects usually go hand-in-hand, that is, a specimen deforms elastically up to a *yield stress* threshold, after which plastic deformation occurs. Despite its great practical importance, no fully satisfactory description of elasto-plasticity has emerged so far. Particularly in the (very nonlinear) regime of large deformations, many challenges remain in the modeling, analysis, and computation.

The theory described in these notes was developed to put the description of the microscopic origins of elasto-plasticity on a sounds theoretical foundation. It is part of a larger effort to develop physically sound, yet mathematically rigorous, models of elasto-plasticity in crystalline solids. In such materials, the change in shape of a specimen is due to a rearrangement of bonds in its crystal lattice, which is prototypically depicted in Fig. 0.1: The lattice “slips” (slides) over the plane H (with normal vector N) in direction of the so-called *Burgers vector* b.

The precise mechanism by which such lattice slip occurs was identified in the groundbreaking work of Taylor and Orowan roughly 100 years ago, building on earlier work of

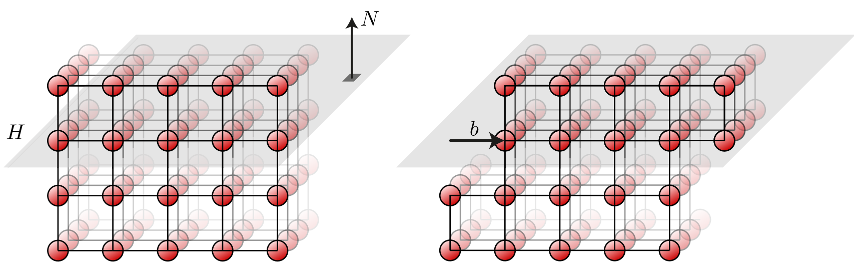

Figure 0.1: Slip over a lattice plane.

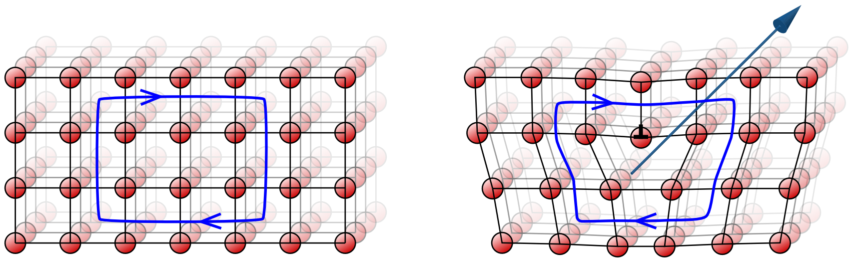

Figure 0.2: A Burgers circuit in a perfect lattice and around an (edge) dislocation.

Volterra, as the motion of so-called *dislocations*. Roughly speaking, dislocations are line defects in the crystal lattice, i.e., deviations of the atomic bonds from a perfect arrangement along a line through the crystal. Such a dislocation is prototypically shown on the right in Fig. 0.2. Note that there is an excess plane of atoms above the dislocation line, which is depicted by the arrow into the paper. The symbol "$\perp$" indicates where this excess plane of atoms ends. Defects like the ones depicted always occur on lines (or curves) through the crystal, which are either closed or end at the boundary of the crystal (or the boundary of a crystal grain in polycrystals).

Consider again the lattice slip depicted in Fig. 0.1. At first sight, it appears that to bring about this slip one would have to break all the vertical bonds crossing the slip plane at once, slide the top of the crystal to the right, and then reconnect the atoms with bonds between their new positions (in this very simplistic, but surprisingly accurate model of interactomic bonds). It turns out, however, that this mode of slip would be very expensive energetically, leading to predictions on the strength of metals that are wildly too large. It was Taylor's seminal discovery that the presence of dislocations in fact greatly reduces the force needed for plastic deformation: Instead of having to break all bonds over the slip plane at once, it suffices to break just *one row* of bonds at the leading edge of the progressing slip, as depicted in Fig. 0.3. While the total number of bonds that have to be broken is the same as in the naive slip motion described before, this progressive mode of slip propagation yields a much reduced activation energy, which is in line with experimental observations [5, 22].

So, to understand how crystalline slip comes about, we need a way to model, and then analyze, dislocation lines that are transported around in a crystal domain. Mathematically, such an analysis is complicated by two pertinent phenomena, namely low regularity and topological change. The first of these refers to dislocation lines having less smoothness than one would hope for to employ the tools of Differential Geometry (which requires relatively regular structures). The second refers to the fact that dislocation lines may split, merge, or otherwise interact in a way that would destroy any attempt to handle them using any form of parametrization.

The basic idea we will pursue in these notes to analyze transport phenomena of curves, and also higher-dimensional geometric structures, is to encode their motion into a single ge-

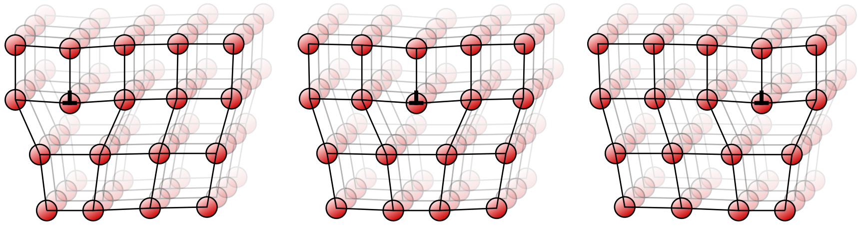

Figure 0.3: Dislocation motion deforming a lattice.

ometric object, the *space-time trajectory* traced out by the motion. This space-time trajectory is defined as the (hyper-)surface obtained by gluing all the objects at individual points in time together along the time axis. For dislocations, this space-time surface is called a *slip trajectory*: a two-dimensional surface, possibly with low regularity, in four-dimensional space-time, whose time-slices recover the dislocation curves observed at individual times.

There are two fundamental origins of these space-time trajectories. One is to declare them as fundamental and work with them from the start, in what is termed the *variational approach* since it originated from minimization problems over spaces of admissible trajectories. Alternatively, we may take a time-indexed family of geometric structures and investigate whether this family can be "glued" together to furnish a space-time trajectory. The latter point of view is called the *transport approach* since it is intimately related to whether or not the objects are in fact "advected" by a velocity field via the so-called Geometric Transport Equation.

These two approaches are closely related, but not equivalent: It turns out that there are reasonable evolutions represented by space-time trajectories that do not represent the advection of a geometric structure by any velocity vector field. Thus, the Geometric Transport Equation should be viewed not as a replacement for the space-time viewpoint, but as its more regular counterpart.

The appropriate mathematical language for the space-time theory as outlined above is supplied by the field of Geometric Measure Theory. Unlike Differential Geometry, which is primarily concerned with smooth geometric structures, Geometric Measure Theory is designed to handle low-regularity structures. Thus, one may consider surfaces with Lipschitz-type regularity and including folds, corners, junctions, or other singular features. These requirements lead naturally to the notion of *normal and integral currents*, which are measure-theoretic analogues of manifolds (and "density fields" of manifolds). The analytic aspects of Geometric Measure Theory are equally important: It provides notions of convergence for geometric structures as well as compactness principles that are strong enough to support existence arguments.

The origins of the theory to be developed can be found in the modeling paper [21] on elasto-plasticity driven by dislocation motion, as well as its mathematical implementation in [28, 29]. Much of the material on the Geometric Transport Equation follows [9, 10].

Finally, some recent ideas from [11] have been employed to streamline the presentation. However, much of the material from these sources has been revised and rearranged for the present notes to create a more coherent presentation.

These notes are meant to be reasonably self-contained for a reader possessing a solid knowledge of basic Functional Analysis and Measure Theory. As such, we develop many concepts from Geometric Measure Theory in detail, without relying on the literature too much. For background, the standard reference for Geometric Measure Theory remains Federer's encyclopaedic monograph [19]; Morgan's "pictorial guide" [27] is a useful companion. A more gentle introduction is provided by Krantz–Parks [23], partly based on Simon's influential lecture notes [32]. For covering and rectifiability theory we refer to Mattila's treatise [26]. The first part of the book [4] gives a related treatment with applications to the Calculus of Variations.

There is a large amount of material here, but the topics of the first two parts are classical, with the recent developments of the space-time theory entirely contained in the third part. Thus, a reader in a hurry, and one who already has some familiarity with Geometric Measure Theory, may jump straight to the third part and only consult the first and second part for reference. No intimate knowledge of the proofs in the first two parts is necessary to understand the third, and the proofs may be safely skipped on first reading.

I would appreciate comments, corrections and remarks about these lecture notes. They can be sent back to me via

`https://tinyurl.com/tgss26`

or by scanning the following QR-Code:

Filip Rindler
August 2026

Part I

Basic Geometric Measure Theory

Chapter 1

Measures

We assume that the reader is familiar with the basics of measure theory, for example at the level of the relevant chapters of [31] or Chapter 1 of [4], whose conventions we follow to a large extent. The encyclopedic works [6, 7] contain a very thorough treatment of all details. This chapter outlines all the required theory, but many proofs are omitted.

1.1 Measure theory

Let X be a normed finite-dimensional vector space or subset thereof (such as $\mathbb{R}^3$, $\mathbb{R}^{1+3}$ or its subsets, together with the Euclidean norm), and let $\mathcal{B}(X)$ be its **Borel σ-algebra**, that is, the smallest collection of subsets of X such that all open sets of X are in $\mathcal{B}(X)$ and for every finite or countable collection $E_1, E_2, \ldots \in \mathcal{B}(X)$ it holds that $X \setminus E_1 \in \mathcal{B}(X)$ and $\bigcup_j E_j \in \mathcal{B}(X)$. The elements of $\mathcal{B}(X)$ are called the **Borel(-measurable) sets**.

A set function

$$\nu \colon \mathcal{B}(X) \to V,$$

where $(V, \|\cdot\|)$ is another finite-dimensional normed vector space, is called a **finite (vector) measure** if $\nu(\emptyset) = 0$ and for every finite or countable collection $E_1, E_2, \ldots \in \mathcal{B}(X)$ of pairwise disjoint sets, the **σ-additivity**

$$\nu\left(\bigcup_n E_n\right) = \sum_n \nu(E_n)$$

holds. If the collection of sets E_n is infinite, then the right-hand side is to be understood as converging in norm, that is, there is $z \in V$ with $\|\sum_{n \leq N} \nu(E_n) - z\| \to 0$ as $N \to \infty$ and $\sum_n \nu(E_n) := z$. We write $\mathcal{M}(X; V)$ for the set of all (vector) measures on X with values in the finite-dimensional normed vector space V. We will often refer to vector measures simply as "measures" in the following.

We will also need a slightly more general notion of (vector) measures: A set function ν defined on the relatively compact Borel subsets of X, with values in a finite-dimensional normed vector space V, and with property that ν's restriction to $\mathcal{B}(K)$ is a (vector) measure for any compact $K \subset X$, is called a **Radon measure**. Note that this includes the finiteness constraint $\nu(K) \in V$ (equivalently, $\|\nu(K)\| < \infty$), whereas "$\nu(X)$" is not necessarily defined. All Radon measures on the space X with values in the finite-dimensional normed vector space V are collected in the set $\mathcal{M}_{\mathrm{loc}}(X; V)$. Let us also remark that the defining feature of "Radon" measures is their inner and outer regularity as in Proposition 1.3 below; here, however, this regularity is automatic since X is always locally compact and separable.

A subset $N \subset Z$ of a Borel set $Z \in \mathcal{B}(X)$ with $\nu(Z) = 0$ is called **ν-negligible** or a **ν-null set**. A set $A \subset X$ that can be written as $A = B \cup N$ with $B \in \mathcal{B}(X)$ a Borel set and N μ-negligible, is called **μ-measurable**. The **μ-completion** $\mathcal{B}_\nu(X)$ of the Borel σ-algebra $\mathcal{B}(X)$ is the σ-algebra of all sets $A \subset X$ that can be written as $A = B \cup N$ with $B \in \mathcal{B}(X)$ a Borel set and N ν-negligible. For a ν-measurable set $A = B \cup N \in \mathcal{B}_\nu(X)$ one sets $\nu(A) := \nu(B)$; this extends the domain of definition of ν to the ν-completion $\mathcal{B}_\nu(X)$.

If μ is a measure for target space $V = \mathbb{R}$ and $\mu(B) \geq 0$ for all Borel sets B, the measure μ is called a **positive finite measure**. We can also allow the value $+\infty$ (for which the above σ-additivity still makes sense) to obtain the set of **positive (Radon) measures** $\mathcal{M}^+(X)$; note that here we still require finiteness of μ on compact sets. If the finiteness on compact sets does not hold, then we speak of a **positive Borel measure**. We call $\mu \in \mathcal{M}^+(X)$ **σ-finite** if there is a finite or countable collection of sets $A_n \in \mathcal{B}(X)$ with $X = \bigcup_n A_n$ and $\mu(A_n) < \infty$ for all n.

The following is a result from general measure theory, whose proof can for instance be found in Theorem 1.49 of [4]:

Theorem 1.1 (Carathéodory). *Let μ be a **(positive) outer measure** on X, that is, μ assigns to any subset $A \subset X$ a value in $[0, \infty]$ and satisfies the following properties:*

(i) $\mu(\emptyset) = 0$.

(ii) *μ is **σ-subadditive**, i.e., for any $A, A_n \subset X$, where $n \in \mathbb{N}$ or from a finite subset thereof, it holds that*

$$A \subset \bigcup_n A_n \quad \textit{implies} \quad \mu(A) \leq \sum_n \mu(A_n).$$

(iii) *μ is **metric additive**, i.e., for any $A, B \subset X$ it holds that*

$$\operatorname{dist}(A, B) > 0 \quad \textit{implies} \quad \mu(A \cup B) = \mu(A) + \mu(B).$$

Then, the restriction of μ to the μ-completion $\mathcal{B}_\mu(X)$ (which includes the Borel σ-algebra $\mathcal{B}(X)$) is a positive Borel measure.

Note that an outer measure is not necessarily finite on compact sets, so we cannot in general improve the conclusion of this theorem to entail that μ is a Radon measure.

The basic theory of vector (Radon) measures is in many respects the same as the theory of positive measures, with only minor adjustments. Indeed, one can often identify V with a normed copy of $\mathbb{R}^N$ for some $N \in \mathbb{N}$ and then argue component-wise. However, new aspects appear as well, some of which we will now discuss.

For a vector measure $\nu \in \mathcal{M}(X; V)$ we define its **total variation measure** $\|\nu\|$ as follows: Set for all Borel sets $A \in \mathcal{B}_\mu(X)$,

$$\|\nu\|(A) := \sup \left\{ \sum_{n=1}^{\infty} \|\nu(A_n)\| \;:\; A = \bigcup_{n=1}^{\infty} A_n,\ A_n \in \mathcal{B}(X),\ A_m \cap A_n = \emptyset \text{ if } m \neq n \right\}.$$

The following are standard results (see [4] for proofs).

Proposition 1.2. *The set function $\|\nu\|$ is a positive and finite measure, $\|\nu\| \in \mathcal{M}^+(X)$, and for all sets $A \in \mathcal{B}_\mu(X)$ it holds that $\|\nu(A)\| \le \|\nu\|(A)$.*

A similar definition and proposition can be given if only $\nu \in \mathcal{M}_{\mathrm{loc}}(X; V)$, in which case $\|\nu\|$ may not be finite.

For a positive measure $\mu \in \mathcal{M}^+(X)$ on a finite-dimensional vector space X, the **support** $\operatorname{supp} \mu$ is the (closed) set of points $x \in X$ such that for every open neighborhood U of x it holds that $\mu(U) > 0$. The support $\operatorname{supp} \nu$ of a vector measure $\nu \in \mathcal{M}_{\mathrm{loc}}(X; V)$ is defined to be the support of $\|\nu\|$. The **restriction** $\mu \llcorner E$ for a set $E \in \mathcal{B}_\mu(X)$ is defined as $(\mu \llcorner E)(A) := \mu(A \cap E)$ for all sets $A \in \mathcal{B}_\mu(X)$.

For a **ν-measurable map** $f\colon X \to \mathbb{R}$, that is, $f^{-1}(A) \in \mathcal{B}_\nu(X)$ for any Borel set $A \subset \mathbb{R}$, and a vector measure $\nu \in \mathcal{M}(X; V)$, the **integral**

$$\int f \,\mathrm{d}\nu \in V$$

is defined as in the standard Lebesgue theory, namely via approximation with step functions. For instance, if $f = \alpha \mathbb{1}_A$, where $\mathbb{1}_A$ is the **indicator function** of a μ-measurable set $A \in \mathcal{B}_\mu(X)$ and $\alpha \in \mathbb{R}$, then $\int f \,\mathrm{d}\nu = \alpha \cdot \nu(A)$, which is well defined since V is a vector space. For the measure $A \in \mathcal{B}_\mu(X) \mapsto \int_A f \,\mathrm{d}\nu$ we also use the notation "$f\nu$".

Proposition 1.3. *Let $\mu \in \mathcal{M}^+(X)$ be **open σ-finite**, that is, there is a sequence of open sets U_n, $n \in \mathbb{N}$, such that $\mu(U_n) < \infty$ and $X = \bigcup_n U_n$. Then, for any μ-measurable set $A \subset X$ the **outer regularity***

$$\mu(A) = \inf\{\, \mu(U) \;:\; U \supset A \text{ open} \,\}$$

*and the **inner regularity***

$$\mu(A) = \sup\{\, \mu(K) \;:\; K \subset A \text{ compact} \,\}$$

hold.

Finally, recall that $\mathrm{C}_c(\Omega)$ denote the Banach spaces of continuous and bounded functions $\varphi\colon \Omega \to \mathbb{R}$ with compact support, and $\mathrm{C}_0(\Omega)$ denotes the Banach space of continuous and bounded functions $\varphi\colon \Omega \to \mathbb{R}$ with $\varphi(x) \to 0$ as $|x| \to \infty$, i.e., the closure of $\mathrm{C}_c(\Omega)$ in $\mathrm{C}(\Omega)$ with respect to the $\|\bullet\|_\infty$-norm. Then the following fundamental result holds:

Theorem 1.4 (Riesz representation theorem). *Let $\Omega \subset \mathbb{R}^d$ be open. The dual space to $\mathrm{C}_0(\Omega)$ can be identified with $\mathcal{M}(\Omega;\mathbb{R})$. More precisely, for any linear functional T on $\mathrm{C}_0(\Omega)$ satisfying $\langle T,\varphi\rangle \le M\|\varphi\|_\infty$ for all $\varphi \in \mathrm{C}_0(\Omega)$ and a constant $M > 0$, it holds that there is a finite measure $\nu \in \mathcal{M}(\Omega;\mathbb{R})$ such that*

$$\bigl\langle T,\varphi\bigr\rangle = \int_\Omega \varphi \,\mathrm{d}\nu, \qquad \varphi \in \mathrm{C}_0(\Omega),$$

and $\|\nu\|(\Omega) \le M$. Likewise, the dual space to $\mathrm{C}_c(\Omega)$ can be identified with $\mathcal{M}_{\mathrm{loc}}(\Omega;\mathbb{R})$.

For later use we also recall how to define a “disintegration” of product (vector) measures on a product measure space $X \times Y$, where both X, Y are normed finite-dimensional spaces or open subsets thereof. A family of vector measures

$$(\nu_x)_{x\in X} \subset \mathcal{M}(Y;V),$$

is called **weakly* measurable** with respect to a positive measure $\lambda \in \mathcal{M}^+(X)$ if for all Carathéodory integrands $f\colon X \times \mathbb{R}^N \to \mathbb{R}$, the compound function

$$x \mapsto \bigl\langle f(x,\bullet),\nu_x\bigr\rangle := \int f(x,A)\,\mathrm{d}\nu_x(A), \qquad x \in X,$$

is λ-measurable.

Theorem 1.5 (Disintegration). *Let $X \subset \mathbb{R}^d$ be open and let $\mu \in \mathcal{M}(X \times \mathbb{R}^N;V)$ be a positive Radon measure. Define the measure $\kappa \in \mathcal{M}(X;V)$ via*

$$\kappa(B) := \|\mu\|(B \times \mathbb{R}^N) \qquad \textit{for any Borel set } B \subset X$$

and assume that it is Radon, that is, $\kappa(K \times Y) < \infty$ for all compact sets $K \subset X$. Then, there exists a κ-weakly measurable family $(\mu_x)_{x\in X} \subset \mathcal{M}(Y;V)$ of measures such that*

$$\|\mu_x\|(Y) = 1 \qquad \textit{for } \kappa\textit{-almost every } x$$

and

$$\mu = \kappa(\mathrm{d}x) \otimes \mu_x,$$

that is,

$$\int f \, \mathrm{d}\mu = \int_X \int f(x, A) \, \mathrm{d}\mu_x(A) \, \mathrm{d}\kappa(x)$$

for all $f \in \mathrm{C}_0(X \times Y)$*. Moreover, also*

$$\|\mu\| = \kappa(\mathrm{d}x) \otimes \|\mu_x\|.$$

Finally, the family $(\mu_x)_{x \in X} \subset \mathcal{M}(Y; V)$ *is* κ*-essentially unique, that is, if another family* $(\mu'_x)_{x \in X} \subset \mathcal{M}(Y; V)$ *also has the properties above, then* $\mu'_x = \mu_x$ *for* κ*-almost every* $x \in X$*.*

A proof of this theorem can be found in Theorem 2.28 and Corollary 2.29 of [4] or, in a slightly simpler case, in Theorem [30].

Remark 1.6. For κ-a.e. $t \in \mathbb{R}$ it holds that

$$\mu_t = \operatorname*{w^*\text{-}lim}_{h \to 0} \frac{\mu \llcorner (t-h, t+h) \times \mathbb{R}^d}{\kappa((t-h, t+h))},$$

as can be seen without too much effort by integrating a test function from a countable dense subset against μ and using the Lebesgue Differentiation Theorem 1.16 below.

Remark 1.7. For the special case $X = \mathbb{R}$, $Y = \mathbb{R}^d$, and $f(t, x) = \mathbf{t}(t, x) := t$, an important variation of the disintegration procedure described in the preceding theorem is the following: Assume that the **measure-theoretic pushforward** $\mathbf{t}_\# \mu := \mu \circ \mathbf{t}$ of $\mu \in \mathcal{M}(\mathbb{R}^{1+d}; V)$ with respect to $\mathbf{t}$, that is,

$$\mathbf{t}_\# \mu(I) := \mu(\mathbf{t}^{-1}(I)) = \mu\bigl(\{ (t, x) \in \mathbb{R}^{1+d} \ : \ t \in I \}\bigr)$$

for any Borel set $I \subset \mathbb{R}$ (or, equivalently, any interval I), satisfies

$$\mathbf{t}_\# \mu \ll \lambda$$

for a positive measure $\lambda \in \mathcal{M}^+(\mathbb{R})$. Then, we could equivalently decompose

$$\mu = \lambda(\mathrm{d}t) \otimes \mu_t$$

with a family $\mu_t \in \mathcal{M}(\{t\} \times \mathbb{R}^d; V)$ that is weakly*-measurable with respect to λ. Technically, this is not a generalized product as above since the μ_t have different domains of definition, namely the "slices" $\{t\} \times \mathbb{R}^d$. Nevertheless, we understand the above product notation to mean

$$(\lambda(\mathrm{d}t) \otimes \mu_t)(A) := \int \int \mathbb{1}_A(t, x) \, \mathrm{d}\mu_t(x) \, \mathrm{d}\lambda(t)$$

for any Borel set $A \subset \mathbb{R}^{1+d}$. This decomposition is also referred to as the "disintegration".

1.2 Coverings and differentiation of measures

The idea of coverings is to select from a family of sets a subfamily with controlled overlap or even a pairwise disjoint or almost disjoint (with respect to a measure) subfamily. Their proofs are more an exercise in geometry than in measure theory, so we here only quote the most important results without proofs.

Let $\mathcal{F}$ be a family of subsets of $\mathbb{R}^d$. Such an $\mathcal{F}$ is called a **disjoint family** of subsets of $\mathbb{R}^d$ if for any $E, F \in \mathcal{F}$ with $E \neq F$ it holds that $E \cap F = \emptyset$. We will also write the **union** of $\mathcal{F}$ as

$$\bigcup \mathcal{F} := \bigcup_{E \in \mathcal{F}} E.$$

The following is a very general covering theorem, whose proof can be found in Theorem 2.7 of [26].

Theorem 1.8 (Besicovitch covering theorem). *There is a constant $\beta_d \in \mathbb{N}$, which only depends on the dimension of the ambient space $\mathbb{R}^d$, such that the following holds for every family $\mathcal{F}$ of closed balls in $\mathbb{R}^d$ with the property that the set A of their centers is bounded: There are β_d disjoint and at most countable subfamilies $\mathcal{F}_1, \dots, \mathcal{F}_{\beta_d} \subset \mathcal{F}$ with*

$$A \subset \bigcup_{n=1}^{\beta_d} \Big(\bigcup \mathcal{F}_n \Big).$$

In particular, there is a finite or countable cover of A consisting of closed balls $\overline{B}_k \in \mathcal{F}$ (that is, $A \subset \bigcup_k \overline{B}_k$) such that every point of $\mathbb{R}^d$ belongs to at most β_d of the balls $\overline{B}_k$.

The next covering result is an improved version of the classical Vitali covering theorem for the Lebesgue measure. Given a set $A \subset \mathbb{R}^d$, a **fine cover** of A is a family $\mathcal{F}$ of closed balls such that every point in A is contained in a ball in $\mathcal{F}$ of arbitrarily small radius.

Theorem 1.9 (Besicovitch–Vitali covering theorem). *Let $A \subset \mathbb{R}^d$ be a bounded Borel set and let $\mathcal{F}$ be a fine cover of A. For every (vector or positive) Radon measure $\mu \in \mathcal{M}(\mathbb{R}^d; V)$ there is a disjoint and countable subfamily $\mathcal{F}' \subset \mathcal{F}$ such that*

$$\mu\Big(A \setminus \bigcup \mathcal{F}' \Big) = 0.$$

A proof can be found in Theorem 2.19 of [4].

Remark 1.10. The Besicovitch–Vitali covering theorem still holds if $\mu \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d; V)$ is σ-finite. Moreover, it remains true with *open* balls for the Lebesgue measure, or more generally if μ does not charge the boundaries of balls, or even if $\mathcal{F}$ has a subcover of μ-almost all of A with more than countably many balls for every center (exercise).

In general, a claim as in the Besicovitch–Vitali covering theorem with open balls instead of closed balls is, however, false:

Example 1.11. Let $A := \{0\} \cup \{ 1/n \ : \ n \in \mathbb{N} \}$ and define the fine cover $\mathcal{F}$ of A to be

$$\mathcal{F} := \left\{ \left(-\frac{1}{m}, \frac{1}{m}\right) \ : \ m \in \mathbb{N} \right\} \cup \left\{ \left(\frac{1}{n} - \frac{1}{m}, \frac{1}{n} + \frac{1}{m}\right) \ : \ m, n \in \mathbb{N}, \ m > n \right\}.$$

Let μ be any positive Radon measure with $\mu(\{a\}) > 0$ for all $a \in A$. Then, one can convince oneself (exercise) that there is no finite or countable disjoint subcover of $\mathcal{F}$ that covers μ-almost all of A. On the other hand, if one instead uses closed intervals in $\mathcal{F}$, then a finite subcover exists (consisting of the single interval $[-1, 1]$).

Consider a positive measures $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ and a (vector) Radon measure $\nu \in \mathcal{M}(\mathbb{R}^d; V)$ valued in a finite-dimensional normed vector space V. We say that ν has a **Radon–Nikodym density** with respect to μ if there is a μ-measurable map $g\colon \mathbb{R}^d \to V$ with $\nu = g\mu$. The goal of this section is to find ways to decide when such a density g exists, perhaps after subtracting a "singular part" from ν, and to compute it.

We first consider the case where also ν is a positive or signed (and possibly infinite) measure. In this case we define the **lower density** and the **upper density** of ν with respect to μ at $x \in \operatorname{supp} \mu$ as, respectively,

$$\mathrm{D}^-_\mu \nu(x) := \liminf_{r \downarrow 0} \frac{\nu(B_r(x))}{\mu(B_r(x))}, \qquad \mathrm{D}^+_\mu \nu(x) := \limsup_{r \downarrow 0} \frac{\nu(B_r(x))}{\mu(B_r(x))}.$$

If the lower and upper density agree, we call them just the **density**. By the regularity of measures (see Proposition 1.3) we may replace open balls by closed balls in these definitions without changing the value. We then have the following density lemma:

Lemma 1.12. *Let $\mu, \nu \in \mathcal{M}^+(\mathbb{R}^d)$ or $\mu, \nu \in \mathcal{M}(\mathbb{R}^d; \mathbb{R})$, let $A \subset \operatorname{supp} \mu$ be a Borel set, and let $t \in [0, \infty)$, or $t \in (-\infty, \infty)$ if mu, ν are signed measures. Then:*

$$\mathrm{D}^-_\mu \nu \le t \text{ on } A \qquad \text{implies} \qquad \nu(A) \le t\mu(A)$$

and

$$\mathrm{D}^+_\mu \nu \ge t \text{ on } A \qquad \text{implies} \qquad \nu(A) \ge t\mu(A).$$

If ν is finite, then $\{ x \in \operatorname{supp} \mu \ : \ \mathrm{D}^+_\mu \nu(x) = \infty \}$ is μ-negligible.

Proof. We only show the first assertion, the other is analogous. Assume without loss of generality that A is bounded, take an open set $U \supset A$, and define for all $\varepsilon > 0$ the family

$$\mathcal{F}_\varepsilon := \left\{ \overline{B}_r(x) \ : \ \overline{B}_r(x) \subset U, \ \nu(\overline{B}_r(x)) \le (t + \varepsilon)\mu(\overline{B}_r(x)) \right\}.$$

If $D_\mu^- \nu \leq t$ on A, then $\mathcal{F}_\varepsilon$ is a fine cover for A. By the Besicovitch–Vitali Covering Theorem 1.9, there exists an at most countable disjoint subfamily $\mathcal{F}'_\varepsilon \subset \mathcal{F}_\varepsilon$, so that

$$\nu\Big(A \setminus \bigcup \mathcal{F}'_\varepsilon\Big) = 0.$$

Then,

$$\nu(A) \leq \sum_{\overline{B} \in \mathcal{F}'_\varepsilon} \nu(\overline{B}) \leq (t+\varepsilon) \sum_{\overline{B} \in \mathcal{F}'_\varepsilon} \mu(\overline{B}) \leq (t+\varepsilon)\mu(U).$$

Letting $\varepsilon \downarrow 0$, we obtain $\nu(A) \leq t\mu(U)$. We conclude by the outer regularity of μ, see Proposition 1.3.

The additional claim follows directly from the estimate involving the upper density (note that we always assume that $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ is Radon, i.e., finite on compact sets). □

The following is the main theorem of this section.

Theorem 1.13 (Besicovitch differentiation theorem). *Let $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ and $\nu \in \mathcal{M}(\mathbb{R}^d; V)$, where $(V, \|\cdot\|)$ is a normed vector space. Then, for μ-almost every $x \in \operatorname{supp} \mu$, the "blow-up quotient" limit*

$$\frac{d\nu}{d\mu}(x) := \lim_{r \downarrow 0} \frac{\nu(B_r(x))}{\mu(B_r(x))} \tag{1.1}$$

exists in V and ν has the ***Radon–Nikodym decomposition***

$$\nu = \frac{d\nu}{d\mu}\,\mu + \nu^s,$$

where the ***singular part*** *$\nu^s \in \mathcal{M}(\mathbb{R}^d; V)$ is given as $\nu^s := \nu \llcorner E$ with*

$$E := (\mathbb{R}^d \setminus \operatorname{supp} \mu) \cup \Big\{ x \in \operatorname{supp} \mu \ :\ \lim_{r \downarrow 0} \frac{\|\nu\|(B_r(x))}{\mu(B_r(x))} = \infty \Big\}. \tag{1.2}$$

Proof. We first observe that we can argue componentwise, so we may assume that $V = \mathbb{R}$ (i.e., ν is a "signed measure") since our set E is the union of the componentwise E's. We also know from Lemma 1.12 that $D_\mu^+ \nu$ is finite μ-almost everywhere.

For any Borel set $A \in \mathcal{B}(\mathbb{R}^d)$ we define

$$\lambda^-(A) := \int_A D_\mu^- \nu(x) \, d\mu(x), \qquad \lambda^+(A) := \int_A D_\mu^+ \nu(x) \, d\mu(x).$$

Fix $t > 1$ and let $A \in \mathcal{B}(\mathbb{R}^d)$ with $A \subset \operatorname{supp} \mu$ and $D_\mu^+ \nu(x) \in (0, \infty)$ for all $x \in A$. Now split A into the disjoint sets

$$A_k := \big\{ x \in A \ :\ D_\mu^+ \nu(x) \in (t^k, t^{k+1}] \big\} \subset \mathbb{R}^d \setminus E, \qquad k \in \mathbb{Z}.$$

Lemma 1.12 then implies

$$\lambda^+(A_k) \le t^{k+1}\mu(A_k) \le t\nu(A_k).$$

Using now the σ-additivity, we conclude that $\lambda^+(A) \le t(\nu \llcorner (\mathbb{R}^d \setminus E))(A)$. This then clearly also holds for any $A \in \mathcal{B}(\mathbb{R}^d)$ with $A \subset \mathbb{R}^d \setminus E$. Letting $t \downarrow 1$ and also using an analogous argument for λ^-, we arrive at

$$\lambda^+ \le \nu \llcorner (\mathbb{R}^d \setminus E) \le \lambda^-.$$

Hence, since $\lambda^- \le \lambda^+$ trivially, we obtain that

$$\mathrm{D}^-_\mu\nu(x) = \mathrm{D}^+_\mu\nu(x) = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) \qquad \text{for } \mu\text{-a.e. } x \in \operatorname{supp}\mu \setminus E$$

and $\nu \llcorner (\mathbb{R}^d \setminus E) = \lambda^\pm = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\,\mu$. □

The preceding theorem has a number of very well-known corollaries.

Theorem 1.14 (Radon–Nikodym). *Let $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ and $\nu \in \mathcal{M}(\mathbb{R}^d; V)$, where $(V, \|\cdot\|)$ is a normed vector space. Assume that ν is **absolutely continuous** with respect to μ, in symbols "$\nu \ll \mu$", which is defined to mean that*

$$\mu(A) = 0 \textit{ for } A \in \mathcal{B}(\mathbb{R}^d) \quad \textit{implies} \quad \|\nu\|(A) = 0.$$

Then,

$$\nu = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\mu$$

with the Radon–Nikodym derivative $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ defined in (1.1).

Proof. Observe that the absolute continuity implies $\operatorname{supp}\nu \subset \operatorname{supp}\mu$. Moreover, if we had $\|\nu\|(E) > 0$ for the set E defined in (1.2), then from Lemma 1.12 it would follow that $\mu(E) = 0$, so by absolutely continuity $\|\nu\|(E) = 0$, a contradiction. Thus, E is ν-negligible and the claim follows from the Besicovitch Differentiation Theorem 1.13. □

Lemma 1.15. *Let $\nu \in \mathcal{M}(\mathbb{R}^d; V)$. Then the **polar***

$$\frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|} \in \mathrm{L}^1(\|\nu\|; V)$$

satisfies

$$\left\| \frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|}(x) \right\| = 1 \qquad \textit{for } \|\nu\|\textit{-a.e. } x.$$

Proof. Since clearly $\nu \ll \|\nu\|$, the Radon–Nikodym derivative $h := \frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|}$ exists. By (1.1) in conjunction with Proposition 1.2 we then see that $\|h\| \leq 1$. Now estimate, for every $A \in \mathcal{B}(\mathbb{R}^d)$,

$$\|\nu\|(A) = \sup \sum_i \left\| \int_{A_i} h \, \mathrm{d}\|\nu\| \right\| \leq \sup \sum_i \int_{A_i} \|h\| \, \mathrm{d}\|\nu\| = \int_A \|h\| \, \mathrm{d}\|\nu\| \leq \|\nu\|(A),$$

where the supremum is over all finite Borel partitions $\{A_i\}_i$ of A. So, all the inequalities are in fact equalities and so $\|h\| = 1$ almsot everywhere with respect to $\|\nu\|$. □

Theorem 1.16 (Lebesgue differentiation theorem). *Let $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ and let $f \in \mathrm{L}^1(\mu)$, that is, $f\colon \mathbb{R}^d \to \mathbb{R}$ is μ-measurable and $\int |f| \, \mathrm{d}\mu < \infty$. Then, μ-almost every x_0 is a* ***Lebesgue point*** *of f with respect to μ, that is,*

$$\lim_{r \downarrow 0} ⨍_{B_r(x_0)} |f(x) - f(x_0)| \, \mathrm{d}\mu(x) := \lim_{r \downarrow 0} \frac{1}{\mu(B_r(x_0))} \int_{B_r(x_0)} |f(x) - f(x_0)| \, \mathrm{d}\mu(x) = 0.$$

Proof. Let $q \in \mathbb{Q}^d$ and apply the Besicovitch Differentiation Theorem 1.13 to $|f - q|\mu$. This yields

$$\lim_{r \downarrow 0} ⨍_{B_r(x_0)} |f(x) - q| \, \mathrm{d}\mu(x) = |f(x_0) - q|$$

for all $x_0 \in \operatorname{supp} \mu \setminus N_q$ with $\mu(N_q) = 0$. Thus, for $x_0 \in \operatorname{supp} \mu \setminus \bigcup_{q \in \mathbb{Q}^d} N_q$, we have

$$\begin{aligned} &\limsup_{r \downarrow 0} ⨍_{B_r(x_0)} |f(x) - f(x_0)| \, \mathrm{d}\mu(x) \\ &\quad \leq \limsup_{r \downarrow 0} ⨍_{B_r(x_0)} |f(x) - q| \, \mathrm{d}\mu(x) + |q - f(x_0)| \\ &\quad = 2|f(x_0) - q| \end{aligned}$$

for any $q \in \mathbb{Q}^d$. By the density of $\mathbb{Q}^d$ in $\mathbb{R}^d$ the right-hand side is zero and the result follows. □

Corollary 1.17. *Let $\mu \in \mathcal{M}^+(\mathbb{R}^d)$. Then, μ-almost every $x_0 \in \operatorname{supp} \mu$ is a* ***density point*** *of μ, that is,*

$$\lim_{r \to 0} \frac{\mu(B_r(x_0) \cap \operatorname{supp} \mu)}{\mu(B_r(x_0))} = 1.$$

Proof. Assuming without loss of generality that μ is finite, apply the preceding theorem to the function $f \equiv 1$. □

We now come to one of the fundamental theorems of Analysis, Rademacher's theorem. Because Lipschitz maps are ubiquitous in Geometric Measure Theory, Rademacher's theorem occupies a central place in the field.

Theorem 1.18 (Rademacher). *Let $f\colon \mathbb{R}^d \to \mathbb{R}^m$ be Lipschitz. Then, at $\mathcal{L}^d$-almost every $x \in \mathbb{R}^d$, the map f is differentiable in the classical sense (limit of difference-quotients). In particular, the* ***gradient*** *of f at x,*

$$\nabla f(x) = \left[\frac{\partial f^i}{\partial x_j}(x)\right]^i_j = \begin{bmatrix} \frac{\partial f^1}{\partial x_1}(x) & \cdots & \frac{\partial f^1}{\partial x_d}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x_1}(x) & \cdots & \frac{\partial f^m}{\partial x_d}(x), \end{bmatrix}$$

exists for such x.

Proof. Without loss of generality we may assume $m = 1$.

For $d = 1$, let $\mu \in \mathcal{M}(\mathbb{R}; \mathbb{R})$ be the signed measure uniquely defined via $\mu((-\infty, t)) := f(t)$ (note that the intervals $(-\infty, t)$ generate the σ-algebra on $\mathbb{R}$). Since $\mu([s, t)) = f(t) - f(s)$ for $s < t$, the measure μ is absolutely continuous with respect to $\mathcal{L}^1$. Then, by the Radon–Nikodym Theorem 1.14, there exists $g \in \mathrm{L}^1(\mathbb{R})$ with

$$f(t) - f(s) = \mu([s, t)) = \int_s^t g(\tau) \,\mathrm{d}\tau$$

for any $s < t$. Then, for $\mathcal{L}^1$-almost all t,

$$\frac{f(t+h) - f(t)}{h} = \fint_t^{t+h} g(\tau) \,\mathrm{d}\tau \to g(t)$$

as $h \to 0$ by the properties of integrals (or by a version of the Lebesgue Differentiation Theorem 1.16 for one-sided intervals). With $f' = g$, this shows the claim for $d = 1$.

For $d > 1$, fix a direction $v \in \mathbb{R}^d$ with $|v| = 1$ and denote by N_v the set of those $x \in \mathbb{R}^d$ for which the directional derivative $\partial_v f$ does not exist at x. It can be shown (exercise) that N_v is $\mathcal{L}^d$-measurable. By the one-dimensional case applied to $s \mapsto f(x + sv)$ for $x \in \mathbb{R}^d$, we get that (as a subset of $\mathbb{R}$)

$$\mathcal{L}^1\bigl(N_v \cap (x + \mathbb{R}v)\bigr) = 0.$$

Hence, Fubini's theorem yields that $\mathcal{L}^d(N_v) = 0$ and we conclude that $\partial_v f$ exists for $\mathcal{L}^d$-almost every $x \in \mathbb{R}^d$. Taking the union $N := N_{\mathrm{e}_1} \cup \cdots \cup N_{\mathrm{e}_d}$, we see that the gradient ∇f exists outside the exceptional set N.

Let $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ be arbitrary, $i \in \{1, \ldots, d\}$, and $x \notin N$. By passing to the limit $h \to 0$ in

$$\int_{\mathbb{R}^d} \frac{f(x + h\mathrm{e}_i) - f(x)}{h} \,\varphi(x) \,\mathrm{d}x = -\int_{\mathbb{R}^d} \frac{\varphi(x) - \varphi(x - h\mathrm{e}_i)}{h} \,f(x) \,\mathrm{d}x$$

via Lebesgue's dominated convergence theorem (we may restrict the integration to $\mathbb{R}^d \setminus N$), we further obtain the following integration-by-parts formula for the almost-everywhere defined derivative $\partial_{\mathrm{e}_i} f$:

$$\int_{\mathbb{R}^d} \varphi(x)\, \partial_{\mathrm{e}_i} f(x)\, \mathrm{d}x = -\int_{\mathbb{R}^d} \partial_{\mathrm{e}_i}\varphi(x)\, f(x)\, \mathrm{d}x. \tag{1.3}$$

Let $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^d)$. Then, for $x \in \mathbb{R}^d \backslash (N_v \cup N)$, $v \in \mathbb{R}^d$ with $|v| = 1$, and $h > 0$, we have by a change of variables, Lebesgue's dominated convergence theorem, and an integration by parts via (1.3),

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{f(x+hv) - f(x)}{h}\, \varphi(x)\, \mathrm{d}x &= -\int_{\mathbb{R}^d} \frac{\varphi(x) - \varphi(x-hv)}{h}\, f(x)\, \mathrm{d}x \\ &\to -\int_{\mathbb{R}^d} \partial_v \varphi(x)\, f(x)\, \mathrm{d}x \\ &= -\int_{\mathbb{R}^d} \sum_{i=1}^d \partial_{\mathrm{e}_i}\varphi(x) v_i\, f(x)\, \mathrm{d}x \\ &= \int_{\mathbb{R}^d} \sum_{i=1}^d \varphi(x)\, \partial_{\mathrm{e}_i} f(x) v_i\, \mathrm{d}x \\ &= \int_{\mathbb{R}^d} \varphi(x)\, \nabla f(x) v\, \mathrm{d}x.\end{aligned}$$

On the other hand,

$$\int_{\mathbb{R}^d} \frac{f(x+hv) - f(x)}{h}\, \varphi(x)\, \mathrm{d}x \to \int_{\mathbb{R}^d} \partial_v f(x)\, \varphi(x)\, \mathrm{d}x$$

As φ was arbitrary, this shows that the directional derivative $\partial_v f(x)$ exists and equals $\nabla f(x) v$ outside an $\mathcal{L}^d$-negligible set $\widetilde{N}_v$.

Now take a countable dense subset $\{v_n\}_n$ of the unit sphere in $\mathbb{R}^d$. It remains to show that f is (totally) differentiable at points $x \in \mathbb{R}^d \setminus \widetilde{N}$, where $\widetilde{N} := \bigcup_n \widetilde{N}_{v_n}$. We can estimate that

$$\left| \left(\frac{f(x+hv) - f(x)}{h} - \nabla f(x) v \right) - \left(\frac{f(x+hv') - f(x)}{h} - \nabla f(x) v' \right) \right| \le (d+1) L |v - v'|$$

for any $x \in \mathbb{R}^d \setminus \widetilde{N}$, $v, v' \in \mathbb{R}^d$ with $|v| = |\widetilde{v}| = 1$, and $h > 0$, where L is the Lipschitz constant of f. Then, a straightforward compactness argument (details as exercise) shows that

$$\frac{f(x+hv) - f(x)}{h} - \nabla f(x) v \to 0 \qquad \text{as } h \to 0,$$

uniformly in these v. Together with the above estimate, this proves the total differentiability (exercise). $\square$

1.3 Weak* convergence of measures

A sequence $(\mu_j)_j \subset \mathcal{M}^+(\mathbb{R}^d)$ **converges weakly*** in $\mathcal{M}^+(\mathbb{R}^d)$ to $\mu \in \mathcal{M}^+(\mathbb{R}^d)$, in symbols

$$\mu_j \overset{*}{\rightharpoonup} \mu,$$

if

$$\langle \psi, \mu_j \rangle \to \langle \psi, \mu \rangle$$

for all $\psi \in \mathrm{C}_0(\mathbb{R}^d)$, where

$$\langle \psi, \mu \rangle := \int \psi \, \mathrm{d}\mu.$$

Likewise, $(\nu_j)_j \subset \mathcal{M}(\mathbb{R}^d; V)$ converges weakly* in $\mathcal{M}(\mathbb{R}^d; V)$ to $\nu \in \mathcal{M}(\mathbb{R}^d; V)$ if for every $\psi \in \mathrm{C}_0(\mathbb{R}^d; V^*)$ it holds that

$$\langle \psi, \nu_j \rangle := \int \left\langle \frac{\mathrm{d}\nu_j}{\mathrm{d}\|\nu_j\|}, \psi \right\rangle \mathrm{d}\|\nu_j\| \to \langle \psi, \nu \rangle.$$

The main compactness result is the following:

Theorem 1.19 (Banach–Alaoglu). *Let $(\mu_j)_j \subset \mathcal{M}^+(\mathbb{R}^d)$ with $\sup_j \mu_j(\mathbb{R}^d) < \infty$. Then there exists a weakly* converging subsequence. Likewise, for $(\nu_j)_j \subset \mathcal{M}(\mathbb{R}^d; V)$ with $\sup_j \|\nu_j\|(\mathbb{R}^d) < \infty$ there is a weakly* converging subsequence.*

This follows from the usual (sequential) Banach–Alaoglu theorem via the Riesz Representation Theorem 1.4.

We also recall a useful convergence lemma:

Lemma 1.20. *Let $\mu_j \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}^d)$. Then for every lower semicontinuous function $g \colon \mathbb{R}^d \to [0, \infty]$ it holds that*

$$\int g \, \mathrm{d}\mu \le \liminf_{j \to \infty} \int g \, \mathrm{d}\mu_j,$$

and for every upper semicontinuous function $h \colon \mathbb{R}^d \to [0, \infty)$ with compact support it holds that

$$\int h \, \mathrm{d}\mu \ge \limsup_{j \to \infty} \int h \, \mathrm{d}\mu_j.$$

In particular, for $U \subset \mathbb{R}^d$ open and $K \subset \mathbb{R}^d$ compact,

$$\mu(U) \le \liminf_{j \to \infty} \mu_j(U) \qquad \textit{and} \qquad \mu(K) \ge \limsup_{j \to \infty} \mu_j(K).$$

A proof can be found in Lemma 1.62 of [4].

Two major convergence theorems for functionals defined on measures are the following:

Theorem 1.21 (Reshetnyak continuity theorem). *Let $g \in \mathrm{C}(\Omega \times \mathbb{S}^{N-1})$ be bounded. For any sequence $(\mu_j)_j \subset \mathcal{M}(\Omega;\mathbb{R}^N)$ with $\mu_j \to \mu$ strictly it holds that*

$$\int g\biggl(x, \frac{\mathrm{d}\mu_j}{\mathrm{d}\|\mu_j\|}(x)\biggr) \,\mathrm{d}\|\mu_j\|(x) \to \int g\biggl(x, \frac{\mathrm{d}\mu}{\mathrm{d}\|\mu\|}(x)\biggr) \,\mathrm{d}\|\mu\|(x).$$

Theorem 1.22 (Reshetnyak lower semicontinuity theorem). *Let $g\colon \Omega\times\mathbb{S}^{N-1} \to [0,\infty]$ be bounded, lower semicontinuous, and convex in the second argument. For any sequence $(\mu_j)_j \subset \mathcal{M}(\Omega;\mathbb{R}^N)$ with $\mu_j \overset{*}{\rightharpoonup} \mu$ it holds that*

$$\int_\Omega g\biggl(x, \frac{\mathrm{d}\mu}{\mathrm{d}\|\mu\|}(x)\biggr) \,\mathrm{d}\|\mu\|(x) \le \liminf_{j\to\infty} \int_\Omega g\biggl(x, \frac{\mathrm{d}\mu_j}{\mathrm{d}\|\mu_j\|}(x)\biggr) \,\mathrm{d}\|\mu_j\|(x).$$

A proof of the first result can be found as Theorem 13.3 in [30]. The proof of the second result is similar and can be found for instance in Theorem 2.38 of [4].

1.4 Hausdorff measure

Recall that the volume of the unit ball in $\mathbb{R}^k$ is given as

$$\omega_k := \frac{\pi^{k/2}}{\Gamma(k/2+1)},$$

where

$$\Gamma(z) := \int_0^\infty t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t, \qquad z > 0,$$

is the **Euler Γ-function**. We can also use the above formula to define the volume of a ball of *real* dimension $k \in [0,\infty)$.

Fix a "dimension" $k \in [0,\infty)$. Then, for any $\delta \in (0,\infty]$ and any set $A \subset \mathbb{R}^d$, we define

$$\mathcal{H}^k_\delta(A) := \frac{\omega_k}{2^k} \cdot \inf\biggl\{ \sum_n \mathrm{diam}(A_n)^k \;:\; A \subset \bigcup_n A_n,\; \mathrm{diam}(A_n) < \delta \biggr\},$$

where the infimum is over all finite or countable covers $\{A_n\}_n$ of A consisting of arbitrary sets $A_n \subset \mathbb{R}^d$. Here, the **diameter** is defined for a set $A \subset \mathbb{R}^d$ as

$$\mathrm{diam}(A) := \sup\{\, |x-y| \;:\; x,y \in A \,\}$$

with the convention $\mathrm{diam}(\emptyset) := 0$. Clearly, $\mathcal{H}^k_\delta(A)$ is increasing as $\delta \downarrow 0$. We may thus define the **k-dimensional Hausdorff (outer) measure** of a set $A \subset \mathbb{R}^d$ as the limit

$$\mathcal{H}^k(A) := \lim_{\delta\downarrow 0} \mathcal{H}^k_\delta(A) \in [0,\infty].$$

Example 1.23. For $k = 0$, we obtain the **counting measure** $\mathcal{H}^0$, which yields the number of elements in a set.

Remark 1.24. One can constrain the sets A_n to be convex, open or closed, and intersecting A, which is often useful. However, restricting the A_n to have a particular shape, e.g., balls or cubes, yields different measures. For instance, only using balls gives the so-called *spherical Hausdorff measure*, which is larger than $\mathcal{H}^k$ for some sets.

Remark 1.25. We have used the "Italian convention" to normalize the Hausdorff measure. There is also the "Finnish convention", which defines the Hausdorff measures without the prefactor $\omega_k/2^k$ (see, e.g., [26]). This avoids constants in several places at the expense that the Hausdorff measure does not agree with the usual notion of area or volume any longer. In any source one should always check which convention is used.

We will now investigate some properties of the Hausdorff measure.

Proposition 1.26. *For every $k \in [0, \infty)$ the k-dimensional Hausdorff measure $\mathcal{H}^k$ is a (positive) outer measure.*

Proof. The first property is trivial. Combining covers for the A_n it is straightforward to see that $\mathcal{H}^k_\delta$ is σ-subadditive for any $\delta > 0$. It is elementary to check that then also the limit as $\delta \downarrow 0$ is σ-subadditive, from which (ii) follows.

Let now $A, B \subset \mathbb{R}^d$ with $\delta_0 := \operatorname{dist}(A, B) > 0$. For $\delta < \delta_0/2$ any partition from the definition of $\mathcal{H}^k_\delta$ applied to $A \cup B$, splits into those A_n that intersect A and those that intersect B (the A_n intersection neither A nor B can be discarded). Thus,

$$\mathcal{H}^k_\delta(A \cup B) \geq \mathcal{H}^k_\delta(A) + \mathcal{H}^k_\delta(B).$$

Passing to the limit $\delta \downarrow 0$ and combining this with the other inequality from (ii), we obtain the metric additivity in property (iii). □

Thus we obtain directly via Carathéodory's Theorem 1.1 the following statement:

Proposition 1.27. *The restriction of $\mathcal{H}^k$ to the set of $\mathcal{H}^k$-measurable sets (which includes the Borel sets) is a positive measure.*

Lemma 1.28. *The k-dimensional Hausdorff measure in $\mathbb{R}^d$ is invariant with respect to translations, that is,*

$$\mathcal{H}^k(A) = \mathcal{H}^k(A + z)$$

for all $A \in \mathcal{B}(\mathbb{R}^d)$ and $z \in \mathbb{R}^d$. Moreover, for all $A \in \mathcal{B}(\mathbb{R}^d)$ and all $\lambda > 0$ it scales via

$$\mathcal{H}^k(\lambda A) = \lambda^k \mathcal{H}^k(A).$$

Proof. The first assertion is trivial and the second one follows directly from the scaling properties of $\mathrm{diam}(\cdot)^k$. □

Lemma 1.29. *Let $0 \le k < k' < \infty$. Then, for all $A \in \mathcal{B}(\mathbb{R}^d)$ it holds that*

$$\mathcal{H}^{k'}(A) > 0 \quad \textit{implies} \quad \mathcal{H}^k(A) = \infty$$

and

$$\mathcal{H}^k(A) < \infty \quad \textit{implies} \quad \mathcal{H}^{k'}(A) = 0.$$

Moreover, for $k > d$, the Hausdorff measure $\mathcal{H}^k$ is identically zero.

Proof. The first claim follows from the inequality

$$\mathcal{H}^{k'}_\delta(A) \le \frac{\omega_{k'}}{\omega_k} 2^{k-k'} \delta^{k'-k} \, \mathcal{H}^k_\delta(A),$$

for any $\delta > 0$. This holds since all sets for a partition that is admissible in $\mathcal{H}^{k'}_\delta$ have diameter at most δ. Since $k' - k > 0$, we may let $\delta \to 0$ to obtain the first and second claims.

For the third claim we observe that the unit cube $Q = (-1/2, 1/2)^d$ can be covered by n^d closed cubes with side length $1/n$ for any $n \in \mathbb{N}$. Thus,

$$\mathcal{H}^k_\delta(Q) \le \omega_k \left(\frac{\sqrt{d}}{2}\right)^k n^{d-k} \qquad \text{if } \delta < \frac{\sqrt{d}}{n}.$$

If $k > d$, this implies $\mathcal{H}^k(Q) = 0$. From the σ-subadditivity and translation-invariance of $\mathcal{H}^k$ we obtain that $\mathcal{H}^k$ is identically zero. □

Lemma 1.30. *Let $f\colon \mathbb{R}^d \to \mathbb{R}^m$ be Lipschitz with Lipschitz constant $L \ge 0$. Then, for all $A \in \mathcal{B}(\mathbb{R}^d)$ it holds that*

$$\mathcal{H}^k(f(A)) \le L^k \mathcal{H}^k(A).$$

In particular, if $\mathcal{H}^k(A) = 0$, then also $\mathcal{H}^k(f(A)) = 0$.

Proof. This is a direct consequence of the inequality $\mathrm{diam}(f(A)) \le L\,\mathrm{diam}(A)$ for any Borel set $A \in \mathcal{B}(\mathbb{R}^d)$. □

An important property of the Hausdorff measure is that it naturally agrees with Lebesgue measure in the case $k = d$:

Proposition 1.31. *For any Borel set $A \in \mathcal{B}(\mathbb{R}^d)$ it holds that*

$$\mathcal{L}^d(A) = \mathcal{H}^d(A).$$

For the proof we will need the following **isodiametric inequality**:

Proposition 1.32. *For any $\mathcal{L}^d$-measurable set $A \subset \mathbb{R}^d$ it holds that*

$$\mathcal{L}^d(A) \le \omega_d \left(\frac{\operatorname{diam}(A)}{2} \right)^d.$$

The proof of the isodiametric inequality will be part of the exercises.

Proof of Proposition 1.31. We first show the inequality $\mathcal{L}^d(A) \ge \mathcal{H}^d(A)$, for which we may assume that A is bounded (by writing it as a union of bounded sets). Repeating the argument of the third claim in Lemma 1.29, we see that $\mathcal{H}^d$ is finite on bounded sets, hence a positive (Radon) measure.

Fix now $\delta > 0$ and an open set $U \supset A$. By the Besicovitch–Vitali Covering Theorem 1.9 we can find a countable family of closed disjoint balls $\overline{B}_n = \overline{B}_{r_n}(x_n) \subset U$ with $r_n < \delta$ such that their union covers $\mathcal{H}^d$-almost all of A (and then also $\mathcal{H}^d_\delta$-almost all of A). Then, by the σ-subadditivity of $\mathcal{H}^d_\delta$,

$$\mathcal{H}^d_\delta(A) \le \sum_n \mathcal{H}^d_\delta(\overline{B}_n) \le \omega_d \sum_n r_n^d = \sum_n \mathcal{L}^d(\overline{B}_n) \le \mathcal{L}^d(U).$$

Hence, letting $\delta \downarrow 0$, we obtain $\mathcal{H}^d(A) \le \mathcal{L}^d(U)$ and we conclude by the outer regularity of $\mathcal{L}^d$ (see Proposition 1.3). This shows $\mathcal{L}^d(A) \ge \mathcal{H}^d(A)$.

For the other inequality $\mathcal{L}^d(A) \le \mathcal{H}^d(A)$, let $\delta > 0$ and take any finite or countable cover $\{A_n\}_n$ of A, that is, $A \subset \bigcup_n A_n$, with $\operatorname{diam}(A_n) < \delta$ for all n. Then, via the isodiametric inequality,

$$\mathcal{L}^d(A) \le \sum_n \mathcal{L}^d(A_n) \le \frac{\omega_d}{2^d} \sum_n \operatorname{diam}(A_n)^d.$$

Thus, $\mathcal{L}^d(A) \le \mathcal{H}^d_\delta(A)$ and we conclude again by letting $\delta \downarrow 0$. □

Remark 1.33. The proof also shows that $\mathcal{H}^d_\delta(A) = \mathcal{H}^d(A)$ for every $\delta > 0$ (even $\delta = \infty$).

1.5 Hausdorff dimension and densities

For a set $A \subset \mathbb{R}^d$ we define its **Hausdorff dimension** to be

$$\begin{aligned} \mathcal{H}\text{-}\dim(A) &:= \inf\{\, k \in [0,\infty) \,:\, \mathcal{H}^k(A) = 0 \,\} \\ &= \sup\{\, k \in [0,\infty) \,:\, \mathcal{H}^k(A) = \infty \,\} \in [0,d]. \end{aligned}$$

The equality and upper estimate follow directly from Lemma 1.29.

Example 1.34. Let $\gamma\colon [0,1] \to \mathbb{R}^d$ be a non-trivial Lipschitz curve. Then it can be shown by a simple covering argument that $\operatorname{img}\gamma$ has Hausdorff dimension 1 (details as exercise).

Example 1.35. Let $C_0 = [0,1]$ and inductively construct C_j from C_{j-1} ($j = 1, 2, \dots$) as follows: The set C_{j-1} consists of a collection of closed intervals. We split each of these intervals into three intervals of the same length, with the middle interval open and the other two intervals closed. Then, we obtain C_j by removing the middle (open) set. The **Cantor set** is then defined to be

$$C_\infty := \bigcap_{j \in \mathbb{N}} C_k.$$

We can estimate its Hausdorff dimension as follows: Every C_j is a cover for C_∞. So, for $\gamma \in (0,1)$ we have

$$\mathcal{H}^\gamma(C_\infty) \le \limsup_{j \to \infty} \frac{\omega_\gamma}{2^\gamma} \cdot 2^j \left(\frac{1}{3}\right)^{j \cdot \gamma}$$

because in C_j there are 2^j intervals of length 3^{-j}. For $\gamma > \ln 2/\ln 3$ the upper limit on the right-hand side is zero, for $\gamma = \ln 2/\ln 3$ it is finite, and for $\gamma < \ln 2/\ln 3$ it is infinity. Thus, the Hausdorff dimension of C_∞ is at most $\ln 2/\ln 3$. In fact, one can show (exercise) that $\dim_{\mathcal{H}} C_\infty = \ln 2/\ln 3 \approx 0.6309$.

Example 1.36. Let $\lambda \in (0, 1/2)$. The construction of the previous example can be generalized as follows: Instead of splitting each interval into three intervals of equal length in the inductive construction, at each stage of the construction we now prescribe the length of the kept intervals to be λa, where a is the length of the interval under consideration, and the length of the removed interval to be $(1-2\lambda)a$. For the resulting set $C_\infty(\lambda)$ we now obtain the estimate

$$\mathcal{H}^\gamma(C_\infty(\lambda)) \le \limsup_{j \to \infty} \frac{\omega_\gamma}{2^\gamma} \cdot 2^j \lambda^{j\gamma},$$

where by the Hausdorff dimension of $C_\infty(\lambda)$ can then be shown to be $\dim_{\mathcal{H}} C_\infty(\lambda) = \ln 2/\ln(1/\lambda)$.

In the following chapters we will often need to integrate maps $f \colon \mathbb{R}^d \to V$ with V a finite-dimensional normed vector space with respect to a Hausdorff measure $\mathcal{H}^k$, where $k \in [0, \infty)$. Usually, integration is only defined for at least σ-finite measures, which $\mathcal{H}^k$ is not (unless $k > d$). This can, however, be fixed easily: Let V be a normed finite-dimensional vector space. We call a Borel-measurable map $f \colon \mathbb{R}^d \to V$ **$\mathcal{H}^k$-integrable** if the restriction $\mathcal{H}^k \llcorner \{f \neq 0\}$ of $\mathcal{H}^k$ to the set $Y_f := \{\, x \in \mathbb{R}^d \ : \ f(x) \neq 0 \,\}$ (that is, $\mathcal{H}^k \llcorner \{f \neq 0\}(A) := \mathcal{H}^k(A \cap \{f \neq 0\})$ for any Borel set A) is σ-finite and the now-defined integral

$$\int_{\{x \,:\, f \neq 0\}} |f(x)| \, \mathrm{d}\mathcal{H}^k(x)$$

is finite. In this case, we may define

$$\int_{\mathbb{R}^d} f \, \mathrm{d}\mathcal{H}^k := \int_{\{f \neq 0\}} f \, \mathrm{d}\mathcal{H}^k.$$

Let $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ and let $k \in [0, \infty)$. We define the **upper k-density** of μ at $x_0 \in \mathbb{R}^d$ as

$$\Theta_k^*(\mu, x_0) := \limsup_{r \downarrow 0} \frac{\mu(B_r(x_0))}{\omega_k r^k}. \tag{1.4}$$

If the $\limsup$ can be replaced by a proper limit, we denote the resulting **k-density** of μ at $x_0 \in \mathbb{R}^d$ by $\Theta_k(\mu, x_0)$. The following *density lemma* is often useful.

Lemma 1.37. *Let $\mu \in \mathcal{M}^+(\mathbb{R}^d)$, let $A \in \mathcal{B}(\mathbb{R}^d)$ be a Borel set, and let $t > 0$. If $\Theta_k^*(\mu, x) \geq t$ for all $x \in A$, then*

$$\mu \llcorner A \geq t\mathcal{H}^k \llcorner A. \tag{1.5}$$

In particular, if $\mu(A) = 0$, then $\Theta_k^*(\mu, x) = \Theta(\mu, x) = 0$ for $\mathcal{H}^k$-almost every $x \in A$.

Proof. Without loss of generality we assume $k > 0$ and that A is bounded. Let $U \supset A$ be open and bounded, and fix $\delta \in (0, t)$ and $r_0 < \delta/2$. We define the set $\mathcal{F}_\delta$ to contain all closed balls $\overline{B}_r(x) \subset U$ for $x \in A$, $r < r_0$, such that

$$\mu(\overline{B}_r(x)) \geq (t - \delta)\omega_k r^k, \qquad \mu(\overline{B}_r(x) \setminus B_r(x)) = 0.$$

Note here that the second condition can always be achieved since the finiteness of μ on compact sets entails that $\mu(\overline{B}_r(x) \setminus B_r(x)) > 0$ for at most countably many r's.

By the Besicovitch Covering Theorem 1.8 we can select a countable subfamily $(\overline{B}_n)_n$ of the balls in $\mathcal{F}_\delta$ that covers A and such that any point in A is contained in at most β_d balls $\overline{B}_n = \overline{B}_{r_n}(x_n)$. Then,

$$\mathcal{H}_\delta^k(A) \leq \sum_n \omega_k r_n^k \leq \frac{1}{t - \delta} \sum_n \mu(\overline{B}_n) \leq \frac{\beta_d}{t - \delta} \mu(U) < \infty.$$

Letting $\delta \downarrow 0$, we have identified $\mathcal{H}^k \llcorner A$ as a finite measure. We can now apply the Besicovitch–Vitali Covering Theorem with the fine cover $\mathcal{F}_\delta$ and select a countable *disjoint* subfamily $(\overline{B}_l)_l$ of $\mathcal{F}_\delta$ consisting of closed ball $\overline{B}_l = \overline{B}_{r_l}(x_l)$ that covers $\mathcal{H}^k$-almost all of A. Hence, we estimate analogously to before

$$\mathcal{H}_\delta^k(A) \leq \sum_l \omega_k r_l^k \leq \frac{1}{t - \delta} \sum_l \mu(\overline{B}_l) \leq \frac{1}{t - \delta} \mu(U)$$

Letting $\delta \downarrow 0$ and then letting U approximate A from the outside (via Proposition 1.3), we obtain $\mu(A) \geq t\mathcal{H}^k(A)$. The same argument applies to Borel subsets of A, whereby we arrive at (1.5). The additional claim is a direct consequence. $\square$

Remark 1.38. There is also a statement about the opposite inequality: If $\Theta^*_k(\mu, x) \leq t$ for all $x \in A$, then $\mu \llcorner A \leq 2^k t \mathcal{H}^k \llcorner A$ (exercise). While for $k \in [0, 1]$ the factor 2^k is optimal, it seems to be unknown if this is the case also for $k > 1$ (some people conjecture the factor 2).

Chapter 2

Rectifiability

The measure-theoretic generalization of a smooth k-dimensional surface (i.e., a manifold) in $\mathbb{R}^d$, possibly with boundary or only immersed (not embedded), is the notion of "$\mathcal{H}^k$-rectifiable" sets. One can think of them as being parameterizable, locally at least, by Lipschitz maps. As in the case of the area and coarea formulas, Lipschitz-type regularity turns out to be just right: For instance, $\mathcal{H}^k$-rectifiable sets still have tangent planes $\mathcal{H}^k$-almost everywhere and can thus still be thought of as locally linearizable (unlike fractal sets, which do not have such approximating tangents). Moreover, there are versions of the area and coarea formula on rectifiable sets. In this chapter, we will explore the properties of these $\mathcal{H}^k$-rectifiable sets, tangential differentiability of Lipschitz maps, and the area and coarea formula on rectifiable sets.

2.1 Area formula

Before we can talk about rectifiable sets, we need to introduce two cornerstones of Geometric Measure Theory, the area and coarea formulas. The first can be seen as a version of the transformation formula for integrals that also applies to lower-dimensional sets. The second is a far-reaching generalization of Fubini's theorem. The basic ideas leading to these formulas are easy to understand for polytopes, where they correspond to elementary geometric observations. The technical part consists of extending these concepts to more general geometric objects via an approximation procedure for Lipschitz maps by piecewise linear ones.

We call a set $P \subset \mathbb{R}^d$ a **k-dimensional parallelotope (anchored at the origin)** if there are $k \in \mathbb{N}$ linearly independent vectors $v_1, \ldots, v_k \in \mathbb{R}^d$ with

$$P = \left\{ \sum_{j=1}^{k} \theta_j v_j \ : \ \theta_j \in [0,1], \ j = 1, \ldots, k \right\}. \tag{2.1}$$

The following can be proved (exercise) using elementary geometry (use Cavalieri's principle and reduce to orthogonal vectors v_j):

Lemma 2.1. *The k-dimensional volume of a parallelotope P as in* (2.1) *is*

$$\sqrt{\det(V^\mathsf{T} V)}, \qquad V := (v_1 \cdots v_k) \in \mathbb{R}^{d \times k},$$

that is, V contains the vectors $v_1, \ldots, v_k$ as its columns.

Let $f\colon \mathbb{R}^k \to \mathbb{R}^m$ be differentiable at $x \in \mathbb{R}^k$. If $k \le m$, define the **k-dimensional Jacobian** of f at x via

$$\mathbf{J}_k f(x) := \sqrt{\det(\nabla f(x)^\mathsf{T} \nabla f(x))}.$$

If $k \ge m$ define the **m-dimensional Jacobian $\mathbf{J}_m f$** via

$$\mathbf{J}_m f(x) := \sqrt{\det(\nabla f(x) \nabla f(x)^\mathsf{T})}.$$

Note that if $k = m$, then both definitions agree and reduce to

$$\mathbf{J}_k f(x) = |\det \nabla f(x)|,$$

which is the factor in the classical transformation formula for integrals.

The first version of the area formula concerns maps defined on the flat space $\mathbb{R}^k$. We will see another, more general, version in the next chapter.

Theorem 2.2 (Area formula). *Let $f\colon \mathbb{R}^k \to \mathbb{R}^m$ be Lipschitz with $k \le m$. Then, for every $\mathcal{L}^k$-measurable map $g\colon \mathbb{R}^k \to \mathbb{R}$, it holds that*

$$\int_{\mathbb{R}^k} g(x)\, \mathbf{J}_k f(x) \,\mathrm{d}x = \int_{\mathbb{R}^m} \sum_{x \in f^{-1}(y)} g(x) \,\mathrm{d}\mathcal{H}^k(y).$$

In particular, if $A \subset \mathbb{R}^k$ is $\mathcal{L}^k$-measurable,

$$\int_A \mathbf{J}_k f(x) \,\mathrm{d}x = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(y)) \,\mathrm{d}\mathcal{H}^k(y) \tag{2.2}$$

and, if f is injective,

$$\int_A \mathbf{J}_k f(x) \,\mathrm{d}x = \mathcal{H}^k(f(A)). \tag{2.3}$$

By the Rademacher Theorem 1.18 we know that f is differentiable $\mathcal{L}^k$-almost everywhere, and so also $\mathbf{J}_k f$ exists $\mathcal{L}^k$-almost everywhere. Hence, the above integrals are well-defined. Note that the integral with respect to $\mathcal{H}^k$ on the right-hand side is understood in the sense explained at the end of Section 1.4.

In order to show the area formula, it suffices to prove (2.2). The general case then follows by approximation with step functions (details as exercise). Note that the statement of (2.2) includes the claim that $y \mapsto \mathcal{H}^0(A \cap f^{-1}(y))$ is $\mathcal{H}^k$-measurable. We will accomplish the proof of (2.2) in several steps, starting from the special case of f being a linear map and then extending to a general Lipschitz map f by approximation.

Proof of Theorem 2.2 for linear maps. 0 We first show the result in the case where $f(x) = Lx$ is linear for a matrix $L \in \mathbb{R}^{m \times k}$. A standard result from linear algebra yields that L has a **polar decomposition**

$$L = OS$$

with $O \in \mathbb{R}^{m \times k}$ orthogonal ($O^\mathsf{T} O = \mathrm{Id}_k$), and $S \in \mathbb{R}^{k \times k}$ symmetric ($S^\mathsf{T} = S$) and positive (semi-)definite ($v^\mathsf{T} S v \geq 0$ for all $v \in \mathbb{R}^k$). We have that $L^\mathsf{T} L = S^2$, whereby

$$\mathbf{J}_k L = \det S \quad (\geq 0).$$

Let π be a k-dimensional subspace of $\mathbb{R}^m$ containing the image of L. Denote by $\mathcal{H}^k_\pi$ the restriction of $\mathcal{H}^k$ to π, for which it holds that $\mathcal{H}^k_\pi(B) = \mathcal{H}^k(B)$ for any Borel set $B \subset \pi$ (details as an exercise, for which one uses $\dim \pi = k$). Moreover, the Hausdorff measure is obviously invariant under orthogonal transformations (isometries), so we have

$$\mathcal{H}^k(L(A)) = \mathcal{H}^k_\pi(L(A)) = \mathcal{H}^k(S(A)).$$

Using also Proposition 1.31 as well as the transformation formula for Lebesgue integrals, we obtain (in $\mathbb{R}^k$)

$$\mathcal{H}^k(S(A)) = \mathcal{L}^k(S(A)) = \det S \cdot \mathcal{L}^k(A) = \int_A \mathbf{J}_k L \,\mathrm{d}x$$

This already yields the claim in the case that f is linear: Assume first that S has full rank and thus that f is injective, whereby

$$\mathcal{H}^k(L(A)) = \int_A \mathbf{J}_k L \,\mathrm{d}x.$$

This is (2.3), which is equivalent to (2.2) in the injective case. Otherwise, S has rank less than k, whence $\det S = 0$ and $\mathcal{H}^k(S(A)) = 0$ (because the image is contained in a subspace of dimension at most $k - 1$). So, both sides of (2.2) are zero. □

The first lemma in the procedure to extend the area formula to Lipschitz maps is the following version of the *Sard theorem*:

Lemma 2.3. *Let $f\colon \mathbb{R}^k \to \mathbb{R}^m$ be Lipschitz with $k \leq m$. Then, for*

$$Z := \{\, x \in \mathbb{R}^k \ :\ \mathbf{J}_k f(x) = 0 \,\},$$

where the existence of $\mathbf{J}_k f(x)$ is part of the condition, it holds that $\mathcal{H}^k(f(Z)) = 0$.

Proof. Fix $\varepsilon \in (0,1)$. For every $x \in Z$ we infer from Taylor's theorem that

$$f(B_r(x)) \subset f(x) + \nabla f(x) B_r + B_{\varepsilon r} \tag{2.4}$$

for $r > 0$ sufficiently small (here B_r is the open unit ball with radius $r > 0$). Since $\nabla f(x) B_r$ is contained in a $(k-1)$-plane π, we can use a cover consisting of cubes with one face parallel to π and side length εr to estimate

$$\mathcal{H}^k_\infty(\nabla f(x) B_r + B_{\varepsilon r}) \leq C \varepsilon r^k$$

with a constant $C > 0$ that only depends on the dimensions and the Lipschitz constant of f (the easy geometric details are an exercise).

Now use the Besicovitch–Vitali Covering Theorem 1.9 (and Remark 1.10) for $\mathcal{L}^k$ and cover Z with countably many disjoint (open) balls $B_{r_n}(x_n)$ such that (2.4) holds. Then,

$$\begin{aligned}
\mathcal{H}^k_\infty(f(Z)) &\leq \sum_n \mathcal{H}^k_\infty(f(B_{r_n}(x_n))) \\
&\leq \sum_n \mathcal{H}^k_\infty(\nabla f(x_n) B_{r_n} + B_{\varepsilon r_n}) \\
&\leq C\varepsilon \sum_n r_n^k \\
&\leq C\varepsilon \mathcal{L}^k(Z + B_1),
\end{aligned}$$

where the constant changes in the last line. Then let $\varepsilon \downarrow 0$ to conclude that $\mathcal{H}^k_\infty(f(Z)) = 0$. Hence also (exercise) $\mathcal{H}^k(f(Z)) = 0$. □

The following lemma shows that we may approximate a Lipschitz function well on the on the set where the Jacobian is positive.

Lemma 2.4. *Let $f \colon \mathbb{R}^k \to \mathbb{R}^m$ be Lipschitz with $k \leq m$, let $\beta > 1$, and let $A \subset \mathbb{R}^k$ be $\mathcal{L}^k$-measurable with*

$$\mathbf{J}_k f > 0 \quad \mathcal{L}^k\text{-a.e. on } A.$$

Then, $\mathcal{L}^k$-almost all of A can be covered with countably many disjoint compact sets $A_n \subset \mathbb{R}^k$ with the following properties for every n:

(i) f is differentiable and $\mathbf{J}_k f > 0$ everywhere on A_n.

(ii) The restriction $f|_{A_n}$ is injective with a Lipschitz inverse on its image.

(iii) There is a matrix $L_n \in \mathbb{R}^{m \times k}$ with $\mathbf{J}_k L_n > 0$ and the Lipschitz constants of the compound maps $f|_{A_n} \circ L_n^{-1}$, $L_n \circ (f|_{A_n})^{-1}$ are bounded from above by β (where we have considered L_n as an invertible linear map from $\mathbb{R}^k$ to its image).

(iv) $\beta^{-k}\mathbf{J}_k L_n \leq \mathbf{J}_k f \leq \beta^k \mathbf{J}_k L_n$ *on* A_n.

Proof. Removing a $\mathcal{L}^k$-negligible set, we can assume that ∇f exists and $\mathbf{J}_k f > 0$ everywhere on A. Moreover, since any $\mathcal{L}^k$-measurable set is equal to a Borel set in union with an $\mathcal{L}^k$-negligible set, we may further assume without loss of generality that A is a Borel set.

Fix $\varepsilon > 0$ with $\beta^{-1} + \varepsilon < 1 < \beta - \varepsilon$, and a countable dense subset $\{x_i\}$ of A as well as a countable dense subset $\{L_j\}$ of the collection of matrices $L \in \mathbb{R}^{m\times m}$ with $\mathbf{J}_k L > 0$. Define $Y_{ij}(r)$ to be the set of all $y \in B_r(x_i)$ such that the following two conditions hold:

$$(\beta^{-1} + \varepsilon)|L_j v| \leq |\nabla f(y)v| \leq (\beta - \varepsilon)|L_j v| \qquad \text{for all } v \in \mathbb{R}^k,$$
$$|f(z) - f(y) - \nabla f(y)(z-y)| \leq \varepsilon |L_j(z-y)| \qquad \text{for all } z \in B_{2r}(y).$$

For every $y \in A$ by density we can choose a j (depending on y) such that the Lipschitz constant of the compound map $\nabla f(y) \circ L_j^{-1}$ is bounded from above by $\beta - \varepsilon$ and the Lipschitz constant of $L_j \circ (\nabla f(y))^{-1}$ is bounded from above by $(\beta^{-1}+\varepsilon)^{-1}$. Now pick $i, m \in \mathbb{N}$ suitably such that $y \in B_{1/m}(x_i)$ and

$$|f(z) - f(y) - \nabla f(y)(z-y)| \leq \varepsilon |L_j(z-y)| \qquad \text{for all } z \in B_{2/m}(y).$$

This is possible, since the right-hand side is of order $\mathrm{o}(1/m)|z-y|$, hence we may bound it by $\varepsilon|L_j(z-y)|$ for m sufficiently large (note that L_j is invertible on its image). Thus, $y \in Y_{ij}(1/m)$ and hence the sets $\{Y_{ij}(1/m)\}_{ijm}$ form a cover of A.

For $y \in Y_{ij}(r)$ and $z \in B_{2r}(y)$ we may also combine the above conditions (use $v := z - y$) to obtain

$$\beta^{-1}|L_j(z-y)| \leq |f(z) - f(y)| \leq \beta|L_j(z-y)|.$$

Note also that $Y_{ij}(1/m) \subset B_{1/m}(x_i) \subset B_{2/m}(y)$ for any $y \in Y_{ij}(1/m)$. As a consequence, f is injective with a Lipschitz inverse on $Y_{ij}(1/m)$.

We can thus construct from the $Y_{ij}(1/m)$ disjoint Borel sets F_n whose union covers A and, using the inner regularity of $\mathcal{L}^k$, see Proposition 1.3, we obtain the desired collection $\{A_n\}$ satisfying (i)–(iii).

To show (iv) we employ once more the polar decomposition: Write, for fixed $x \in A_n$,

$$\nabla f(x) = OS, \qquad L_n = O_n S_n$$

with $O, O_n \in \mathbb{R}^{m\times k}$ orthogonal and $S, S_n \in \mathbb{R}^{k\times k}$ symmetric and positive definite. Then,

$$\beta^{-1}|O_n S_n v| \leq |OSv| \leq \beta|O_n S_n v|, \qquad v \in \mathbb{R}^k.$$

Hence,

$$\beta^{-1}|w| \leq |SS_n^{-1}w| \leq \beta|w| \qquad \text{for all } w \text{ in the image of } S_n.$$

This implies $\beta^{-k} \leq \det(SS_n^{-1}) \leq \beta^k$, from which (iv) follows. □

We are now ready to utilize these lemmas to complete our proof of the area formula.

Proof of Theorem 2.2. As explained before, it suffices to show (2.2).

By Rademacher's Theorem 1.18 we may assume that everywhere on A the gradient ∇f exists. Moreover, by Lemma 2.3 we see that for $Z := \{\mathbf{J}_k f = 0\}$ it holds that

$$\mathcal{H}^0(Z \cap f^{-1}(y)) = 0 \qquad \text{for } \mathcal{H}^k\text{-a.e. } y \in \mathbb{R}^m.$$

So, we may also assume that $\mathbf{J}_k f > 0$ everywhere on A.

Via Lemma 2.4 we obtain for any fixed $\beta > 1$ a countable collection of disjoint compact sets $A_n \in \mathcal{B}(\mathbb{R}^k)$ covering $\mathcal{L}^k$-almost all of A, and matrices $L_n \in \mathbb{R}^{m\times m}$ with $\mathbf{J}_k L_n > 0$ such that

(i) f is differentiable and $\mathbf{J}_k f > 0$ everywhere on A_n.

(ii) The restriction $f|_{A_n}$ is injective with a Lipschitz inverse on its image.

(iii) The Lipschitz constants of the compound maps $f|_{A_n} \circ L_n^{-1}$, $L_n \circ (f|_{A_n})^{-1}$ are bounded from above by β.

(iv) $\beta^{-k}\mathbf{J}_k L_n \le \mathbf{J}_k f \le \beta^k \mathbf{J}_k L_n$ on A_n.

We first observe that for $\mathcal{H}^k$-a.e. $y \in \mathbb{R}^m$ it holds that

$$\mathcal{H}^0(A \cap f^{-1}(y)) = \sum_n \mathcal{H}^0(A_n \cap f^{-1}(y)). \tag{2.5}$$

Since the A_n are compact, so are the sets $f(A_n)$, and thus the functions $y \mapsto \mathcal{H}^0(A_n \cap f^{-1}(y)) = \mathbb{1}_{f(A_n)}(y)$ are upper semicontinuous, hence $\mathcal{H}^k$-measurable. Then also the sum in (2.5) is $\mathcal{H}^k$- measurable. This implies that $y \mapsto \mathcal{H}^0(A \cap f^{-1}(y))$ (which agrees with the right-hand side $\mathcal{H}^k$-almost everywhere) is $\mathcal{H}^k$-measurable.

It remains to prove (2.2). By (iii) and Lemma 1.30 we observe

$$\begin{aligned}\mathcal{H}^k(f(A_n)) &= \mathcal{H}^k\big(f|_{A_n} \circ L_n^{-1} \circ L_n\big) \le \beta^k \mathcal{H}^k(L_n(A_n)),\\ \mathcal{H}^k(L_n(A_n)) &= \mathcal{H}^k\big(L_n \circ (f|_{A_n})^{-1} \circ f|_{A_n}\big) \le \beta^k \mathcal{H}^k(f(A_n)).\end{aligned}$$

We now estimate for every n, using the area formula for linear maps and (iv),

$$\beta^{-2k}\mathcal{H}^k(f(A_n)) \le \beta^{-k}\mathcal{H}^k(L_n(A_n)) = \beta^{-k}\int_{A_n} \mathbf{J}_k L_n \,\mathrm{d}x \le \int_{A_n} \mathbf{J}_k f(x)\,\mathrm{d}x.$$

Similarly,

$$\int_{A_n} \mathbf{J}_k f(x)\,\mathrm{d}x \le \beta^k \int_{A_n} \mathbf{J}_k L_n\,\mathrm{d}x = \beta^k \mathcal{H}^k(L_n(A_n)) \le \beta^{2k}\mathcal{H}^k(f(A_n)).$$

Now sum over n and use (2.5) as well as the injectivity of f restricted to A_n, to obtain

$$\begin{aligned}
\beta^{-2k} \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(y)) \, \mathrm{d}\mathcal{H}^k(y) &= \beta^{-2k} \sum_n \mathcal{H}^k(f(A_n)) \\
&\le \int_A \mathbf{J}_k f(x) \, \mathrm{d}x \\
&\le \beta^{2k} \sum_n \mathcal{H}^k(f(A_n)) \\
&= \beta^{2k} \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(y)) \, \mathrm{d}\mathcal{H}^k(y).
\end{aligned}$$

Letting $\beta \downarrow 1$, we arrive at (2.2) and the proof of the area formula is complete. □

Example 2.5. Let $\varphi \colon U \to \mathbb{R}$ be a Lipschitz map, where $U \subset \mathbb{R}^k$ is open. To compute the (k-)area of the graph of φ over U, namely

$$\Gamma := \{ (x, \varphi(x)) \ : \ x \in U \},$$

we can use the area formula: Set $f(x) := (x, \varphi(x))$, which is a Lipschitz map from U to $\mathbb{R}^{k+1}$. Then,

$$\mathcal{H}^k(\Gamma) = \mathcal{H}^k(f(U)) = \int_U \mathbf{J}_k f(x) \, \mathrm{d}x = \int_U \mathbf{T}_k \varphi(x) \, \mathrm{d}x.$$

with the **metric tensor**

$$\mathbf{T}_k \varphi(x) := \sqrt{1 + \sum_{M \text{ minor}} M(\nabla \varphi(x))^2},$$

where the summation is over the minors (i.e., determinants of square submatrices) of any order. The last equality follows from the Cauchy–Binet theorem, which asserts that

$$\mathbf{J}_k L = \sqrt{\sum_{M \ k\text{-minor}} M(L)^2},$$

where a k-minor is any $(k \times k)$-subdeterminant in orthogonal coordinates of the linear map L. Its verification belongs to the domain of linear algebra and can for instance be found in Proposition 2.69 of [4].

Example 2.6. If $k = 1$ in the previous example, we obtain that for an injective Lipschitz map $\varphi \colon (a, b) \to \mathbb{R}$ ($a < b$), the length of the graph is

$$\int_a^b \sqrt{1 + \varphi'(t)^2} \, \mathrm{d}t,$$

in agreement with a classical fact for continuously differentiable φ. The injectivity can also be dropped if we count overlapping parts with their multiplicity.

2.2 Coarea formula

The coarea formula is a generalization of Fubini's theorem from flat slices to level sets of a Lipschitz map.

Theorem 2.7 (Coarea formula). *Let $f\colon \mathbb{R}^k \to \mathbb{R}^m$ be Lipschitz with $k \geq m$. Then, for every $\mathcal{L}^k$-measurable map $g\colon \mathbb{R}^k \to \mathbb{R}$, it holds that*

$$\int_{\mathbb{R}^k} g(x)\,\mathbf{J}_m f(x)\,\mathrm{d}x = \int_{\mathbb{R}^m} \int_{f^{-1}(y)} g(x)\,\mathrm{d}\mathcal{H}^{k-m}(x)\,\mathrm{d}y.$$

In particular, if $A \subset \mathbb{R}^k$ is $\mathcal{L}^k$-measurable,

$$\int_A \mathbf{J}_m f(x)\,\mathrm{d}x = \int_{\mathbb{R}^m} \mathcal{H}^{k-m}(A \cap f^{-1}(y))\,\mathrm{d}y. \tag{2.6}$$

Like in the proof of the area formula, we only need to show (2.6) and then conclude the full theorem via an approximation with step functions. Furthermore, we again first prove the coarea formula for linear f, where it is essentially a corollary to Fubini's theorem.

Proof of Theorem 2.7 for linear maps. Let $f(x) = Lx$ for a matrix $L \in \mathbb{R}^{m\times k}$ with polar decomposition $L = SO^\mathsf{T}$, where $O \in \mathbb{R}^{k\times m}$ is orthogonal ($O^\mathsf{T}O = \mathrm{Id}_m$) and $S \in \mathbb{R}^{m\times m}$ symmetric and positive semidefinite. Consequently,

$$\mathbf{J}_m L = \det S.$$

If $\det S = 0$, then (2.6) holds trivially since the integrand on the right-hand side can be non-zero only on an $\mathcal{L}^m$-negligible set since the image of A under f is $\mathcal{L}^m$-negligible by Lemma 2.1 (or the area formula).

So, assume $\det S > 0$, whereby f is surjective onto $\mathbb{R}^m$ and thus we may change variables from $y \in \mathbb{R}^m$ to $z = S^{-1}y \in \mathbb{R}^m$ (via the classical transformation formula) to obtain

$$\begin{aligned}\int_{\mathbb{R}^m} \mathcal{H}^{k-m}(A \cap L^{-1}(y))\,\mathrm{d}y &= \int_{\mathbb{R}^m} \mathcal{H}^{k-m}\big(A \cap [O^\mathsf{T}]^{-1}(S^{-1}y)\big)\,\mathrm{d}y \\ &= \det S \cdot \int_{\mathbb{R}^m} \mathcal{H}^{k-m}\big(A \cap [O^\mathsf{T}]^{-1}(z)\big)\,\mathrm{d}z.\end{aligned}$$

Here we use the notation $L^{-1}(y)$ for the $(k-m)$-plane of all $x \in \mathbb{R}^k$ with $Lx = z$ and likewise $[O^\mathsf{T}]^{-1}(z)$ for the $(k-m)$-plane of all $x \in \mathbb{R}^k$ with $O^\mathsf{T}x = z$. We remark that the measurability of the right-hand side is a consequence of the transformation formula.

Furthermore, by Fubini's theorem (more precisely, Cavalieri's principle with respect to the $(k-m)$-planes $[O^\mathsf{T}]^{-1}(z)$ for $z \in \mathbb{R}^m$),

$$\int_{\mathbb{R}^m} \mathcal{H}^{k-m}\big(A \cap [O^\mathsf{T}]^{-1}(z)\big)\,\mathrm{d}z = \mathcal{L}^k(A).$$

Combining,

$$\int_{\mathbb{R}^m} \mathcal{H}^{k-m}(A \cap L^{-1}(y))\,\mathrm{d}y = \det S \cdot \mathcal{L}^k(A) = \int_A \mathbf{J}_m f(x)\,\mathrm{d}x.$$

The proof in the linear case is thus complete. □

In order to show the coarea formula in full generality, we again need an approximation procedure for Lipschitz maps, contained in the following lemmas.

Lemma 2.8. *Let $f\colon \mathbb{R}^k \to \mathbb{R}^m$ be Lipschitz with $k \geq m$ and let $N \subset \mathbb{R}^k$ be $\mathcal{H}^k$-negligible (equivalently, $\mathcal{L}^k$-negligible). Then,*

$$\mathcal{H}^{k-m}(N \cap f^{-1}(y)) = 0 \qquad \textit{for } \mathcal{L}^m\textit{-a.e. } y \in \mathbb{R}^m.$$

Proof. Fix $\varepsilon > 0$. Since N is $\mathcal{H}^k$-negligible, there are (at most) countably many sets $A_n \subset \mathbb{R}^k$ such that $N \subset \bigcup_n A_n$, $\operatorname{diam}(A_n) < \varepsilon$ for all n, and

$$\sum_n \operatorname{diam}(A_n)^k < \varepsilon.$$

We may furthermore assume without loss of generality that the A_n are all compact. Now set for $y \in \mathbb{R}^m$,

$$h_\varepsilon(y) := \sum_n \left(\frac{\operatorname{diam}(A_n)}{2}\right)^{k-m} \mathbb{1}_{f(A_n)}(y).$$

Then,

$$\mathcal{H}^{k-m}_\varepsilon(N \cap f^{-1}(y)) \leq \omega_{k-m} h_\varepsilon(y), \qquad y \in \mathbb{R}^m.$$

By the isodiametric inequality, Proposition 1.32, and Lemma 1.30,

$$\begin{aligned}
\int_{\mathbb{R}^m} h_\varepsilon(y)\,\mathrm{d}y &= \sum_n \left(\frac{\operatorname{diam}(A_n)}{2}\right)^{k-m} \mathcal{L}^m(f(A_n)) \\
&\leq L^m \sum_n \left(\frac{\operatorname{diam}(A_n)}{2}\right)^{k-m} \mathcal{L}^m(A_n) \\
&\leq C \sum_n \operatorname{diam}(A_n)^k \\
&< C\varepsilon,
\end{aligned}$$

where L is the Lipschitz constant of f, which is later absorbed into a dimensional constant $C > 0$. So, $h_\varepsilon \to 0$ in L^1 and we may find a sequence $\varepsilon_j \downarrow 0$ so that $h_{\varepsilon_j} \to 0$ almost everywhere. Thus, for $\mathcal{H}^m$-almost every $y \in \mathbb{R}^m$,

$$\mathcal{H}^{k-m}_{\varepsilon_j}(N \cap f^{-1}(y)) \leq \omega_{k-m} h_{\varepsilon_j}(y) \to 0$$

as $j \to \infty$, yielding the claim. □

Lemma 2.9. *Let $f\colon \mathbb{R}^k \to \mathbb{R}^m$ be Lipschitz with $k \geq m$, and let $A \subset \mathbb{R}^k$ be $\mathcal{L}^k$-measurable. Then, $y \mapsto \mathcal{H}^{k-m}(A \cap f^{-1}(y))$ is $\mathcal{L}^m$-measurable.*

Proof. It can be shown directly from the definition that for all compact sets $K \subset \mathbb{R}^k$ it holds that

$$y \mapsto \mathcal{H}^{k-m}(K \cap f^{-1}(y))$$

is upper semicontinuous (details as exercise), hence measurable. From the σ-finiteness of $\mathcal{L}^k$ we see that there is an increasing sequence of compact sets $K_n \subset \mathbb{R}^k$ with $\mathcal{L}^k(A \setminus \bigcup_n K_n) = 0$. Thus, by inner regularity and Lemma 2.8,

$$\mathcal{H}^{k-m}(A \cap f^{-1}(y)) = \lim_{n\to\infty} \mathcal{H}^{k-m}(K_n \cap f^{-1}(y)) \qquad \text{for } \mathcal{L}^m\text{-a.e. } y \in \mathbb{R}^m.$$

The functions on the right-hand side are measurable, hence so is their almost everywhere limit. □

Lemma 2.10. *Let $f\colon \mathbb{R}^k \to \mathbb{R}^m$ be Lipschitz with $k \geq m$. Then, for*

$$Z := \left\{ x \in \mathbb{R}^k \ : \ \mathbf{J}_m f(x) = 0 \right\},$$

where the existence of $\mathbf{J}_m f(x)$ is part of the condition, it holds that

$$\mathcal{H}^{k-m}(Z \cap f^{-1}(y)) = 0 \qquad \textit{for } \mathcal{L}^m\textit{-a.e. } y \in \mathbb{R}^m.$$

Proof. For a compact set $K \subset \mathbb{R}^k$ define the set function

$$\nu(K) := \int_{\mathbb{R}^m} \mathcal{H}^{k-m}_\infty(K \cap f^{-1}(y)) \,\mathrm{d}y,$$

which is well-defined by the same proof as the one for Lemma 2.9. It is straightforward to see that ν is σ-subadditive, i.e., for any at most countable family of compact sets K_n such that also $\bigcup_n K_n$ is compact, it holds that

$$\nu\biggl(\bigcup_n K_n\biggr) \leq \sum_n \nu(K_n).$$

Fix now $x_0 \in Z$ and define for $r > 0$ the **blow-up map**

$$f_{x_0,r}(z) := \frac{f(x_0 + rz) - f(x_0)}{r}, \qquad z \in \mathbb{R}^k.$$

With this at hand we can compute

$$\begin{aligned}
\nu(\overline{B}_r(x_0)) &= \int_{\mathbb{R}^m} \mathcal{H}^{k-m}_\infty(\overline{B}_r(x_0) \cap f^{-1}(y))\,\mathrm{d}y \\
&= \int_{\mathbb{R}^m} \mathcal{H}^{k-m}_\infty\big(\{\, x_0 + rz \;:\; f(x_0+rz) = y,\ z \in \overline{B}_1 \,\}\big)\,\mathrm{d}y \\
&= r^{k-m} \int_{\mathbb{R}^m} \mathcal{H}^{k-m}_\infty\big(\{\, z \in \overline{B}_1 \;:\; f_{x_0,r}(z) = (y - f(x_0))/r \,\}\big)\,\mathrm{d}y \\
&= r^k \int_{\mathbb{R}^m} \mathcal{H}^{k-m}_\infty(\overline{B}_1 \cap [f_{x_0,r}]^{-1}(y'))\,\mathrm{d}y'
\end{aligned}$$

by the change of variables $x = x_0 + rz$ (use Lemma 1.28) and $y = f(x_0) + ry'$.

By the differentiability (which is entailed by the existence of $\mathbf{J}_m f(x_0)$), $f_{x_0,r}$ converges uniformly on compact sets to $\nabla f(x_0)$ as $r \downarrow 0$. The image of $\nabla f(x_0)$ is contained in an $(m-1)$-plane $\pi(x_0)$ since $\mathbf{J}_m f(x_0) = 0$. If $y' \notin \pi(x_0)$, then $\overline{B}_1 \cap [f_{x_0,r}]^{-1}(y')$ is empty for r sufficiently small. On the other hand, the same conclusion holds for $|y'|$ larger than the Lipschitz constant of f. Thus, the integrand converges to zero almost everywhere and is zero outside a compact set. The dominated convergence theorem thus yields

$$r^{-k}\nu(\overline{B}_r(x_0)) \to 0 \qquad \text{as } r \downarrow 0.$$

For a compact set $K \subset Z$ and $\varepsilon > 0$ we observe that the closed balls $\overline{B}_r(x)$ with $x \in K$ and $r > 0$ so small that $\nu(\overline{B}_r(x)) < \varepsilon\omega_k r^k$ form a fine cover of K. Thus, by the Besicovitch Covering Theorem 1.8 we may find an at most countable collection of these balls $\overline{B}_n = \overline{B}_{r_n}(x_n)$ covering K such that every $x' \in K$ is contained in at most $\beta_k \in \mathbb{N}$ such balls. Hence, using the σ-subadditivity of ν,

$$\nu(K) \le \sum_n \nu(\overline{B}_n) \le \varepsilon \sum_n \omega_k r_n^k = \varepsilon \sum_n \mathcal{L}^k(\overline{B}_n) \le \varepsilon\beta_k\mathcal{L}^k(K + B_1).$$

Since $\varepsilon > 0$ was arbitrary, $\nu(K) = 0$. Consequently, $\mathcal{H}^{k-m}_\infty(K \cap f^{-1}(y)) = 0$ for $\mathcal{L}^m$-almost every y.

Finally, $\mathcal{L}^k$-almost all of Z can be covered by an at most countable family of compact sets $K_n \subset Z$. Since $\mathcal{H}^{k-m}_\infty(K_n \cap f^{-1}(y)) = 0$ implies $\mathcal{H}^{k-m}(K_n \cap f^{-1}(y)) = 0$, we thus obtain

$$\mathcal{H}^{k-m}\Big(\bigcup_n K_n \cap f^{-1}(y)\Big) = 0 \qquad \text{for } \mathcal{L}^m\text{-a.e. } y \in \mathbb{R}^m.$$

On the other hand, by Lemma 2.8, we have

$$\mathcal{H}^{k-m}\Big(\Big(Z \setminus \bigcup_n K_n\Big) \cap f^{-1}(y)\Big) = 0 \qquad \text{for } \mathcal{L}^m\text{-a.e. } y \in \mathbb{R}^m,$$

so the proof is complete. □

Lemma 2.11. *Let $f\colon \mathbb{R}^k \to \mathbb{R}^m$ be Lipschitz with $k \geq m$, let $\beta > 1$, and let $A \subset \mathbb{R}^k$ be $\mathcal{L}^k$-measurable with*

$$\mathbf{J}_m f > 0 \quad \mathcal{L}^k\text{-a.e. on } A.$$

Then, $\mathcal{L}^k$-almost all of A can be covered with countably many disjoint compact sets $A_n \subset \mathbb{R}^k$ with the following properties for every n:

(i) f is differentiable and $\mathbf{J}_m f > 0$ everywhere on A_n.

(ii) There is a matrix $P_n \in \mathbb{R}^{(k-m)\times k}$ such that $F_n(x) := (f(x), P_n x)$ is injective on A_n.

(iii) There is an invertible matrix $L_n \in \mathbb{R}^{k\times k}$ such that the Lipschitz constants of the compound maps $L_n^{-1} \circ F_n$, $F_n^{-1} \circ L_n$ on A_n are bounded from above by β.

(iv) $\beta^{-m}\mathbf{J}_m L_n \leq \mathbf{J}_m f \leq \beta^m \mathbf{J}_m L_n$ on A_n.

Proof. Analogously to the proof of Lemma 2.4, we may assume that ∇f exists and $\mathbf{J}_m f > 0$ everywhere on the Borel set A.

Fix a countable dense subset $\{P_j\}$ of the matrix space $\mathbb{R}^{(k-m)\times k}$. Then, for every $x \in A$ there is a $j(x)$ such that the map $F\colon A \to \mathbb{R}^k$ given by $F(x) := (f(x), P_{j(x)}x)$ has a full-rank gradient at x, or equivalently $|\det \nabla F(x)| > 0$. Chopping A into corresponding Borel measurable pieces A_j, we may in the following assume that there is one $P \in \mathbb{R}^{(k-m)\times k}$ such that with $F(x) = (f(x), Px)$ it holds that $|\det \nabla F| > 0$ everywhere on A.

By a similar procedure as in the Lipschitz-approximation lemma for the area formula, Lemma 2.4, we may partition A further to assume that F is injective (on the restricted A), its inverse F^{-1} (defined on F's range $F(A)$) is Lipschitz, and $\mathbf{J}_k F = |\det \nabla F| > 0$ on A. By the Kirszbraun theorem (or a weaker variant), we can extend F^{-1} to a Lipschitz map G on the whole of $\mathbb{R}^k$. Moreover, invoking Lemma 2.3, we see that also for G it holds that

$$\mathbf{J}_k G = |\det \nabla G| > 0 \qquad \mathcal{L}^k\text{-a.e. on } F(A).$$

Now we can apply the approximation procedure used in the proof of the area formula, Lemma 2.4, to G to obtain that $F(A)$ can be covered with an $\mathcal{L}^k$-negligible set $B_0 \subset \mathbb{R}^k$ and countably many disjoint compact sets $B_n \subset \mathbb{R}^k$ with the following properties for every n:

(a) G is differentiable and $\mathbf{J}_k G > 0$ everywhere on B_n.

(b) The restriction $G|_{B_n}$ is injective with a Lipschitz inverse on its image.

(c) There is an invertible matrix $M_n \in \mathbb{R}^{k\times k}$ with $\mathbf{J}_k M_n = |\det M_n| > 0$ and the Lipschitz constants of the compound maps $G|_{B_n} \circ M_n^{-1}$, $M_n \circ (G|_{B_n})^{-1}$ are bounded from above by β.

(d) $\beta^{-k}\mathbf{J}_k M_n \le \mathbf{J}_k G \le \beta^k \mathbf{J}_k M_n$ on B_n.

For $A_0 := G(B_0)$, $A_n := G(B_n)$, $P_n := P$, $L_n := M_n^{-1}$ we have that A_0 is $\mathcal{L}^k$-negligible (because G is Lipschitz) and that (i), (ii) are satisfied directly by construction. The claim (iii) follows from (c) since

$$L_n^{-1} \circ F_n = M_n \circ (G|_{B_n})^{-1}, \qquad F_n^{-1} \circ L_n = G|_{B_n} \circ M_n^{-1}.$$

If $\mathbf{p}\colon \mathbb{R}^m \times \mathbb{R}^{k-m} \to \mathbb{R}^m$, $\mathbf{p}(v, w) := v$, denotes the projection onto the first m variables, then

$$\nabla f = \mathbf{p} \circ \nabla F = \mathbf{p} \circ L_n \circ \big(M_n \circ \nabla F\big) \qquad \text{on } A_n.$$

Then, it is only a matter of some estimates to conclude (iv) from (d) above. □

We are now in a position to prove the coarea formula in the general case.

Proof of Theorem 2.7. We have already seen in Lemma 2.9 that $y \mapsto \mathcal{H}^{k-m}(A \cap f^{-1}(y))$ is $\mathcal{L}^m$-measurable. As in the proof of the area formula, we may assume that everywhere on A the gradient ∇f exists and (using Lemma 2.10 in place of Lemma 2.3) that $\mathbf{J}_m f > 0$ everywhere on A.

Fix $\beta > 1$. Lemma 2.11 implies that $\mathcal{L}^k$-almost all of A can be covered with countably many disjoint compact sets $A_n \subset \mathbb{R}^k$ with the following properties for every n:

(i) f is differentiable and $\mathbf{J}_m f > 0$ everywhere on A_n.

(ii) There is a matrix $P_n \in \mathbb{R}^{(k-m)\times k}$ such that $F_n(x) := (f(x), P_n x)$ is injective on A_n.

(iii) There is an invertible matrix $L_n \in \mathbb{R}^{k\times k}$ such that the Lipschitz constants of the compound maps $L_n^{-1} \circ F_n$, $F_n^{-1} \circ L_n$ on A_n are bounded from above by β.

(iv) $\beta^{-m}\mathbf{J}_m L_n \le \mathbf{J}_m f \le \beta^m \mathbf{J}_m L_n$ on A_n.

Denote again by $\mathbf{p}\colon \mathbb{R}^m \times \mathbb{R}^{k-m} \to \mathbb{R}^m$, $\mathbf{p}(v, w) := v$, the projection onto the first m variables. Also split $L_n = (X_n, Y_n)$ in the same way with $X_n \in \mathbb{R}^{m\times k}$ and $Y_n \in \mathbb{R}^{(k-m)\times k}$. One observes that $\mathbf{J}_m X_n = \mathbf{J}_m L_n$. Then, using the above properties as well as the coarea formula for linear maps, we estimate

$$\begin{aligned}
&\int_{\mathbb{R}^m} \mathcal{H}^{k-m}(A_n \cap f^{-1}(y)) \,\mathrm{d}y \\
&\quad = \int_{\mathbb{R}^m} \mathcal{H}^{k-m}\big(F_n^{-1}\big(F_n(A_n) \cap \mathbf{p}^{-1}(y)\big)\big) \,\mathrm{d}y \\
&\quad = \int_{\mathbb{R}^m} \mathcal{H}^{k-m}\big((F_n^{-1} \circ L_n)\big(L_n^{-1}(F_n(A_n) \cap \mathbf{p}^{-1}(y))\big)\big) \,\mathrm{d}y \\
&\quad \le \beta^{k-m} \int_{\mathbb{R}^m} \mathcal{H}^{k-m}\big(L_n^{-1}(F_n(A_n) \cap \mathbf{p}^{-1}(y))\big) \,\mathrm{d}y \\
&\quad = \beta^{k-m} \int_{\mathbb{R}^m} \mathcal{H}^{k-m}\big((L_n^{-1} \circ F_n)(A_n) \cap X_n^{-1}(y))\big) \,\mathrm{d}y \\
&\quad = \beta^{k-m} \mathbf{J}_m X_n \cdot \mathcal{L}^k\big((L_n^{-1} \circ F_n)(A_n)\big) \\
&\quad \le \beta^{2k-m} \mathbf{J}_m X_n \cdot \mathcal{L}^k(A_n) \\
&\quad \le \beta^{2k} \int_{A_n} \mathbf{J}_m f(x) \,\mathrm{d}x.
\end{aligned}$$

On the other hand, we can also read the above from bottom to top, exchanging "$\le$" to "$\ge$", and inverting the powers of β, to obtain

$$\int_{A_n} \mathbf{J}_m f(x) \,\mathrm{d}x \le \beta^{2k} \int_{\mathbb{R}^k} \mathcal{H}^{k-m}(A_n \cap f^{-1}(y)) \,\mathrm{d}y.$$

Now sum these estimates over n, use that $\mathcal{L}^m(A \setminus \bigcup_n A_n) = 0$ and Lemma 2.10, to arrive at

$$\beta^{2k} \int_{\mathbb{R}^m} \mathcal{H}^{k-m}(A \cap f^{-1}(y)) \,\mathrm{d}y \le \int_A \mathbf{J}_m f(x) \,\mathrm{d}x \le \beta^{2k} \int_{\mathbb{R}^m} \mathcal{H}^{k-m}(A \cap f^{-1}(y)) \,\mathrm{d}y.$$

Now let $\beta \downarrow 1$ to conclude the proof of the coarea formula in the version of (2.6). $\square$

2.3 Rectifiable sets

For $k \in \{0, \ldots, d\}$ an $\mathcal{H}^k$-measurable set $R \subset \mathbb{R}^d$ is called **countably $\mathcal{H}^k$-rectifiable** if

$$R \subset N \cup \bigcup_n h_n(\mathbb{R}^k),$$

where $h_n \colon \mathbb{R}^k \to \mathbb{R}^d$ are Lipschitz maps and $N \subset \mathbb{R}^d$ is $\mathcal{H}^k$-negligible. We call R **$\mathcal{H}^k$-rectifiable** if R is countably $\mathcal{H}^k$-rectifiable and $\mathcal{H}^k(R) < \infty$.

Remark 2.12. One can find slight variations of these definitions in the literature. For instance, [23] and [26] call the above sets "(countably) k-rectifiable", which, however, in [4] and [19] means that the set N above is empty. Often, also countably $\mathcal{H}^k$-rectifiable sets are simply called "$\mathcal{H}^k$-rectifiable". Another variation is that sometimes one allows the h_n to be defined on sets $A_n \subset \mathbb{R}^d$ instead of the whole of $\mathbb{R}^d$, with or without replacing "$\subset$" by "$=$" above . This, however, is equivalent by the usual Lipschitz extension theorems (e.g., the Kirzsbraun theorem).

Example 2.13. An important special case is the following: A set $\Gamma \subset \mathbb{R}^d$ is a **Lipschitz (or C^1-regular) k-graph** ($k \in \{0, \ldots, d\}$) if

$$\Gamma = \{ x \in \mathbb{R}^d \ : \ \varphi(\pi x) = \pi^\perp x \}$$

for an orthogonal projection π onto a k-plane, which by a common convention we also denote by π, with complementary projection $\pi^\perp$, and a Lipschitz (or C^1-regular) map $\varphi \colon \pi \to \pi^\perp$. Any such Lipschitz k-graph is countably $\mathcal{H}^k$-rectifiable. Indeed, take an orthogonal basis $\{v_1, \ldots, v_k\}$ of π and define the Lipschitz map $h \colon \mathbb{R}^k \to \mathbb{R}^d$ via

$$h(z_1, \ldots, z_k) := \sum_{j=1}^k z_j v_j + \varphi\left(\sum_{j=1}^k z_j v_j\right).$$

Then, $\Gamma = h(\mathbb{R}^k)$ and thus Γ is countably $\mathcal{H}^k$-rectifiable.

Example 2.14. Let $\{q_n\}_n$ be a countable dense set of rational points in the unit ball B_1 in $\mathbb{R}^2$. Define

$$S_\infty := \bigcup_{n=1}^\infty S_n,$$

where $S_n := \partial B_{2^{-n}}(q_n)$ denotes the sphere around q_n with radius 2^{-n}. The set S_∞ is $\mathcal{H}^1$-rectifiable (the definition is evidently true).

Lemma 2.15. *Let $R \subset \mathbb{R}^d$ be (countably) $\mathcal{H}^k$-rectifiable.*

(i) For any $\mathcal{H}^k$-measurable set $A \subset R$, the set $R \cap A$ is (countably) $\mathcal{H}^k$-rectifiable.

(ii) For any Lipschitz map $f \colon \mathbb{R}^d \to \mathbb{R}^m$ be Lipschitz, the set $f(R)$ is (countably) $\mathcal{H}^k$-rectifiable.

Proof. These claims follow directly from the definition. □

The following is a *rectifiability theorem*, which allows one to check if a given set is rectifiable by looking at its local behavior only. To set up the statement, we define for a k-plane $\pi \subset \mathbb{R}^d$ (identified with its orthogonal projection) and $M > 0$ the **cone** around π with opening M via

$$C(\pi, M) := \{ x \in \mathbb{R}^d \ : \ |\pi^\perp x| \leq M |\pi x| \},$$

where $\pi^\perp$ is the orthogonal projection onto the orthogonal complement of π.

Theorem 2.16. *Let $A \subset \mathbb{R}^d$ be $\mathcal{H}^k$-measurable with the property that for all $x \in A$ there is a radius $r_x > 0$, a k-plane $\pi_x \subset \mathbb{R}^d$, and $M_x > 0$ such that*

$$A \cap B_{r_x}(x) \subset x + C(\pi_x, M_x). \tag{2.7}$$

Then, A is countably $\mathcal{H}^k$-rectifiable. In fact, A is even the countable union of Lipschitz k-graphs with Lipschitz constants uniformly bounded by $2\sup_{x\in A} M_x$.

Proof. We will show the stronger statement that

$$A = \bigcup_n h_n(\mathbb{R}^k)$$

for countably many Lipschitz maps $h_n\colon \mathbb{R}^k \to \mathbb{R}^d$ with Lipschitz constants bounded from above by $2\sup_{x\in A} M_x$.

Splitting A into countably many subsets, we may assume that $M := \sup_{x\in A} M_x < \infty$. Fix $\delta > 0$, to be chosen later, and pick a finite set of k-planes $\widehat{\pi}_n$ such that $\min_n|\pi - \widehat{\pi}_n| < \delta$ for any k-plane π (the norm being the Frobenius norm of the matrix representations of the projections). Define

$$A_{mn} := \left\{ x \in A \ :\ r_x > \frac{1}{m},\ |\pi_x - \widehat{\pi}_n| < \delta \right\}.$$

Let $B \subset A_{mn}$ with $\operatorname{diam}(B) < 1/m$. We will show that B is contained in a Lipschitz k-graph with Lipschitz constant less than $2M$. This suffices to show the theorem.

So, for $x, y \in B$ we estimate, using the fact that $\widehat{\pi}_n^\perp - \pi_x^\perp = \pi_x - \widehat{\pi}_n$ and the assumption (2.7), as follows:

$$\begin{aligned}
|\widehat{\pi}_n^\perp(x-y)| &\le |\pi_x^\perp(x-y)| + \delta|x-y| \\
&\le M|\pi_x(x-y)| + \delta|x-y| \\
&\le M|\widehat{\pi}_n(x-y)| + \delta(M+1)|x-y| \\
&\le (M + \delta(M+1))|\widehat{\pi}_n(x-y)| + \delta(M+1)|\widehat{\pi}_n^\perp(x-y)|.
\end{aligned}$$

Now choose $\delta > 0$ so small (how small only depending on M) that $\delta(M+1) < 1$ and that $\frac{M+\delta(M+1)}{1-\delta(M+1)} \le 2M$. Then absorb the second term on the right-hand side into the left-hand side to obtain

$$B \subset x + C(\widehat{\pi}_n, 2M).$$

In particular, $\widehat{\pi}_n x = \widehat{\pi}_n x'$ implies $x = x'$ for any $x, x' \in B$. For $y \in \widehat{\pi}_n[B]$ we then let $\varphi_n(y)$ be the unique $z \in \widehat{\pi}_n^\perp$ with $y + z \in B$. It is easily seen that the Lipschitz constant of φ_n is bounded from above by $2M$. Hence, B is contained in a Lipschitz k-graph. Thus we have written A as the countable union of Lipschitz k-graphs. The rectifiability of B then follows from the preceding Example 2.13. □

One application of this result is the following:

Proposition 2.17. *Let $K \subset \mathbb{R}^k$ be compact and let $h\colon \mathbb{R}^k \to \mathbb{R}^d$ be a Lipschitz map that is injective on K and that has the property that for all $z \in K$ the differential $\nabla h(z)$ exists and $\operatorname{rank}(\nabla h(z)) = k$. Then, $h(K)$ is the countable union of Lipschitz (or C^1-regular) k-graphs with Lipschitz constants uniformly bounded by an arbitrary $\varepsilon > 0$. In particular, $h(K)$ is countably $\mathcal{H}^k$-rectifiable.*

Proof. It can be checked from Taylor's theorem that for any $x \in h(K)$ the assumptions of Theorem 2.16 hold with $\pi_x := \nabla h(h^{-1}(x))[\mathbb{R}^k]$, $M_x = \varepsilon/2$ with $\varepsilon > 0$ freely chosen, and $r_x > 0$ sufficiently small. □

The next result characterizes rectifiability in terms of a decomposition into graphs.

Proposition 2.18. *A $\mathcal{H}^k$-measurable set $R \subset \mathbb{R}^d$ is countably $\mathcal{H}^k$-rectifiable if and only if there exist countably many Lipschitz or C^1-regular k-graphs Γ_n such that*

$$\mathcal{H}^k\left(R \setminus \bigcup_n \Gamma_n\right) = 0.$$

Moreover, the Lipschitz constants of the graphs Γ_n may be assumed to be uniformly bounded by an arbitrary $\varepsilon > 0$.

Proof. The "if" direction follows immediately from Example 2.13.

For the "only if" direction, we need only show that for a Lipschitz map $h\colon \mathbb{R}^k \to \mathbb{R}^d$ we can find countably many Lipschitz (or C^1-regular) k-graphs Γ_n such that

$$\mathcal{H}^k\left(h(\mathbb{R}^k) \setminus \bigcup_n \Gamma_n\right) = 0.$$

We will only prove the result for Lipschitz regularity. The result with C^1-regularity can be inferred from this using a Lusin-type approximation result for Lipschitz maps by C^1-maps (see Theorem 5.1.12 in [23], details as an exercise).

Define

$$A := \left\{ z \in \mathbb{R}^k \ : \ \nabla h(z) \text{ exists and } \mathbf{J}_k h(z) > 0 \right\}.$$

By Lemma 2.4 on Lipschitz approximations, we can cover $\mathcal{H}^k$-almost all of A with countably many disjoint compact sets $A_n \subset \mathbb{R}^k$ such that h is differentiable and injective with $\mathbf{J}_k h > 0$ everywhere on A_n. By Proposition 2.17, $h(A_n)$ can be written as a countable union of Lipschitz k-graphs with Lipschitz constants uniformly bounded by an arbitrary $\varepsilon > 0$. Hence, there are countably many Lipschitz k-graphs Γ_n such that (using also Lemma 1.30 to show that $\mathcal{H}^k$-null sets are mapped to $\mathcal{H}^k$-null sets under Lipschitz maps)

$$\mathcal{H}^k\left(h(A) \setminus \bigcup_n \Gamma_n\right) = 0.$$

On the other hand, by the Rademacher Theorem 1.18 and Lemma 2.3,

$$\mathcal{H}^k\big(h(\mathbb{R}^k \setminus A)\big) = 0.$$

Thus, $\mathcal{H}^k$-almost all of $h(\mathbb{R}^k)$ is covered by the Lipschitz k-graphs Γ_n and our claim, hence the "only if" direction, is established. $\square$

Remark 2.19. In fact, in Proposition 2.18 the Γ_n may also be chosen as embedded C^1-manifolds by a straighforward exhaustion argument. See Lemma 5.4.2 in [23] for the details.

2.4 Approximate tangent spaces

Let $A \subset \mathbb{R}^d$ be $\mathcal{H}^k$-measurable. We say that a k-dimensional linear subspace $\pi \subset \mathbb{R}^d$ is the **approximate tangent space** to A at $x_0 \in \mathbb{R}^d$ if

$$\lim_{r\downarrow 0} \int_{r^{-1}(A-x_0)} \varphi(y) \,\mathrm{d}\mathcal{H}^k(y) = \int_\pi \varphi(y) \,\mathrm{d}\mathcal{H}^k(y) \qquad \text{for all } \varphi \in \mathrm{C}_c^\infty(\mathbb{R}^d). \tag{2.8}$$

If such a π exists, it is obviously unique, and we denote it by $\mathrm{T}_{x_0} A$.

Proposition 2.20. *Let $R \subset \mathbb{R}^d$ be $\mathcal{H}^k$-rectifiable. Then, R has an approximate tangent space at $\mathcal{H}^k$-almost every point of R. If $R = h(K)$ with $K \subset \mathbb{R}^k$ compact and $h\colon K \to \mathbb{R}^d$ injective and Lipschitz, then*

$$\mathrm{T}_{x_0} R = \nabla h(h^{-1}(x_0))[\mathbb{R}^k] \qquad \textit{for } \mathcal{H}^k\textit{-a.e. } x_0 \in R. \tag{2.9}$$

Moreover, the tangent space only depends on R, not the Lipschitz maps involved in the definition of rectifiability.

Note that here we need $\mathcal{H}^k$-rectifiability and not just *countable* $\mathcal{H}^k$-rectifiability since otherwise the left-hand side in (2.8) could be infinite (for instance, if we take countably many lines with a fixed direction that concentrate at a line), but the right-hand side is always finite. Alternatively, one could require that $\mathcal{H}^k(R \cap K) < \infty$ for every compact set $K \subset \mathbb{R}^d$.

Proof. Step 1. Let us first assume that $R = h(K)$ for a compact set $K \subset \mathbb{R}^k$ and an injective Lipschitz map $h\colon K \to \mathbb{R}^d$. Fix now $z_0 \in K$ such that

(a) $\mathbf{J}_k h(z_0) > 0$;

(b) $\Theta_k(K, z_0) := \lim_{r\downarrow 0} \dfrac{\mathcal{H}^k(K \cap B_r(z_0))}{\omega_k r^k} = 1$;

(c) $\lim_{r\downarrow 0} r^{-k} \displaystyle\int_{B_r(z_0)} |\mathbf{J}_k h(z) - \mathbf{J}_k h(z_0)| \,\mathrm{d}z = 0.$

The first condition holds at $\mathcal{L}^k$-almost every $z_0 \in K$ by the injectivity of h. The other two conditions hold at $\mathcal{H}^k$-almost every z_0 by Corollary 1.17 on density points and the Lebesgue Differentiation Theorem 1.16, respectively.

Let $x_0 := h(z_0)$ and $\pi := \nabla h(z_0)[\mathbb{R}^k]$, which is a k-dimensional subspace of $\mathbb{R}^d$ by (a). Clearly,

$$h_r(y) := \frac{h(z_0 + ry) - h(z_0)}{r}, \qquad y \in K_r := r^{-1}(K - z_0),$$

parameterizes the set $r^{-1}(R - x_0)$. We now use the area formula, Theorem 2.2 and a change of variables to compute

$$\begin{aligned}\int_{r^{-1}(R-x_0)} \varphi(x) \, \mathrm{d}\mathcal{H}^k(x) &= \int_{K_r} \varphi(h_r(y)) \, \mathbf{J}_k h_r(y) \, \mathrm{d}y \\ &= \int_{K_r} \varphi\biggl(\frac{h(z_0 + ry) - h(z_0)}{r}\biggr) \, (\mathbf{J}_k h)(z_0 + ry) \, \mathrm{d}y \\ &= r^{-k} \int_K \varphi\biggl(\frac{h(z) - h(z_0)}{r}\biggr) \, \mathbf{J}_k h(z) \, \mathrm{d}z.\end{aligned}$$

By condition (c) together with the fact that φ is bounded, we see that

$$\lim_{r \downarrow 0} \int_{r^{-1}(R-x_0)} \varphi(x) \, \mathrm{d}\mathcal{H}^k(x) = \mathbf{J}_k h(z_0) \cdot \lim_{r \downarrow 0} \int_{K_r} \varphi\biggl(\frac{h(z_0 + ry) - h(z_0)}{r}\biggr) \, \mathrm{d}y.$$

Then, since $\nabla h(z_0)$ has full rank, one observes that the supports of the maps

$$y \mapsto \varphi\biggl(\frac{h(z_0 + ry) - h(z_0)}{r}\biggr)$$

are uniformly bounded for r small enough. Moreover, these maps converge pointwise to $\varphi(\nabla h(z_0) y)$. Also noting that, by (b), $\mathbb{1}_{K_r}$ converges to $\mathbb{1}_{\mathbb{R}^k}$ pointwise almost everywhere, we can use the dominated convergence theorem to see that

$$\begin{aligned}\lim_{r \downarrow 0} \int_{r^{-1}(R-x_0)} \varphi(x) \, \mathrm{d}\mathcal{H}^k(x) &= \mathbf{J}_k h(z_0) \cdot \lim_{r \downarrow 0} \int \mathbb{1}_{K_r}(y) \varphi\biggl(\frac{h(z_0 + ry) - h(z_0)}{r}\biggr) \, \mathrm{d}y \\ &= \int_{\mathbb{R}^k} \varphi(\nabla h(z_0) y) \, \mathbf{J}_k h(z_0) \, \mathrm{d}y \\ &= \int_\pi \varphi(y) \, \mathrm{d}\mathcal{H}^k,\end{aligned}$$

where the last equality follows again from the area formula (in fact, the version for linear maps suffices). This shows (2.9).

Step 2. By Proposition 2.18 we know that

$$\mathcal{H}^k\biggl(R \setminus \bigcup_n h_n(K_n)\biggr) = 0.$$

for countably many compact sets $K_n \subset \mathbb{R}^k$ and injective Lipschitz maps $h_n\colon K_n \to \mathbb{R}^d$. Moreover, By Lemma 1.37 we know that

$$\Theta_k(R \setminus h(K_n), x) = 0 \qquad \text{for } \mathcal{H}^k\text{-a.e. } x \in h_n(K_n)\ (n \in \mathbb{N}).$$

Hence, the approximate tangent can be computed separately in all the pieces $h_n(K_n)$, for which we can apply Step 1.

Step 3. The uniqueness statements follow directly from the explicit formula (2.9) for the approximate tangent space. □

Example 2.21. The previous result in particular implies that any embedded (boundaryless) k-dimensional C^1-submanifold M of $\mathbb{R}^d$ has an approximate tangent space $\mathrm{T}_y\, M$ at $\mathcal{H}^k$-almost every $y \in M$; in fact, the proof above shows that this is the case for *all* $y \in M$. It is easy to see that $\mathrm{T}_y\, M$ agrees with the classical tangent space to M at y (how easy depending on one's choice for a definition of the tangent space).

Example 2.22. Let $C_\infty(1/4)$ be the λ-Cantor set of Example 1.36 for $\lambda = 1/4$ (note that $\dim_{\mathcal{H}} C_\infty(1/4) = 1/2$). Define $F := C_\infty(1/4) \times C_\infty(1/4) \subset \mathbb{R}^2$. It can be shown (exercise) that $0 < \mathcal{H}^1(F) < \infty$, but F does not have an approximate tangent space for $\mathcal{H}^1$-almost every point (exercise). Hence, F is not $\mathcal{H}^1$-rectifiable.

Lipschitz maps have even better differentiability properties than implied by the Rademacher Theorem 1.18, namely the following *tangential differentiability*:

Proposition 2.23. *Let $f\colon \mathbb{R}^d \to \mathbb{R}^m$ be Lipschitz and let $R \subset \mathbb{R}^d$ be an $\mathcal{H}^k$-rectifiable set. Then, f is* ***tangentially differentiable*** *at $\mathcal{H}^k$-almost every $x_0 \in R$, that is, the restriction of f to $x_0 + \mathrm{T}_{x_0} R$ (assuming that $\mathrm{T}_{x_0} R$ exists at x_0) is differentiable at x_0 with* ***tangential differential***

$$\mathrm{D}^R f(x_0)\colon\ \mathrm{T}_{x_0} R \to \mathbb{R}^m.$$

Proof. By Proposition 2.18 we may assume that $R = h(K)$ for some compact set $K \subset \mathbb{R}^k$ and an injective Lipschitz map $h\colon K \to \mathbb{R}^d$. Define the set

$$G := \bigl\{\, z \in K \ :\ \nabla h(z), \nabla(f \circ h)(z) \text{ exist and } \mathbf{J}_k h(z) > 0 \,\bigr\}.$$

By the Rademacher Theorem 1.18, Lemma 1.30, and Lemma 2.3 we infer that

$$\mathcal{H}^k(R \setminus h(G)) = 0.$$

We will show the claim for all $x_0 = h(z_0)$ with $z_0 \in G$, for which we also we know from Lemma 2.20 that $\mathrm{T}_{x_0} R = \nabla h(z_0)[\mathbb{R}^k]$.

Let $w := \nabla h(z_0)v \in \mathrm{T}_{x_0} R$ for any $v \in \mathbb{R}^k$. Set $x_t := x_0 + tw = h(z_0) + t\nabla h(z_0)v$, $y_t := h(z_0 + tv)$ ($t \in \mathbb{R}$). By the differentiability of h at z_0,

$$\lim_{t \to 0} \frac{|y_t - x_t|}{t} = 0.$$

Then we use the Lipschitz continuity of f to see

$$\lim_{t \to 0} \frac{f(x_t) - f(x_0)}{t} = \lim_{t \to 0} \frac{f(y_t) - f(x_0)}{t} = \lim_{t \to 0} \frac{f(h(z_0 + tv)) - f(h(z_0))}{t},$$

from which we conclude that

$$\frac{\partial f}{\partial w}(x_0) = \nabla(f \circ h)(z_0)[(\nabla h(x_0))^{-1} w]. \tag{2.10}$$

As a consequence of the existence of the total differentials, the directional derivative $\frac{\partial f}{\partial w}(x_0)$ is linear in w. We may then use a similar density argument like the one at the end of Rademacher's Theorem 1.18, which also uses the Lipschitz continuity of f, to obtain the (total) differentiability of f at x_0 (details as exercise). □

2.5 Area and coarea formula on rectifiable sets

Let $R \subset \mathbb{R}^d$ be an $\mathcal{H}^k$-rectifiable set and let $f \colon \mathbb{R}^d \to \mathbb{R}^m$ be Lipschitz. For $\mathcal{H}^k$-almost every $x \in R$ the approximate tangent space $\mathrm{T}_x R$ is defined via Proposition 2.20, and the restriction $\mathrm{D}^R f(x)$ of the differential $\mathrm{D}f(x)$ to $\mathrm{T}_x R$ exists by Proposition 2.23. Moreover, we may identify $\mathrm{D}^R f(x)$ with an $(m \times k)$-matrix with respect to an orthonormal basis $\{v_i(x)\}_i$ of $\mathrm{T}_x R$.

Then, define the **k-dimensional Jacobian $\mathbf{J}_k^R f$ of f relative to R** for $k \leq m$ via

$$\mathbf{J}_k^R f(x) := \sqrt{\det(\mathrm{D}^R f(x)^\mathsf{T} \mathrm{D}^R f(x))}$$

and for $k \geq m$ define the **m-dimensional Jacobian $\mathbf{J}_m^R f$** of f relative to R as

$$\mathbf{J}_m^R f(x) := \sqrt{\det(\mathrm{D}^R f(x) \mathrm{D}^R f(x)^\mathsf{T})}.$$

It is easy to see that the above formulas do not depend on the choice of the orthonormal basis $\{v_i\}$ since the application of orthogonal change-of-basis matrices cancels in the formulas for $\mathbf{J}_k^R f(x)$, $\mathbf{J}_m^R f(x)$.

We can now extend the "flat" area formula in Theorem 2.2 to the following version on rectifiable sets:

Theorem 2.24 (Area formula on rectifiable sets). *Let $R \subset \mathbb{R}^d$ be a $\mathcal{H}^k$-rectifiable set, and let $f\colon \mathbb{R}^d \to \mathbb{R}^m$ be Lipschitz with $k \le m$. Then, for every $\mathcal{H}^k$-measurable map $g\colon R \to \mathbb{R}$, it holds that*

$$\int_R g(x)\, \mathbf{J}_k^R f(x) \, \mathrm{d}\mathcal{H}^k(x) = \int_{\mathbb{R}^m} \sum_{x \in R \cap f^{-1}(y)} g(x) \, \mathrm{d}\mathcal{H}^k(y).$$

In particular,

$$\int_R \mathbf{J}_k^R f(x) \, \mathrm{d}\mathcal{H}^k(x) = \int_{\mathbb{R}^m} \mathcal{H}^0(R \cap f^{-1}(y)) \, \mathrm{d}\mathcal{H}^k(y). \tag{2.11}$$

and, if f is injective,

$$\int_R \mathbf{J}_k^R f(x) \, \mathrm{d}\mathcal{H}^k = \mathcal{H}^k(f(R)).$$

Proof. By the usual approximation arguments we only need to prove (2.11). In fact, by Proposition 2.18 it suffices to show (2.11) under the additional assumption that $R = h(B)$ with $B \subset \mathbb{R}^k$ a bounded Borel set, and $h\colon \mathbb{R}^k \to \mathbb{R}^d$ a Lipschitz map that is injective when restricted to B. As in the proof of Proposition 2.23 we furthermore assume that

$$B \subset G := \bigl\{\, z \in B \;:\; \nabla h(z), \nabla(f \circ h)(z) \text{ exist and } \mathbf{J}_k h(z) > 0 \,\bigr\}.$$

Let us now express (2.10) as

$$\nabla(f \circ h)(h^{-1}(x)) = \mathrm{D}^R f(x) \circ \mathrm{D}h(h^{-1}(x)) \qquad \text{for } \mathcal{H}^k\text{-a.e. } x \in R,$$

where we have written $\mathrm{D}h(z) = \mathrm{D}^{\mathbb{R}^k} h(z)\colon \mathbb{R}^k \to \mathrm{T}_{h(z)}\, R$ for the (total) derivative of h. Since the determinant is multiplicative with respect to the composition of linear maps, a polar decomposition argument (exercise) yields

$$\mathbf{J}_k(f \circ h)(h^{-1}(x)) = \mathbf{J}_k^R f(x) \cdot \mathbf{J}_k h(h^{-1}(x)) \qquad \text{for } \mathcal{H}^k\text{-a.e. } x \in R. \tag{2.12}$$

Using the flat area formula, Theorem 2.2, we have

$$\int_B \mathbf{J}_k(f \circ h)(z) \, \mathrm{d}z = \int_{\mathbb{R}^m} \mathcal{H}^0(B \cap (f \circ h)^{-1}(y)) \, \mathrm{d}\mathcal{H}^k(y).$$

The right-hand side is equal to

$$\int_{\mathbb{R}^m} \mathcal{H}^0(R \cap f^{-1}(y)) \, \mathrm{d}\mathcal{H}^k(y),$$

whereas for the left-hand side we compute, using $\mathbf{J}_k h > 0$ $\mathcal{H}^k$-almost everywhere, the flat area formula again, and (2.12),

$$\begin{aligned}\int_B \mathbf{J}_k(f \circ h)(z)\, \mathrm{d}z &= \int_B \frac{\mathbf{J}_k(f \circ h)(z)}{\mathbf{J}_k h(z)}\, \mathbf{J}_k h(z)\, \mathrm{d}z \\ &= \int_R \frac{\mathbf{J}_k(f \circ h)(h^{-1}(x))}{\mathbf{J}_k h(h^{-1}(x))}\, \mathrm{d}\mathcal{H}^k(x) \\ &= \int_R \mathbf{J}_k^R f(x)\, \mathrm{d}\mathcal{H}^k(x).\end{aligned}$$

Combining these statements, we obtain (2.11) and the proof is complete. □

Also the coarea formula has a version for rectifiable sets:

Theorem 2.25 (Coarea formula). *Let $R \subset \mathbb{R}^d$ be a $\mathcal{H}^k$-rectifiable set, and let $f\colon \mathbb{R}^d \to \mathbb{R}^m$ be Lipschitz with $k \geq m$. Then $R \cap f^{-1}(y)$ is $\mathcal{H}^{k-m}$-rectifiable for $\mathcal{L}^m$-almost every $y \in \mathbb{R}^m$ and for every $\mathcal{H}^k$-measurable map $g\colon R \to \mathbb{R}$, it holds that*

$$\int_R g(x)\, \mathbf{J}_m^R f(x)\, \mathrm{d}\mathcal{H}^k(x) = \int_{\mathbb{R}^m} \int_{R \cap f^{-1}(y)} g(x)\, \mathrm{d}\mathcal{H}^{k-m}(x)\, \mathrm{d}y. \tag{2.13}$$

In particular,

$$\int_R \mathbf{J}_m^R f(x)\, \mathrm{d}\mathcal{H}^k(x) = \int_{\mathbb{R}^m} \mathcal{H}^{k-m}(R \cap f^{-1}(y))\, \mathrm{d}y. \tag{2.14}$$

Proof. We first prove the asserted rectifiability of the level sets. By Proposition 2.18, there are countably many Lipschitz k-graphs Γ_j such that, for

$$N := R \setminus \bigcup_{j=1}^{\infty} \Gamma_j,$$

it holds that $\mathcal{H}^k(N) = 0$. By Lemma 2.8,

$$\mathcal{H}^{k-m}(N \cap f^{-1}(y)) = 0 \qquad \text{for } \mathcal{L}^m\text{-a.e. } y \in \mathbb{R}^m. \tag{2.15}$$

It therefore suffices to prove rectifiability of the level sets on each Lipschitz graph. Thus let $\Gamma = \psi(U)$ be a Lipschitz k-graph, where $U \subset \mathbb{R}^k$ is Borel and $\psi\colon U \to \mathbb{R}^d$ is Lipschitz. Define $F := f \circ \psi\colon U \to \mathbb{R}^m$. By the flat coarea formula, Theorem 2.7, for $\mathcal{L}^m$-almost every $y \in \mathbb{R}^m$ the set $U \cap F^{-1}(y)$ is $\mathcal{H}^{k-m}$-rectifiable. Since Lipschitz maps send rectifiable sets to rectifiable sets, see Lemma 2.15 (ii),

$$\Gamma \cap f^{-1}(y) = \psi\big(U \cap F^{-1}(y)\big)$$

is $\mathcal{H}^{k-m}$-rectifiable for $\mathcal{L}^m$-almost every y. Taking the countable union over the graphs Γ_j and using (2.15) proves that $R \cap f^{-1}(y)$ is $\mathcal{H}^{k-m}$-rectifiable for $\mathcal{L}^m$-almost every y.

It remains to prove the coarea formula (2.13). In fact, we will only show (2.14), from which the general case follows by approximation with step functions.

As before we may assume, by virtue of Proposition 2.18, that R is actually a graph (also see Example 2.13), that is,

$$R = \{ z \in \mathbb{R}^d \ : \ \varphi(\pi z) = \pi^\perp z \}$$

for an orthogonal projection π onto a k-plane with complementary projection $\pi^\perp$, and a Lipschitz map $\varphi \colon \pi \to \pi^\perp$ with Lipschitz constant less than a given $\varepsilon > 0$. Set $\psi(z) := z + \varphi(z)$ for $z \in \pi$ and define

$$B := \psi^{-1}(R) \subset \pi, \qquad B_y := B \cap (f \circ \psi)^{-1}(y), \qquad y \in \mathbb{R}^m.$$

For $\mathcal{L}^m$-almost every y, the set B_y is $\mathcal{H}^{k-m}$-rectifiable by the flat coarea formula. The area formula on rectifiable sets, see Theorem 2.24, yields

$$\mathcal{H}^{k-m}(R \cap f^{-1}(y)) = \int_{B_y} \mathbf{J}^{B_y}_{k-m}\psi(z) \, \mathrm{d}\mathcal{H}^{k-m}(z).$$

As we assumed the Lipschitz constant of φ to be bounded by ε, we then see that

$$\mathcal{H}^{k-m}(R \cap f^{-1}(y)) = \mathcal{H}^{k-m}(B_y)(1 + \mathrm{O}(\varepsilon)).$$

Also apply the flat coarea formula, Theorem 2.7, on π, and a chain-rule formula for the Jacobians that is analogous to (2.12), to see

$$\int_{\mathbb{R}^m} \mathcal{H}^{k-m}(B_y) \, \mathrm{d}y = \int_B \mathbf{J}_m(f \circ \psi)(z) \, \mathrm{d}\mathcal{L}^k_\pi(z) = \int_B \mathbf{J}^R_m f(\psi(z)) \cdot \mathbf{J}_k\psi(z) \, \mathrm{d}\mathcal{L}^k_\pi(z),$$

where we have denoted the k-dimensional Lebesgue measure on π by $\mathcal{L}^k_\pi$. Then change variables from z to $x = \psi(z)$ via the area formula in the form of Theorem 2.24 applied to $\psi \colon \pi \to R \subset \mathbb{R}^d$ and $g(z) = \mathbf{J}^R_m f(\psi(z))$ to obtain

$$\int_B \mathbf{J}^R_m f(\psi(z)) \cdot \mathbf{J}_k\psi(z) \, \mathrm{d}\mathcal{L}^k_\pi(z) = \int_R \mathbf{J}^R_m f(x) \, \mathrm{d}\mathcal{H}^k(x).$$

Combining and letting $\varepsilon \to 0$,

$$\int_{\mathbb{R}^m} \mathcal{H}^{k-m}(R \cap f^{-1}(y)) \, \mathrm{d}y = \int_R \mathbf{J}^R_m f(x) \, \mathrm{d}\mathcal{H}^k(x).$$

This proves (2.14). □

2.6 Rectifiability theorems

If one cannot prove directly that a given set is rectifiable by writing it as the union of Lipschitz graphs, one needs to rely on other rectifiability criteria. We have already seen Theorem 2.16 in this direction: In conjunction with Proposition 2.20 it states roughly that a set is countably $\mathcal{H}^k$-rectifiable if and only if it has approximate tangent planes at $\mathcal{H}^k$-almost every point. In this section we briefly survey some stronger rectifiability theorems, without giving the (very long and intricate) proofs; see [26] for many more details on this topic.

Theorem 2.26 (Besicovitch–Marstrand–Mattila). *Let $A \in \mathcal{B}(\mathbb{R}^d)$ with $\mathcal{H}^k(A) < \infty$. Then, A is $\mathcal{H}^k$-rectifiable if (and only if)*

$$\Theta_k(A,x) := \lim_{r\downarrow 0} \frac{\mathcal{H}^k(A \cap B_r(x))}{\omega_k r^k} = 1 \qquad \text{for } \mathcal{H}^k\text{-a.e. } x \in A.$$

We call $\Theta(A,x)$ the k**-density** of A at x. In fact, there is an even stronger version of the above theorem:

Theorem 2.27 (Preiss). *Let $A \in \mathcal{B}(\mathbb{R}^d)$ with $\mathcal{H}^k(A) < \infty$. Then, A is $\mathcal{H}^k$-rectifiable if (and only if) for $\mathcal{H}^k$-almost every $x \in A$ the k-density $\Theta_k(A,x)$ exists (as a limit) in $(0,\infty)$.*

Note that if the hypothesis of the theorem is true, then automatically $\Theta_k(A,x) = 1$ for $\mathcal{H}^k$-almost every $x \in A$ by Corollary 1.17.

Another approach to rectifiability is furnished by the following idea. Call a set $A \subset \mathbb{R}^d$ **purely $\mathcal{H}^k$-unrectifiable** if $\mathcal{H}^k(A \cap R) = 0$ for every $\mathcal{H}^k$-rectifiable set $R \subset \mathbb{R}^d$. Then, we have the following splitting result:

Proposition 2.28. *Let $A \in \mathcal{B}(\mathbb{R}^d)$ with $\mathcal{H}^k(A) < \infty$. Then,*

$$A = R \cup U$$

with R a $\mathcal{H}^k$-rectifiable set and U a purely $\mathcal{H}^k$-unrectifiable set. This decomposition is unique up to $\mathcal{H}^k$-negligible sets.

This result is then combined with the following deep rectifiability result, whose proof is in Section 18 of [26].

Theorem 2.29 (Besicovitch–Federer). *Let $A \in \mathcal{B}(\mathbb{R}^d)$ with $0 < \mathcal{H}^k(A) < \infty$. Then:*

(i) A is $\mathcal{H}^k$-rectifiable if and only if $\mathcal{H}^k(\pi_V[B]) > 0$ for $\gamma_{d,k}$-almost all k-hyperplanes V and all $\mathcal{H}^k$-measurable subsets $B \subset A$.

(ii) A is purely $\mathcal{H}^k$-unrectifiable if and only if $\mathcal{H}^k(\pi_V[A]) = 0$ for $\gamma_{d,k}$-almost all k-hyperplanes V.

Here, $\pi_V : \mathbb{R}^d \to V$ is the orthogonal projection onto V and $\gamma_{d,k}$ is the Haar measure on the (Grassmannian) manifold of k-hyperplanes in $\mathbb{R}^d$, i.e., the unique (up to constants) measure that is invariant with respect to the action of orthogonal transformations.

Corollary 2.30. *Let $A \in \mathcal{B}(\mathbb{R}^d)$ with $\mathcal{H}^k(A) < \infty$ have the property that for any $\mathcal{H}^k$-measurable subset $B \subset A$ with $\mathcal{H}^k(\pi_V[B]) = 0$ for $\gamma_{d,k}$-almost all k-hyperplanes V, it holds that $\mathcal{H}^k(B) = 0$. Then, A is $\mathcal{H}^k$-rectifiable.*

Proof. According to Proposition 2.28 we may write $A = R \cup U$ with R $\mathcal{H}^k$-rectifiable and U purely $\mathcal{H}^k$-unrectifiable. Via the Besicovitch–Federer Rectifiability Theorem 2.29 we know that $\mathcal{H}^k(\pi_V[U]) = 0$ for $\gamma_{d,k}$-almost all k-hyperplanes V. Thus, by assumption, $\mathcal{H}^k(U) = 0$. We infer that A and R agree up to an $\mathcal{H}^k$-null set and A has been shown to be rectifiable. $\square$

Part II

Introduction to Currents

Chapter 3

Differential Forms

Differential forms are the natural objects that can be "integrated" over manifolds. Currents will be introduced in the next chapter as *dual* to differential forms. Since we will only discuss currents in a Euclidean ambient space, we likewise only needs differential forms in Euclidean spaces. This simplifies some things, but nevertheless we need to first discuss the language of multi-linear algebra as an efficient means to describe planes. Then we will discuss differential forms and prove the fundamental Stokes theorem, which relates (oriented) boundaries to the exterior differential of a form. Finally, as a preparation for later, we discuss Lie derivatives and Hodge duality.

3.1 Multi-linear algebra

In 3-dimensional space, the tangent spaces to oriented (2-dimensional) surfaces can be specified by prescribing a unit normal vector at every point. The tangent space is then the 2-dimensional plane orthogonal to this normal vector. Additionally, the normal vector induces an orientation for the tangent space: If v_1, v_2 span the tangent space, then we prescribe that the normal needs to be the cross-product $v_1 \times v_2$ (and not $v_2 \times v_1$), hence ordering v_1, v_2. However, this approach is highly dependent on the ambient dimension being three (or two): Even 2-dimensional planes in four ambient dimensions cannot be described by one normal vector only. Hence, we need a better way of specifying oriented planes in arbitrary dimensions. This leads to the concept of multi-vectors. In the following, it is best to keep the intuition in mind that multi-vectors represent oriented and "weighted" planes when parsing the algebraic definitions, which can be a bit cumbersome. Luckily, in practice the actual definition of multi-vectors is largely irrelevant and one only needs to work with the computation rules, which are relatively easy to remember.

Let $k = 1, 2, \ldots, n$ and let V be a finite-dimensional real vector space. On the k-fold product vector space V^k we define an equivalence relation "$\sim$" as follows: Denoting the equivalence class of $(v_1, \ldots, v_k) \in V^k$ under $\sim$ by $v_1 \wedge \cdots \wedge v_k$ (read "$\wedge$" as "wedge"), we define $\sim$ as the smallest equivalence relation obeying the following three "determinant-type" rules:

$$v_1 \wedge \cdots \wedge (\lambda v_i) \wedge \cdots \wedge v_j \wedge \cdots \wedge v_k = v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge (\lambda v_j) \wedge \cdots \wedge v_k,$$
$$v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots \wedge v_k = v_1 \wedge \cdots \wedge (v_i + \lambda v_j) \wedge \cdots \wedge v_j \wedge \cdots \wedge v_k,$$
$$v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots \wedge v_k = v_1 \wedge \cdots \wedge (-v_j) \wedge \cdots \wedge v_i \wedge \cdots \wedge v_k,$$

where $v_1, \ldots, v_k \in V$ and $\lambda \in \mathbb{R}$. Note that the first rule is to be understood as

$$(v_1, \ldots, \lambda v_i, \ldots, v_j, \ldots, v_k) \sim (v_1, \ldots, v_i, \ldots, \lambda v_j, \ldots, v_k)$$

and analogously for the others.

Next, we consider the vector space of formal linear combinations of elements in the quotient $V^k/\sim$ of V^k by $\sim$ and define another equivalence relation "$\approx$" as the linear extension of the rules

$$\lambda(v_1 \wedge \cdots \wedge v_k) = (\lambda v_1) \wedge \cdots \wedge v_k,$$
$$v_1 \wedge v_2 \wedge \cdots \wedge v_k + v_1' \wedge v_2 \wedge \cdots \wedge v_k = (v_1 + v_1') \wedge v_2 \wedge \cdots \wedge v_k,$$

where again $v_1, v_1', v_2, \ldots, v_k \in V$ and $\lambda \in \mathbb{R}$. This quotient by $\approx$ is called the vector space of k-**(multi-)vectors** over V and denoted by $\bigwedge_k V$. Note that $\bigwedge_1 V$ can be canonically identified with V itself and $\bigwedge_k V = \{0\}$ by the above rules if $k > \dim V$. We also set $\bigwedge_0 V := \mathbb{R}$. If an element $\xi \in \bigwedge_k V$ can be written as (i.e., is $\approx$-equivalent to) an element of the form $v_1 \wedge \cdots \wedge v_k$, then the k-vector ξ is called **simple**.

Example 3.1. In $\mathbb{R}^4$, the 2-vector $\mathrm{e}_1 \wedge \mathrm{e}_2 + \mathrm{e}_3 \wedge \mathrm{e}_4$ is not simple since

$$(\mathrm{e}_1 \wedge \mathrm{e}_2 + \mathrm{e}_3 \wedge \mathrm{e}_4) \wedge (\mathrm{e}_1 \wedge \mathrm{e}_2 + \mathrm{e}_3 \wedge \mathrm{e}_4) = 2\mathrm{e}_1 \wedge \mathrm{e}_2 \wedge \mathrm{e}_3 \wedge \mathrm{e}_4 \neq 0,$$

but the wedge-square of any simple vector must vanish by the calculation rules for the wedge product (see the "nilpotence" property in Lemma 3.4 below). One can also show that in $\mathbb{R}^3$ all multi-vectors (of any order) are in fact simple.

Next, one extends the notation "$\wedge$" (which, as defined above, is only a way to designate equivalence classes of the relations $\sim$ and $\approx$) to the **wedge product** (or **exterior product**) of a k-vector $\xi \in \bigwedge_k V$ and an l-vector $\eta \in \bigwedge_l V$ as follows: If $\xi = v_1 \wedge \cdots \wedge v_k$ and $\eta = w_1 \wedge \cdots \wedge w_l$, that is, both ξ and η are simple, then we define

$$\xi \wedge \eta := v_1 \wedge \cdots \wedge v_k \wedge w_1 \wedge \cdots \wedge w_l \in \textstyle\bigwedge_{k+l} V.$$

Observe that is indeed consistent with the previous meaning of "$\wedge$". For general $\xi \in \bigwedge_k V$ and $\eta \in \bigwedge_l V$ we pick any decompositions

$$\xi = \sum_m \xi_m, \qquad \eta = \sum_n \eta_n,$$

with ξ_m, η_n simple k-vectors and l-vectors, respectively, for finite indices m, n (this decomposition is always possible by the definition). Then set

$$\xi \wedge \eta := \sum_m \sum_n \xi_m \wedge \eta_n.$$

Of course, for this to be well-defined, one needs to show that this definition does not depend on the chosen decompositions (which are not unique). This can be accomplished by expressing every ξ_m further as a wedge product of basis vectors and using the computation rules.

Together with the (associative) wedge product, the direct sum of the spaces $\bigwedge_k V$, $k = 0, \dots, d$, is referred to as the **exterior algebra** $\bigwedge_* V$.

Remark 3.2. Connoisseurs of algebra may also think of $\bigwedge_* V$ as the space of alternating contravariant tensors (of any order k) over V, which is also isomorphic to $(\bigotimes_* V)/I$, where $\bigotimes_* V := \bigoplus_{k=0}^\infty V^{\otimes k}$ and $I \subset V^{\otimes 2}$ is the ideal generated by all tensors of the form $v \otimes v$ ($v \in V$). The wedge product $v_1 \wedge \dots \wedge v_k$ ($v_i \in V$) is then the image of $v_1 \otimes \dots \otimes v_k$ under the canonical homomorphism.

We sum up this construction in the following proposition:

Proposition 3.3. *For any finite-dimensional vector space V (e.g., $\mathbb{R}^3$ or $\mathbb{R}^{1+3}$) and any $k = 0, \dots, \dim V$, there exists a vector space*

$$\textstyle\bigwedge_k V$$

together with an operation

$$\wedge\colon \underbrace{V \times \dots \times V}_{k \text{ times}} \to \textstyle\bigwedge_k V$$

that is linear in every entry and such that for $v_1, \dots, v_k, v_1' \in V$ and $\lambda \in \mathbb{R}$ the following identities hold:

$$\begin{aligned} v_1 \wedge \dots \wedge (\lambda v_i) \wedge \dots \wedge v_j \wedge \dots \wedge v_k &= v_1 \wedge \dots \wedge v_i \wedge \dots \wedge (\lambda v_j) \wedge \dots \wedge v_k, \\ v_1 \wedge \dots \wedge v_i \wedge \dots \wedge v_j \wedge \dots \wedge v_k &= v_1 \wedge \dots \wedge (v_i + \lambda v_j) \wedge \dots \wedge v_j \wedge \dots \wedge v_k, \\ v_1 \wedge \dots \wedge v_i \wedge \dots \wedge v_j \wedge \dots \wedge v_k &= v_1 \wedge \dots \wedge (-v_j) \wedge \dots \wedge v_i \wedge \dots \wedge v_k, \\ \lambda(v_1 \wedge \dots \wedge v_k) &= (\lambda v_1) \wedge \dots \wedge v_k, \\ v_1 \wedge v_2 \wedge \dots \wedge v_k + v_1' \wedge v_2 \wedge \dots \wedge v_k &= (v_1 + v_1') \wedge v_2 \wedge \dots \wedge v_k. \end{aligned}$$

The following are some computation rules for wedge product, whose proofs follow directly from the definitions:

Lemma 3.4. *Let $\xi \in \bigwedge_k V$, $\eta, \varrho \in \bigwedge_l V$ (where $k, l \in \{0, \dots, d\}$) and let $\lambda \in \mathbb{R}$. Then:*

(i) ***Linearity:*** $\xi \wedge (\eta + \varrho) = \xi \wedge \eta + \xi \wedge \varrho$ *and* $(\lambda\xi) \wedge \eta = \xi \wedge (\lambda\xi) = \lambda(\xi \wedge \eta)$.

(ii) ***Skew-commutativity:*** $\xi \wedge \eta = (-1)^{kl}\eta \wedge \xi$*; in particular* $\xi \wedge \xi = 0$ *if* k *is odd.*

(iii) ***Nilpotence:*** $\xi \wedge \xi = 0$ *if* ξ *is simple; in particular* $v \wedge v = 0$ *for* $v \in V$.

The main reason for the introduction of k-vectors is that they allow one to specify oriented vector sub-spaces. For this, we define the **span** of a simple k-vector $\xi = v_1 \wedge \cdots \wedge v_k \in \bigwedge_k V$ by

$$\operatorname{span} \xi := \operatorname{span}\{v_1, \ldots, v_k\}.$$

For a non-simple k-vector $\xi = \sum_m \xi_m$ with ξ_m a simple k-vector for a finite index m, we set

$$\operatorname{span} \xi := \operatorname{span}\Big(\bigcup_m \operatorname{span} \xi_m\Big),$$

which turns out to be well-defined.

Lemma 3.5. *For* $\xi, \eta \in \bigwedge_k V$ *simple, it holds that* $\operatorname{span} \xi = \operatorname{span} \eta$ *if and only if* $\eta = \lambda\xi$ *for some* $\lambda \neq 0$.

Proof. For the non-trivial direction, we assume that $\xi, \eta \neq 0$ and $\operatorname{span} \xi = \operatorname{span} \eta$. If $\xi = v_1 \wedge \cdots \wedge v_k$ and $\eta = w_1 \wedge \cdots \wedge w_k$, this implies $w_1, \ldots, w_k \in \operatorname{span} \xi$, so $w_j = \sum_i \lambda_j^i v_i$. Thus,

$$\eta = \Big(\sum_i \lambda_1^i v_i\Big) \wedge \cdots \wedge \Big(\sum_i \lambda_k^i v_i\Big).$$

By a repeated application of the calculation rules for wedge products (exchanging factors and linearity), we obtain that η is a non-zero multiple of $v_1 \wedge \cdots \wedge v_k = \xi$. □

Thus, as a consequence of the preceding lemma we obtain that a simple k-vector can indeed be interpreted as a weighted oriented k-dimensional subspace of the ambient vector space. As for the meaning of the weighting, let us from now on assume that V has a scalar product that turns it into a Hilbert space. Then call a simple k-vector ξ a **unit** if it can be expressed as $\xi = v_1 \wedge \cdots \wedge v_k$ with the $\{v_i\}$ forming an orthonormal basis of $\operatorname{span} \xi$. For $V = \mathbb{R}^N$ we have the following interpretation of the "length" of a multi-vector:

Lemma 3.6. *For* $\xi = v_1 \wedge \cdots \wedge v_k \in \bigwedge_k \mathbb{R}^N$ *simple, assume that* ξ *can be written as* $\xi = \lambda\xi_1$ *with a unit k-vector* ξ_1 *and* $\lambda > 0$*. Then,* λ *is the (k-dimensional) volume of the parallelotope spanned by the vectors* v_j*,* $j = 1, \ldots, k$.

Proof. Let $\xi_1 = w_1 \wedge \cdots \wedge w_k$, where $\{w_j\}$ is an orthonormal basis of $\operatorname{span}\xi = \operatorname{span}\xi_1$. We write $v_j = \sum_i \lambda^i_j w_i$ and collect all the λ^i_j into a matrix $\Lambda = (\lambda^i_j)^i_j \in \mathbb{R}^{k\times k}$ (where i is the row index and j is the column index). We can now bring the matrix to diagonal form by the Gaussian elimination procedure. Note that every operation in the Gaussian elimination corresponds precisely to one of the above computation rules for wedge products. Geometrically, the Gaussian operations correspond to shearing the parallelotope, which does not change its volume. The resulting diagonal matrix corresponds to a k-dimensional cube with volume equal to the product of the obtained diagonal entries. Pulling these entries in front of the matrix yields that λ is precisely the volume of this cube, hence also the volume of the parallelotope spanned by the original vectors $\{v_j\}$. □

A scalar product $(\bullet, \bullet)$ on V extends to the **scalar product** of simple k-vectors $\xi = v_1 \wedge \cdots \wedge v_k$, $\eta = w_1 \wedge \cdots \wedge w_k$ by setting

$$(\xi, \eta) := \det \begin{pmatrix} v_1 \cdot w_1 & \cdots & v_1 \cdot w_k \\ \vdots & \ddots & \vdots \\ v_k \cdot w_1 & \cdots & v_k \cdot w_k \end{pmatrix}.$$

This definition is then extended to general k-vectors by linearity. This means, that if $\xi = \xi_1 + \xi_2$ and $\eta = \eta_1 + \eta_2$ with $\xi_1, \xi_2, \eta_1, \eta_2$ simple, then by "expanding the product",

$$(\xi, \eta) := (\xi_1, \eta_1) + (\xi_1, \eta_2) + (\xi_2, \eta_1) + (\xi_2, \eta_2),$$

with the individual products defined as above. By using a basis, for instance by first defining this for wedges of the unit basis, one can show that the result is well-defined, meaning that the value does not depend on which decomposition of ξ and η one chooses.

The **(Euclidean) norm** of k-vectors is then defined via

$$|\xi| := (\xi, \xi)^{1/2}, \qquad \xi \in \textstyle\bigwedge_k \mathbb{R}^N.$$

In particular, $|\xi| = 1$ for any unit k-vector ξ.

If $L\colon V \to W$ is linear, where V, W are real finite-dimensional vector spaces, we define the linear map $\bigwedge^k S\colon \bigwedge_k V \to \bigwedge_k W$ by setting, for $v_1, \ldots, v_k \in V$,

$$(\textstyle\bigwedge^k L)(v_1 \wedge \cdots \wedge v_k) := (Lv_1) \wedge \cdots \wedge (Lv_k)$$

and extending by (multi-)linearity to $\bigwedge_k V$. We will usually write simply L also for $\bigwedge^k L$.

3.2 Covectors

We will also need multi-vectors based on the dual space V^* of a vector spaces V, which are called **k-(multi-)covectors** and denoted by

$$\textstyle\bigwedge^k V := \bigwedge_k V^*.$$

Once again we set $\bigwedge^0 V := \mathbb{R}$ (or, more precisely, the dual of $\mathbb{R}$) and define the wedge product on covectors similarly to before, yielding $\bigwedge^* V$. As k-covectors are k-vectors in the sense of the preceding section, all the theory developed there applies.

The space $\bigwedge^k V$ can be identified with the dual space to $\bigwedge^k V$ via the following duality pairing: $u_1, \ldots, u_k \in V$ and $\alpha^1, \ldots, \alpha^k \in V^*$ one sets

$$\bigl\langle u_1 \wedge \cdots \wedge u_k, \alpha^1 \wedge \cdots \wedge \alpha^k \bigr\rangle := \det\bigl[\bigl\langle u_i, \alpha^j \bigr\rangle\bigr]^i_j.$$

Then, the definition is extended bilinearly. This duality product is well-defined because the right-hand side is alternating in $u_1, \ldots, u_k$ (swapping two u_i negates the determinant) and alternating in $\alpha^1, \ldots, \alpha^k$ likewise. In particular, it vanishes whenever two vectors or two covectors coincide. Using the Leibniz expansion of the determinant, we may derive the formula

$$\bigl\langle u_1 \wedge \cdots \wedge u_k, \alpha^1 \wedge \cdots \wedge \alpha^k \bigr\rangle = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \prod_{i=1}^k \alpha^{\sigma(i)}(u_i).$$

Furthermore, $\bigwedge^k V$ can be identified with the space of k-multilinear (linear in every argument) and alternating (switches sign when exchanging two arguments) functions on V^k.

In $V = \mathbb{R}^N$ the standard dual basis is usually denoted by $\mathrm{d}x^1, \ldots, \mathrm{d}x^N$ (this can be identified with the exterior differential of the coordinate functions, as we will see later, motivating the notation).

One way to combine k-vectors and k-covectors is the following: For $\eta \in \bigwedge_k V$ and $\alpha \in \bigwedge^l V$ we define the **restriction** $\eta \llcorner \alpha \in \bigwedge_{k-l} V$ with $k \geq l$ (read "$\llcorner$" as "restricted to") via

$$\bigl\langle \eta \llcorner \alpha, \beta \bigr\rangle := \bigl\langle \eta, \alpha \wedge \beta \bigr\rangle, \qquad \beta \in \textstyle\bigwedge^{k-l} V.$$

The **metric dual** $\xi^* \in V^*$ of a unit vector $\xi \in V$ is defined by

$$\xi^*(v) := (\xi, v), \qquad v \in V.$$

Lemma 3.7. *The following calculation rules hold:*

(i) For any vectors $v_1, \ldots, v_n \in V$ and a covector $\alpha \in \bigwedge^1 V$,

$$(v_1 \wedge \cdots \wedge v_n) \llcorner \alpha = \sum_{i=1}^n (-1)^{i-1} \alpha(v_i)\, v_1 \wedge \cdots \wedge \widehat{v}_i \wedge \cdots \wedge v_n,$$

where $\widehat{v}_i$ denotes omission of v_i.

(ii) For any $\varphi \in \bigwedge_k V$, $\psi \in \bigwedge_l V$, and $\alpha \in \bigwedge^1 V$,

$$(\varphi \wedge \psi) \llcorner \alpha = (\varphi \llcorner \alpha) \wedge \psi + (-1)^k \varphi \wedge (\psi \llcorner \alpha).$$

(iii) Let $\xi \in V$ be a unit vector and let $\tau \in \bigwedge_{1+k} V$ with $\xi \wedge \tau = 0$. Then,

$$\xi \wedge (\tau \llcorner \xi^*) = \tau.$$

Proof. Ad (i). Let $\omega := v_1 \wedge \cdots \wedge v_n$ and $\beta \in \bigwedge^{n-1} V$. Using the definition of the duality product via the determinant,

$$\big\langle \omega \llcorner \alpha,\, \beta \big\rangle = \big\langle \omega,\, \alpha \wedge \beta \big\rangle = \big\langle v_1 \wedge \cdots \wedge v_n,\, \alpha \wedge \beta \big\rangle.$$

Expanding the determinant along the column corresponding to α, only the term with α evaluated at v_i and β evaluated at the remaining vectors survives. Hence,

$$\big\langle \omega \llcorner \alpha,\, \beta \big\rangle = \sum_{i=1}^{n} (-1)^{i-1}\, \alpha(v_i)\, \big\langle v_1 \wedge \cdots \wedge \widehat{v}_i \wedge \cdots \wedge v_n,\, \beta \big\rangle.$$

Since this holds for all β, part (i) follows.

Ad (ii). By multilinearity it suffices to take $\varphi = v_1 \wedge \cdots \wedge v_k$ and $\psi = v_{k+1} \wedge \cdots \wedge v_{k+l}$ for $v_1, \ldots, v_{k+l} \in V$. Applying (i) to $\varphi \wedge \psi = v_1 \wedge \cdots \wedge v_{k+l}$ gives

$$\begin{aligned}(\varphi \wedge \psi) \llcorner \alpha = &\sum_{i=1}^{k} (-1)^{i-1}\, \alpha(v_i)\, v_1 \wedge \cdots \wedge \widehat{v}_i \wedge \cdots \wedge v_{k+l} \\ &+ \sum_{i=k+1}^{k+l} (-1)^{i-1}\, \alpha(v_i)\, v_1 \wedge \cdots \wedge \widehat{v}_i \wedge \cdots \wedge v_{k+l}.\end{aligned}$$

By (i), the first sum equals $(\varphi \llcorner \alpha) \wedge \psi$. In the second sum, factor out $\varphi = v_1 \wedge \cdots \wedge v_k$; each remaining sign satisfies $(-1)^{i-1} = (-1)^k \cdot (-1)^{i-k-1}$, so by (i) the second sum equals $(-1)^k\, \varphi \wedge (\psi \llcorner \alpha)$. This shows (ii).

Ad (iii). Apply (ii) with $\varphi := \xi \in \bigwedge_1 V$, $\psi := \tau$, and $\alpha := \xi^*$ to see that

$$(\xi \wedge \tau) \llcorner \xi^* = (\xi \llcorner \xi^*) \wedge \tau - \xi \wedge (\tau \llcorner \xi^*) = \tau - \xi \wedge (\tau \llcorner \xi^*).$$

Here, for the second equality we used that $\xi \llcorner \xi^* = \xi^*(\xi) = \langle \xi, \xi \rangle = 1$ since ξ is a unit vector. The hypothesis $\xi \wedge \tau = 0$ makes the left-hand side vanish, so that (iii) follows by rearranging. $\square$

The following is a useful technical fact:

Lemma 3.8. *Let $\tau \in \bigwedge_1 V$, $\sigma \in \bigwedge_k V$ and $\alpha \in \bigwedge^1 V$, $\beta \in \bigwedge^k V$, $\eta \in \bigwedge^{k+1} V$.*

(i) Suppose that $\langle w, \alpha \rangle = 0$ for every $w \in \operatorname{span} \sigma$. Then,

$$\big\langle \tau \wedge \sigma, \alpha \wedge \beta \big\rangle = \big\langle \tau, \alpha \big\rangle \cdot \big\langle \sigma, \beta \big\rangle.$$

(ii) Suppose that $\tau \perp \operatorname{span}\eta$. Then,

$$\langle \tau \wedge \sigma, \eta \rangle = 0.$$

Proof. We only show (i); the proof of (ii) is similar.

Let $W := \operatorname{span}\sigma$, and let $w_1, \dots, w_m$ be a basis of W. We can write σ as

$$\sigma = \sum_I \sigma_I w_I,$$

where $I = \{i_1, \dots, i_k\}$ ranges over all k-multi-indices of $\{1, \dots, m\}$ and we have set $w_I := w_{i_1} \wedge \cdots \wedge w_{i_k}$. By linearity, it is sufficient to prove the claim when $\sigma = w_I$ for some k-multi-index I and without loss of generality we may assume that $I = \{1, \dots, k\}$. Moreover, it suffices to consider $\beta = \beta_1 \wedge \cdots \wedge \beta_k$ for some $\beta_1, \dots, \beta_k \in V^*$. We also define, for notational simplicity $w_0 := \tau$ and $\beta_0 := \alpha$. Then,

$$\begin{aligned}
\langle \tau \wedge \sigma, \alpha \wedge \beta \rangle &= \langle w_0 \wedge w_1 \wedge \cdots \wedge w_k, \beta_0 \wedge \beta_1 \wedge \cdots \wedge \beta_k \rangle \\
&= \det \big[\langle w_i, \beta_j \rangle \big]_{j=0,\dots,k}^{i=0,\dots,k} \\
&= \langle w_0, \beta_0 \rangle \cdot \big[\langle w_i, \beta_j \rangle \big]_{j=1,\dots,k}^{i=1,\dots,k} \\
&= \langle \tau, \alpha \rangle \cdot \langle \sigma, \beta \rangle,
\end{aligned}$$

where expanded the determinant on the first column ($j = 0$) and used the assumption that $\langle w_i, \alpha \rangle = 0$ for $i = 1, \dots, k$. $\square$

Using the vector–covector duality one can then define the **mass norm** of $\xi \in \bigwedge_k V$ and the **comass norm** of $\eta \in \bigwedge^k V$ via

$$\begin{aligned}
\|\xi\| &:= \sup\big\{ |\langle \xi, \eta \rangle| \ : \ \eta \in \textstyle\bigwedge^k V, \ \|\eta\| = 1 \big\}, \\
\|\eta\| &:= \sup\big\{ |\langle \xi, \eta \rangle| \ : \ \xi \in \textstyle\bigwedge_k V \text{ simple, unit} \big\}
\end{aligned}$$

Lemma 3.9. *For all $\xi \in \bigwedge_k V$ and $\alpha \in \bigwedge^k V$,*

(i) $|\xi| \le \|\xi\|$, with equality if ξ is simple,

(ii) $\|\alpha\| \le |\alpha|$, with equality if α is simple.

Proof. The claims are trivial for $\xi = 0$ and $\alpha = 0$ respectively, so assume both are nonzero.

Ad (ii). Via the inner product we identify $\bigwedge^k V \cong \bigwedge_k V$, so the comass norm reads

$$\|\alpha\| = \sup\big\{ |(\alpha, \beta)| \ : \ \beta \in \textstyle\bigwedge^k V, \ \beta \text{ simple, unit} \big\}.$$

Every simple unit β satisfies $|(\alpha, \beta)| \le |\alpha|$ by Cauchy–Schwarz, so taking the supremum yields $\|\alpha\| \le |\alpha|$. If α is simple, then $\beta := \alpha/|\alpha|$ is a simple unit competitor, giving $\|\alpha\| \ge |\alpha|^{-1}(\alpha, \alpha) = |\alpha|$, hence we obtain equality.

Ad (i). Let $\alpha_0 \in \bigwedge^k V$ be the metric dual of $\xi/|\xi|$, so that $|\alpha_0| = 1$ and $\langle \xi, \alpha_0 \rangle = |\xi|$. By part (ii), $\|\alpha_0\| \le |\alpha_0| = 1$, so $\alpha_1 := \alpha_0/\|\alpha_0\|$ satisfies $\|\alpha_1\| = 1$ and is a valid competitor in the supremum defining $\|\xi\|$, giving

$$\|\xi\| \ge \big|\langle \xi, \alpha_1 \rangle\big| = \frac{|\xi|}{\|\alpha_0\|} \ge |\xi|,$$

where the last inequality uses $\|\alpha_0\| \le 1$.

If ξ is simple, let $\xi_1 := \xi/|\xi|$, which is simple and unit. For every $\alpha \in \bigwedge^k V$ with $\|\alpha\| = 1$, the definition of the comass norm gives $|\langle \xi_1, \alpha \rangle| \le \|\alpha\| = 1$, and therefore

$$\big|\langle \xi, \alpha \rangle\big| = |\xi| \cdot \big|\langle \xi_1, \alpha \rangle\big| \le |\xi|.$$

Taking the supremum over all such α gives $\|\xi\| \le |\xi|$, and combined with the inequality above we obtain $\|\xi\| = |\xi|$. □

3.3 Differential forms

Let $\Omega \subset \mathbb{R}^d$ be an open set (the ambient dimension d being fixed). As usual we denote by $\mathrm{C}(\Omega) = \mathrm{C}^0(\Omega), \mathrm{C}^k(\Omega)$, $k = 1, 2, \ldots$, the spaces of continuous, bounded, and k-times continuously differentiable functions with bounded derivatives. As norms on these spaces we have

$$\|u\|_{\mathrm{C}^k} := \sum_{|\alpha| \le k} \|\partial^\alpha u\|_\infty, \qquad u \in \mathrm{C}^k(\Omega), \qquad k = 0, 1, 2, \ldots,$$

where $\|\cdot\|_\infty$ is the supremum norm. Here, the sum is over all multi-indices $\alpha \in \mathbb{N}_0^d$ ($\mathbb{N}_0 := \mathbb{N} \cup \{0\}$) with $|\alpha| := \alpha_1 + \cdots + \alpha_d \le k$ and

$$\partial^\alpha := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d}$$

is the α-derivative operator.

We define the set $\mathrm{C}^\infty(\Omega) := \bigcap_{k=0}^\infty \mathrm{C}^k(\Omega)$ of infinitely-often differentiable functions (but this is not a Banach space anymore). A subscript "$_c$" indicates that all functions u in the respective function space (e.g. $\mathrm{C}_c^\infty(\Omega)$) must have their support

$$\operatorname{supp} u := \overline{\{\, x \in \Omega \,:\, u(x) \neq 0 \,\}}$$

compactly contained in Ω (so, $\operatorname{supp} u \subset \Omega$ and $\operatorname{supp} u$ compact).

The space $\mathrm{C}_c^\infty(\Omega)$, that is, the set of all $\varphi\colon \Omega \to \mathbb{R}$ that are infinitely differentiable and compactly supported, is called the space of **test functions** (on Ω). Denote further by $\mathcal{D}^k(\Omega)$ ($k \in \mathbb{N} \cup \{0\}$) the space of **(smooth, compactly supported) differential k-forms**, that is,

$$\mathcal{D}^k(\Omega) := \mathrm{C}_c^\infty(\Omega; \textstyle\bigwedge^k \mathbb{R}^d),$$

where $\mathrm{C}_c^\infty(\Omega; W)$ contains all smooth maps that take values in the finite-dimensional normed vector space W and that are compactly supported in Ω. Since $\bigwedge^0 \mathbb{R}^d$ is identified with $\mathbb{R}$, we have that $\mathcal{D}^0$ contains the smooth and compactly supported functions with values in $\mathbb{R}$. In addition to the vector space structure of $\mathcal{D}^k(\Omega)$, we also define the **wedge product** of two differential forms: For $\omega \in \mathcal{D}^k(\Omega)$, $\eta \in \mathcal{D}^l(\Omega)$, we define $\omega \wedge \eta \in \mathcal{D}^{k+l}(\Omega)$ by setting

$$(\omega \wedge \eta)(x) := \omega(x) \wedge \eta(x), \qquad x \in \Omega.$$

One can think of a differential k-form ω as specifying a way to measure the size of k-dimensional objects at any point of the ambient space. For instance, if we follow our intuition that a simple k-vector $\xi = v_1 \wedge \cdots \wedge v_k \in \bigwedge_k \mathbb{R}^d$ specifies the k-dimensional parallelotope spanned by the vectors $v_1, \ldots, v_k$, then pairing it with the k-covector $\omega(x) \in \bigwedge^k \mathbb{R}^d$ via the duality pairing $\langle \xi, \omega(x) \rangle$ yields a real number, which one may interpret as the "oriented volume" assigned to ξ (or, more precisely, the k-parallelotope associated with ξ) at x. In particular, let $\xi = \lambda \xi_1 \in \bigwedge_k \mathbb{R}^d$ for $\xi_1 = \mathrm{e}_1 \wedge \cdots \wedge \mathrm{e}_k$, where we have denoted the canonical basis vectors in $\mathbb{R}^d$ by $\mathrm{e}_1, \ldots, \mathrm{e}_d$. Denoting by $\mathrm{d}x^1, \ldots, \mathrm{d}x^d$ are the canonical basis elements in $\bigwedge^1 \mathbb{R}^d$ (so that $\langle \mathrm{e}_i, \mathrm{d}x^j \rangle = \delta_{ij}$),

$$\big\langle \xi, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^k \big\rangle = \lambda \big\langle \mathrm{e}_1 \wedge \cdots \wedge \mathrm{e}_k, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^k \big\rangle = \lambda$$

is the standard volume of the parallelotope specified by ξ.

The **exterior differential** of $\omega \in \mathcal{D}^k(\Omega)$ is denoted by $\mathrm{d}\omega \in \mathcal{D}^{k+1}(\Omega)$ and defined as follows: For a 0-form $f \in \mathcal{D}^0(\Omega) \cong \mathrm{C}_c^\infty(\Omega)$, we set

$$\mathrm{d}f := \frac{\partial f}{\partial x^1}\, \mathrm{d}x^1 + \cdots + \frac{\partial f}{\partial x^d}\, \mathrm{d}x^d.$$

Here, one should think of $\mathrm{d}x^j$ as the map that takes a (1-)vector and yields its j's component (recall that we are in the ambient space $\mathbb{R}^d$ here and have canonical coordinates). For a k-form $\omega \in \mathcal{D}^k(\Omega)$ that can be written as $\omega = f\, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$ for $f \in \mathrm{C}_c^\infty(\Omega)$ and indices $i_1, \ldots, i_k \in \{1, \ldots, d\}$, we set

$$\mathrm{d}\omega := \mathrm{d}f \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}.$$

If $\omega \in \mathcal{D}^k(\Omega)$ is not of the above simple form, then it is a linear combination of such simple forms (by the definition of k-covectors, the $\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$ form a basis of $\bigwedge^k \mathbb{R}^d$) and we extend this definition by linearity, e.g., for $\omega = \omega_1 + \omega_2$ with $\omega_1, \omega_2 \in \mathcal{D}^k(\Omega)$ simple, we set $\mathrm{d}\omega := \mathrm{d}\omega_1 + \mathrm{d}\omega_2$, and so on. One can show easily that this definition is indeed well-defined.

It is somewhat difficult to interpret the exterior differential. In fact, this interpretation is best done in the context of Stokes's theorem in the next section. Let us just remark that the exterior differential of a 0-form (a function) corresponds to a *gradient*, the exterior differential of a $(d-1)$-form corresponds to a *divergence* and the exterior differential of a 1-form in three ambient dimensions corresponds to a *curl*. In this way, the usual Vector Analysis operations are contained in the exterior calculus, which, however, is more systematic. For instance, we have only one Leibniz rule in the following lemma, and not a confusingly large number of identities for the Vector Analysis operations.

Lemma 3.10. *The exterior differential is linear and the* ***Leibniz rule*** *holds for* $\omega \in \mathcal{D}^k(\Omega)$*,* $\eta \in \mathcal{D}^l(\Omega)$*:*

$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^k \, \omega \wedge \mathrm{d}\eta.$$

Proof. The linearity is obvious, so let us only show the Leibniz rule: By linearity it suffices to consider the case where $\omega = f \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$ and $\eta = g \, \mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_l}$ for $f, g \in \mathrm{C}_c^\infty(\Omega)$ and indices $i_1, \ldots, i_k, \ j_1, \ldots, j_l \in \{1, \ldots, d\}$. Then,

$$\begin{aligned}
\mathrm{d}(\omega \wedge \eta) &= \mathrm{d}(fg) \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \wedge \mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_l} \\
&= \sum_i \left(\frac{\partial f}{\partial x^i} \cdot g + f \cdot \frac{\partial g}{\partial x^i} \right) \mathrm{d}x^i \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \wedge \mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_l} \\
&= \left[\sum_i \frac{\partial f}{\partial x^i} \, \mathrm{d}x^i \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \right] \wedge \eta \\
&\qquad + \omega \wedge \left[(-1)^k \sum_i \frac{\partial g}{\partial x^i} \, \mathrm{d}x^i \wedge \mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_l} \right] \\
&= \mathrm{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathrm{d}\eta,
\end{aligned}$$

where the factor $(-1)^k$ originates from the operation of pushing $\mathrm{d}x^i$ through every one of the $\mathrm{d}x^{i_1}, \ldots, \mathrm{d}x^{i_k}$. □

One crucial property of the exterior differential is its *nilpotence*:

Proposition 3.11. *For* $\omega \in \mathcal{D}^k(\Omega)$ *it holds that* $\mathrm{d}^2\omega = \mathrm{d}(\mathrm{d}\omega) = 0$*.*

Proof. Assume that $\omega = f \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$ for $f \in \mathrm{C}_c^\infty(\Omega)$ and indices $i_1, \ldots, i_k \in \{1, \ldots, d\}$. Then,

$$\begin{aligned}
\mathrm{d}^2\omega &= \sum_{ij} \frac{\partial}{\partial x^j} \left(\frac{\partial f}{\partial x^i} \right) \mathrm{d}x^j \wedge \mathrm{d}x^i \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \\
&= \sum_{i<j} \left[\frac{\partial}{\partial x^i} \left(\frac{\partial f}{\partial x^j} \right) - \frac{\partial}{\partial x^j} \left(\frac{\partial f}{\partial x^i} \right) \right] \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \\
&= 0
\end{aligned}$$

since for smooth maps second derivatives commute. The case of general forms follows by a decomposition into simple forms and the linearity. □

We call a k-form $\omega \in \mathcal{D}^k(\Omega)$ **closed** if $\mathrm{d}\omega = 0$ and **exact** if $\omega = \mathrm{d}\psi$ for some "potential" $\psi \in \mathcal{D}^{k-1}(\Omega)$. By the previous proposition, exact forms are closed, but the converse does not hold. Quantifying the difference is the content of Cohomology Theory. However, for simply connected sets Ω, it is true that closed forms are also exact by Poincaré's lemma.

Let $\theta \colon \Omega \to \Omega'$ be a smooth map between open sets $\Omega \subset \mathbb{R}^d$, $\Omega' \subset \mathbb{R}^m$ that is **proper**, meaning that preimages of compact sets are themselves compact. Let $\omega \in \mathcal{D}^k(\Omega')$. Then, the **pullback** of ω under θ is the k-form $\theta^*\omega \in \mathcal{D}^k(\Omega)$ defined at $x \in \mathbb{R}^d$ via

$$\big\langle \xi, (\theta^*\omega)(x) \big\rangle := \big\langle \mathrm{D}\theta(x)[\xi], \omega(\theta(x)) \big\rangle, \qquad \xi \in \textstyle\bigwedge_k \mathbb{R}^d.$$

Here, $\mathrm{D}\theta(x) = \bigwedge^k \mathrm{D}\theta(x)$ denotes the extension of the differential $\mathrm{D}\theta(x)[v] := \nabla\theta(x)v$ to k-vectors, that is, if $\xi = v_1 \wedge \cdots \wedge v_k$, then

$$\mathrm{D}\theta(x)[\xi] = \mathrm{D}\theta(x)[v_1] \wedge \cdots \wedge \mathrm{D}F(x)[v_k].$$

Note that the pullback $\theta^*\omega$ as defined above is indeed a smooth, compactly supported differential k-form on Ω by the smoothness and properness of θ (without the properness the above definition may not be sensible since $\omega \circ \theta$ may then not have compact support).

Lemma 3.12. *Let $\theta \colon \Omega \to \Omega'$ be a smooth and proper map between open sets $\Omega \subset \mathbb{R}^d$, $\Omega' \subset \mathbb{R}^m$, and let $\omega \in \mathcal{D}^k(\Omega')$. Then,*

$$\mathrm{d}(\theta^*\omega) = \theta^*(\mathrm{d}\omega).$$

Proof. Let first treat the case $k = 0$, in which $\theta^*\omega = \omega \circ \theta$. Then, for all $x \in \Omega$ and all $v \in \mathbb{R}^d \cong \bigwedge_1 \mathbb{R}^d$,

$$\begin{aligned}
\big\langle v, \mathrm{d}(\theta^*\omega)(x) \big\rangle &= \nabla(\omega \circ \theta)(x)v \\
&= \nabla\omega(\theta(x))\nabla\theta(x)v \\
&= \big\langle \mathrm{D}\theta(x)[v], (\mathrm{d}\omega)(\theta(x)) \big\rangle \\
&= \big\langle v, (\theta^*(\mathrm{d}\omega))(x) \big\rangle,
\end{aligned}$$

showing the claim for $k = 0$.

Next, if $k = 1$ and $\omega = \mathrm{d}\psi$ for $\psi \in \mathcal{D}^0(\mathbb{R}^d)$, then

$$\mathrm{d}(\theta^*\omega) = \mathrm{d}(\theta^*(\mathrm{d}\psi)) = \mathrm{d}^2(\theta^*\psi) = 0 = \theta^*(\mathrm{d}^2\psi) = \theta^*(\mathrm{d}\psi),$$

where we have used the case $k = 0$ and Proposition 3.11.

For the general case, we use the linearity and the Leibniz rule from Lemma 3.10 to reduce to the above two cases by induction (details as exercise). □

3.4 Stokes's theorem

Roughly, Stokes theorem states that the exterior differential is dual to taking the (oriented) boundary of a set. We will consider a version that is less than the most general one (which involves manifolds), but which will give a good intuition about the central claim. The boundary operator of currents will later yield a far-reaching generalization.

For $k \in \mathbb{N}$, let $R \subset \mathbb{R}^k$ be a k**-box**, that is,

$$R = [a_1, b_1] \times \cdots \times [a_k, b_k],$$

where $a_i, b_i \in \mathbb{R}$ with $a_i < b_i$ for $i = 1, \ldots, k$. Then, we can define the **integral** of a k-form $\omega \in \mathcal{D}^k(\mathbb{R}^k)$ over R via

$$\int_R \omega = \int_{\oplus R} \omega := \int_R \big\langle \mathrm{e}_1 \wedge \cdots \wedge \mathrm{e}_k, \omega(x) \big\rangle \, \mathrm{d}x.$$

The notation "$\oplus R$" expresses that we have picked the "positive orientation" of R by pairing $\omega(x)$ with the d-vector $\mathrm{e}_1 \wedge \cdots \wedge \mathrm{e}_k$. We could have also chosen the k-vector $-\mathrm{e}_1 \wedge \cdots \wedge \mathrm{e}_k$ (note that these two are the only unit k-vectors in $\mathbb{R}^k$). In this case we write

$$\int_{\ominus R} \omega := -\int_R \big\langle \mathrm{e}_1 \wedge \cdots \wedge \mathrm{e}_k, \omega(x) \big\rangle \, \mathrm{d}x.$$

We can extend this definition to the integral of a $(k-1)$-form on a **(closed)** $(k-1)$**-box**

$$R = [a_1, b_1] \times \cdots \times [a_{i-1}, b_{i-1}] \times \{c\} \times [a_{i+1}, b_{i+1}] \times \cdots \times [a_k, b_k],$$

with $a_i, b_i \in \mathbb{R}$ with $a_i < b_i$ for $i = 1, \ldots, i-1, i+1, \ldots, k$ and $c \in \mathbb{R}$, as follows: For a $(k-1)$-form $\omega \in \mathcal{D}^{k-1}(\mathbb{R}^k)$ define

$$\int_R \omega = \int_{\oplus R} \omega := \int_R \big\langle \mathrm{e}_1 \wedge \cdots \wedge \mathrm{e}_{i-1} \wedge \mathrm{e}_{i+1} \wedge \cdots \wedge \mathrm{e}_k, \omega(x) \big\rangle \, \mathrm{d}\mathcal{H}^{k-1}(x)$$

and

$$\int_{\ominus R} \omega := -\int_R \big\langle \mathrm{e}_1 \wedge \cdots \wedge \mathrm{e}_{i-1} \wedge \mathrm{e}_{i+1} \wedge \cdots \wedge \mathrm{e}_k, \omega(x) \big\rangle \, \mathrm{d}\mathcal{H}^{k-1}(x),$$

which is to be understood as the integral with respect to $\mathcal{H}^{k-1} \llcorner R$. Note that in the case $k = 1$, the above integrals are point-evaluations.

The symbols "$\oplus$", "$\ominus$" can be used to form **chains** of boxes by linearity: For two k- or $(k-1)$-boxes R_1, R_2 and a corresponding form ω we set

$$\int_{R_1 \oplus R_2} \omega := \int_{R_1} \omega + \int_{R_2} \omega, \qquad \int_{R_1 \ominus R_2} \omega := \int_{R_1} \omega + \int_{\ominus R_2} \omega = \int_{R_1} \omega - \int_{R_2} \omega,$$

and so on.

Let now $F\colon U \to \mathbb{R}^d$ be a smooth and proper (but not necessarily injective!) map, where $U \subset \mathbb{R}^k$ is open. For a k- or $(k-1)$-box $R \subset U$ and $\omega \in \mathcal{D}^k(\mathbb{R}^d)$ (if R is top-dimensional) or $\omega \in \mathcal{D}^{k-1}(\mathbb{R}^d)$ (if R is codimension-1), we set

$$\int_{F_*R} \omega := \int_R F^*\omega.$$

Here, $F_*R := F(R)$ is called the **pushforward** of R under F. This definition extends to chains by (formal) linearity: For boxes R_1, R_2 as before, we set

$$\int_{F_*[R_1 \oplus R_2]} := \int_{F_*R_1} \omega + \int_{F_*R_2} \omega, \qquad \int_{F_*[R_1 \ominus R_2]} := \int_{F_*R_1} \omega - \int_{F_*R_2} \omega,$$

and so on. Formally, we can then define chains of pushforwards via $F_*[R_1 \oplus R_2] := F_*R_1 \oplus F_*R_2$, $F_*[R_1 \ominus R_2] := F_*R_1 \ominus F_*R_2$, etc.

For a k-box $R = [a_1, b_1] \times \cdots \times [a_k, b_k]$ we can define an **(oriented) boundary** as follows:

$$\partial R := \bigoplus_{i=1}^{d} \ominus^{i-1}(R_i^+ \ominus R_i^-) = \bigoplus_{i=1}^{d} (\ominus^{i-1} R_i^+) \oplus (\ominus^i R_i^-).$$

where

$$\begin{aligned} R_i^- &:= [a_1, b_1] \times \cdots [a_{i-1}, b_{i-1}] \times \{a_i\} \times [a_{i+1}, b_{i+1}] \times \cdots \times [a_k, b_k], \\ R_i^+ &:= [a_1, b_1] \times \cdots [a_{i-1}, b_{i-1}] \times \{b_i\} \times [a_{i+1}, b_{i+1}] \times \cdots \times [a_k, b_k], \end{aligned}$$

and the powers of "$\ominus$" are defined as follows:

$$\ominus^i := \begin{cases} \oplus & \text{if } i \text{ is even,} \\ \ominus & \text{if } i \text{ is odd.} \end{cases}$$

Note that this extends the definition of the classical (topological) boundary of R to a suitably oriented chain of $(k-1)$-boxes. This orientation is called the **induced orientation**.

Example 3.13. If $k = 1$ and $R = [a, b]$, then $\partial R = \{b\} \ominus \{a\}$.

Example 3.14. If $k = 2$ and $R = [a_1, b_1] \times [a_2, b_2]$, then

$$\partial R = (\{b_1\} \times [a_2, b_2]) \ominus (\{a_1\} \times [a_2, b_2]) \ominus ([a_1, b_1] \times \{b_2\}) \oplus ([a_1, b_1] \times \{a_2\}).$$

Also the notion of boundary extends to pushforwards: Let $F\colon U \to \mathbb{R}^d$ be a smooth and proper map, where $\Omega \subset \mathbb{R}^k$ is open. For a k-box $R \subset U$ we set

$$\partial(F_*R) := \bigoplus_{i=1}^{d} \ominus^{i-1}(F_*R_i^+ \ominus F_*R_i^-) = F_*(\partial R),$$

where we interpret $R_i^{\pm}$ to lie in their own copies of $\mathbb{R}^{k-1}$.

Comparing the relation $\partial(F_*R) = F_*(\partial R)$, which holds by definition, with the relation $\mathrm{d}(F^*\omega) = F^*(\mathrm{d}\omega)$, proved in Lemma 3.12, hints at a duality between the exterior differential and the oriented boundary. This duality is the content of the Stokes theorem:

Theorem 3.15 (Stokes). *Let R be a k-box in $\mathbb{R}^k$ and let $F\colon U \to \mathbb{R}^d$ be smooth and proper, where $U \supset R$ is open. Then, for any $\omega \in \mathcal{D}^{k-1}(V)$, where $V \supset F(U)$ is open, it holds that*

$$\int_{F_*R} \mathrm{d}\omega = \int_{\partial(F_*R)} \omega.$$

Proof. It suffices to show that for all $\eta \in \mathcal{D}^{k-1}(\mathbb{R}^k)$ it holds that

$$\int_R \mathrm{d}\eta = \int_{\partial R} \eta. \tag{3.1}$$

Then, using Lemma 3.12,

$$\int_{F_*R} \mathrm{d}\omega = \int_R F^*(\mathrm{d}\omega) = \int_R \mathrm{d}(F^*\omega) = \int_{\partial R} F^*\omega = \int_{F^*(\partial R)} \omega = \int_{\partial(F_*R)} \omega$$

and the claim of the theorem follows.

To establish (3.1) we write

$$\eta = \sum_{i=1}^{k} \eta_i \,\widehat{\mathrm{d}x^i},$$

where $\eta_i \in \mathrm{C}_c^\infty(\mathbb{R}^k)$ for $i = 1, \ldots, k$, and

$$\widehat{\mathrm{d}x^i} := \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge \cdots \wedge \mathrm{d}x^k$$

This is always possible since the $(k-1)$-covectors $\widehat{\mathrm{d}x^i}$, $i = 1, \ldots, k$, span $\bigwedge^{k-1} \mathbb{R}^k$ (exercise). By linearity we need only to show

$$\int_R \mathrm{d}\big[\eta_i \,\widehat{\mathrm{d}x^i}\big] = \int_{\partial R} \eta_i \,\widehat{\mathrm{d}x^i} \tag{3.2}$$

for all $i = 1, \ldots, k$. For this, we compute

$$\begin{aligned}
\mathrm{d}\big[\eta_i \,\widehat{\mathrm{d}x^i}\big] &= \sum_j \frac{\partial \eta_i}{\partial x^j} \,\mathrm{d}x^j \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge \cdots \wedge \mathrm{d}x^k \\
&= \frac{\partial \eta_i}{\partial x^i} \,\mathrm{d}x^i \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge \cdots \wedge \mathrm{d}x^k \\
&= (-1)^{i-1} \frac{\partial \eta_i}{\partial x^i} \,\mathrm{d}x^i \wedge \cdots \wedge \mathrm{d}x^k.
\end{aligned}$$

Assume that $R = [a_1, b_1] \times \cdots \times [a_k, b_k]$. Then, using the definition of the integral over a box and Fubini's theorem,

$$\begin{aligned}
&\int_R \mathrm{d}\big[\eta_i \,\widehat{\mathrm{d}x^i}\big] \\
&\quad = (-1)^{i-1} \int_R \frac{\partial \eta_i}{\partial x^i} \left\langle \mathrm{e}_1 \wedge \cdots \wedge \mathrm{e}_k, \mathrm{d}x^i \wedge \cdots \wedge \mathrm{d}x^k \right\rangle \mathrm{d}x \\
&\quad = (-1)^{i-1} \int_{[a_1,b_1]\times\cdots[a_{i-1},b_{i-1}]\times[a_{i+1},b_{i+1}]\times\cdots\times[a_k,b_k]} \int_{a_i}^{b_i} \frac{\partial \eta_i}{\partial x^i} \,\mathrm{d}x^i \,\widehat{\mathrm{d}x^i} \\
&\quad = (-1)^{i-1} \bigg(\int_{R_i^+} \eta_i \,\mathrm{d}\mathcal{H}^{k-1} - \int_{R_i^-} \eta_i \,\mathrm{d}\mathcal{H}^{k-1} \bigg) \\
&\quad = \int_{\ominus^{i-1}(R_i^+ \ominus R_i^-)} \eta_i \,\widehat{\mathrm{d}x^i},
\end{aligned}$$

where we have abbreviated $\widehat{\mathrm{d}x^i} := \mathrm{d}(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^k)$ (here just in the sense of integration in $\mathbb{R}^{k-1}$). On the other hand, for $j \neq i$, we write

$$\widehat{\mathrm{e}}_j := \mathrm{e}_1 \wedge \cdots \wedge \mathrm{e}_{j-1} \wedge \mathrm{e}_{j+1} \wedge \cdots \wedge \mathrm{e}_k$$

and observe

$$\int_{R_j^\pm} \eta_i \,\widehat{\mathrm{d}x^i} = \int_{R_j^\pm} \eta_i \left\langle \widehat{\mathrm{e}}_j, \widehat{\mathrm{d}x^i} \right\rangle \mathrm{d}x = 0$$

since $\langle \widehat{\mathrm{e}}_j, \widehat{\mathrm{d}x^i} \rangle = 0$ for $i \neq j$. Thus, by the definition of the boundary of a box, we obtain

$$\int_R \mathrm{d}\big[\eta_i \,\widehat{\mathrm{d}x^i}\big] = \int_{\ominus^{i-1}(R_i^+ \ominus R_i^-)} \eta_i \,\widehat{\mathrm{d}x^i} = \int_{\partial R} \eta_i \,\widehat{\mathrm{d}x^i}.$$

Thus, (3.2) holds, which implies (3.1), and then the claim of the theorem. □

3.5 Lie derivative

Suppose that $v \colon \mathbb{R}^d \to \mathbb{R}^d$ is a Lipschitz and globally bounded (time-independent) vector field. It is a classical result from the theory of ODEs that there exists a unique **flow** of v, that is, a family of maps $\Phi \colon \mathbb{R}^{1+d} \to \mathbb{R}^d$ such that

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi(t,x) = v(\Phi(t,x)), & (t,x) \in \mathbb{R}^{1+d}, \\ \Phi(0,x) = x, & x \in \mathbb{R}^d. \end{cases}$$

We will often write $\Phi^t(x) := \Phi(t,x)$ for $t \in \mathbb{R}$. The following proposition collects some well-known facts:

Proposition 3.16. *Let $v\colon \mathbb{R}^d \to \mathbb{R}^d$ is a Lipschitz and globally bounded. Then:*

(i) For each $t \in \mathbb{R}$, the map $\Phi^t\colon \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz.

(ii) For every $t \in \mathbb{R}$ the map Φ^t is invertible and $(\Phi^t)^{-1} = \Phi^{-t}$.

(iii) The family $(\Phi^t)_{t\in\mathbb{R}}$ forms a group under composition, that is,

$$\Phi^t \circ \Phi^s = \Phi^{t+s} = \Phi^s \circ \Phi^t \qquad \text{for all } s,t \in \mathbb{R}. \tag{3.3}$$

(iv) If $v \in \mathrm{C}^\infty(\mathbb{R}^d;\mathbb{R}^d)$ is smooth, then for every $t \in \mathbb{R}$ the map Φ^t is a C^∞-diffeomorphism and the map $t \mapsto \Phi^t$ is Lipschitz with values in $\mathrm{C}^1(\mathbb{R}^d;\mathbb{R}^d)$.

Proof. All these statements follow from the classical Cauchy–Lipschitz theory, see, for instance Chapter 9 of [24]. □

We also define the **extended flow** $\overline{\Phi}\colon \mathbb{R}^{1+d} \to \mathbb{R}^{1+d}$ given by

$$\overline{\Phi}(t,x) := (t, \Phi^t(x)), \qquad (t,x) \in \mathbb{R}^{1+d},$$

for which

$$\overline{\Phi}^{-1}(t,y) = (t, (\Phi^t)^{-1}(y)) = (t, \Phi^{-t}(y)), \qquad (t,y) \in \mathbb{R}^{1+d}.$$

Lemma 3.17. *Let $v\colon \mathbb{R}^d \to \mathbb{R}^d$ is a Lipschitz and globally bounded. Then:*

(i) $\mathrm{D}\Phi^t(x)[v(x)] = v(\Phi^t(x))$ for every $(t,x) \in \mathbb{R}^{1+d}$;

(ii) $\mathrm{D}\overline{\Phi}^{-1}(s,y)[(1,v(y))] = (1,0)$ for every $(s,y) \in \mathbb{R}^{1+d}$;

(iii) $\mathrm{D}\overline{\Phi}(t,x)[(1,0)] = (1, v(\Phi^t(x)))$ for every $(t,x) \in \mathbb{R}^{1+d}$.

Here, in vectors like $(1,0)$ the "0" is understood as the zero vector in $\mathbb{R}^d$ and thus $(1,0) \in \mathbb{R}^{1+d}$.

Proof. We only show (i); the proofs of (ii) and (iii) follow a similar pattern. Denote by $\mathrm{Lip}(v)$ the Lipschitz constant of v. For any $(t,x) \in \mathbb{R}^{1+d}$ we estimate

$$\begin{aligned}
\left|\Phi^h(x) - x - hv(x)\right| &= \left|\int_0^h v(\Phi^\tau(x)) - v(x) \,\mathrm{d}\tau\right| \\
&\le \int_0^h \left|v(\Phi^\tau(x)) - v(\Phi^0(x))\right| \,\mathrm{d}\tau \\
&\le \mathrm{Lip}(v) \cdot h \cdot \sup_{0\le\tau\le h} \left|\Phi^\tau(x) - \Phi^0(x)\right| \\
&\le \mathrm{Lip}(v) \cdot h \cdot \sup_{0\le\tau\le h} \left|\int_0^\tau v(\Phi^\sigma(x)) \,\mathrm{d}\sigma\right|,
\end{aligned}$$

whence

$$\left|\Phi^h(x) - x - hv(x)\right| \le \mathrm{Lip}(v)\|v\|_\infty h^2.$$

Combining this with the group law and the Lipschitz continuity of Φ, see Proposition 3.16,

$$\begin{aligned}
\left|\Phi^t(x + hv(x)) - \Phi^{t+h}(x)\right| &= \left|\Phi^t(x + hv(x)) - \Phi^t(\Phi^h(x))\right| \\
&\le \mathrm{Lip}(\Phi^t)\left|x + hv(x) - \Phi^h(x)\right| \\
&\le \mathrm{Lip}(\Phi^t)\mathrm{Lip}(v)\|v\|_\infty h^2.
\end{aligned}$$

Thus,

$$\lim_{h\to 0} \frac{\Phi^t(x + hv(x)) - \Phi^{t+h}(x)}{h} = 0.$$

So,

$$\begin{aligned}
\mathrm{D}\Phi^t(x)[v(x)] &= \lim_{h\to 0} \frac{\Phi^t(x + hv(x)) - \Phi^t(x))}{h} \\
&= \lim_{h\to 0} \left[\frac{\Phi^t(x + hv(x)) - \Phi^{t+h}(x)}{h} + \frac{\Phi^{t+h}(x) - \Phi^t(x)}{h}\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Phi^t(x) \\
&= v(\Phi^t(x)),
\end{aligned}$$

which is the claim (i). □

We can now use the flow map to define a sort of "directional derivative" of a differential form: The **Lie derivative** of a smooth k-form $\omega \in \mathcal{D}^k(\mathbb{R}^d)$ with respect to the smooth vector field $v \in \mathrm{C}^\infty(\mathbb{R}^d; \mathbb{R}^d)$ is the smooth k-form

$$\mathcal{L}_v\omega := \lim_{h\to 0} \frac{\Phi^{h*}\omega - \omega}{h}. \tag{3.4}$$

This is understood to mean that for every $\eta \in \bigwedge_k \mathbb{R}^d$ and every $x \in \mathbb{R}^d$,

$$\left\langle \eta, (\mathcal{L}_v\omega)(x)\right\rangle := \lim_{h\to 0} \frac{1}{h}\left[\left\langle \eta, (\Phi^{h*}\omega)(x)\right\rangle - \left\langle \eta, \omega(x)\right\rangle\right]$$

exists. Here we recall that $(\Phi^t)^*\omega$ is the pullback of ω under Φ^t, that is,

$$\left\langle \eta, (\Phi^{h*}\omega)(x)\right\rangle = \left\langle \mathrm{D}\Phi^h(x)[\eta], \omega(\Phi^h(x))\right\rangle.$$

It is straightforward to see that the so-defined $\mathcal{L}_v\omega$ is indeed a smooth k-form on $\mathbb{R}^d$. One should understand $\mathcal{L}_v\omega$ as a "directional derivative" of ω along the flow of v. In particular, for $f \in \mathcal{D}^0(\mathbb{R}^d) \simeq \mathrm{C}_c^\infty(\mathbb{R}^d)$, we have

$$(\mathcal{L}_v f)(x) = \lim_{h\to 0} \frac{f(\Phi^h(x)) - f(x)}{h} = \frac{\partial}{\partial v} f(x). \tag{3.5}$$

The Lie derivative also satisfies a Leibniz rule:

Lemma 3.18. *Let $\alpha \in \mathcal{D}^k(\mathbb{R}^d)$, $\beta \in \mathcal{D}^l(\mathbb{R}^d)$ and let $v \in \mathrm{C}^\infty(\mathbb{R}^d;\mathbb{R}^d)$. Then,*

$$\mathcal{L}_v(\alpha \wedge \beta) = (\mathcal{L}_v\alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_v\beta).$$

Proof. Since $\Phi^{h*}(\alpha \wedge \beta) = (\Phi^{h*}\alpha) \wedge (\Phi^{h*}\beta)$, we compute

$$\begin{aligned}\Phi^{h*}(\alpha \wedge \beta) - \alpha \wedge \beta &= (\Phi^{h*}\alpha) \wedge (\Phi^{h*}\beta) - \alpha \wedge \beta \\ &= (\Phi^{h*}\alpha - \alpha) \wedge (\Phi^{h*}\beta) + \alpha \wedge (\Phi^{h*}\beta - \beta).\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{L}_v(\alpha \wedge \beta) &= \lim_{h\to 0} \frac{\Phi^{h*}(\alpha \wedge \beta) - \alpha \wedge \beta}{h} \\ &= \lim_{h\to 0} \frac{(\Phi^{h*}\alpha - \alpha) \wedge (\Phi^{h*}\beta)}{h} + \lim_{h\to 0} \frac{\alpha \wedge (\Phi^{h*}\beta - \beta)}{h}.\end{aligned}$$

Since $\Phi^{h*}\beta \to \beta$ smoothly as $h \to 0$, the first limit gives $(\mathcal{L}_v\alpha) \wedge \beta$, and the second gives $\alpha \wedge (\mathcal{L}_v\beta)$. □

The Lie derivative can be expressed in a more useful way using the **interior product** $\eta \,\lrcorner\, \alpha \in \bigwedge^{l-k} V$ of $\eta \in \bigwedge_k V$ and $\alpha \in \bigwedge^l V$ (V a finite-dimensional space) with $k \leq l$ (read "$\lrcorner$" as "into"), which is defined via

$$\langle \sigma, \eta \,\lrcorner\, \alpha \rangle := \langle \sigma \wedge \eta, \alpha \rangle, \qquad \sigma \in \textstyle\bigwedge_{l-k} V. \tag{3.6}$$

This operation is somewhat similar to the restriction "$\llcorner$" defined in Section 3.1, but whereas the restriction modifies a multi-vector with a multi-covector, here it is the other way around.

For a k-covector $\alpha \in \bigwedge^k V$ and a 1-vector $v \in \bigwedge_1 V$ we define the **contraction** (in analogy to what is usually done in differential geometry, see, e.g., [24]) as

$$\iota_v\alpha := (-1)^k \, v \,\lrcorner\, \alpha \in \textstyle\bigwedge^{k-1} V, \tag{3.7}$$

with $\iota_v\alpha := 0$ if $k = 0$ (since there no (-1)-forms). The contraction satisfies

$$\langle \eta, \iota_v\alpha \rangle = \langle v \wedge \eta, \alpha \rangle, \qquad \eta \in \textstyle\bigwedge_k V. \tag{3.8}$$

and, with $\beta \in \bigwedge^l V$,

$$\iota_v(\alpha \wedge \beta) = (\iota_v\alpha) \wedge \beta + (-1)^k \, \alpha \wedge (\iota_v\beta), \tag{3.9}$$

which both follow in an elementary way from the properties of the wedge product.

The following formula is a central result in the theory of differential forms. It expresses the Lie derivative without resorting to a limit procedure.

Proposition 3.19 (Cartan's Magic Formula). *For $\omega \in \mathcal{D}^k(\mathbb{R}^d)$ and $v \in \mathrm{C}^\infty(\mathbb{R}^d;\mathbb{R}^d)$ it holds that*

$$\mathcal{L}_v\omega = \mathrm{d}(\iota_v\omega) + \iota_v(\mathrm{d}\omega). \tag{3.10}$$

This result is well-known and can be found in many texts on differential geometry, see, for instance, Theorem 14.35 in [24]. For the sake of completeness, and because it will be of crucial importance when we define the Lie derivative of currents, we give a complete proof.

Proof. We prove (3.10) by induction on the degree k.

Anchor $k = 0$. Let $f \in \mathcal{D}^0(\mathbb{R}^d) = \mathrm{C}^\infty(\mathbb{R}^d)$ be a smooth function. Then, $\iota_v f = 0$ and so, using (3.8) with $\eta = 1 \in \bigwedge_0 \mathbb{R}^d$, as well as (3.5), gives

$$\big\langle 1, \mathrm{d}(\iota_v f) + \iota_v(\mathrm{d}f)\big\rangle = \big\langle v,\, \mathrm{d}f\big\rangle = \frac{\partial}{\partial v} f = \mathcal{L}_v f,$$

establishing the induction anchor.

Inductive step $(k-1) \to k$. Let $k \geq 1$ and suppose (3.10) holds for all smooth $(k-1)$-forms. By Lemma 3.18 and (3.9), and since all operators involved are linear, it suffices to verify (3.10) for $\omega = \mathrm{d}u \wedge \beta$, where $u \in \mathrm{C}^\infty(\mathbb{R}^d)$ and $\beta \in \mathcal{D}^{k-1}(\mathbb{R}^d)$.

Using Lemma 3.18, the induction anchor case in the form $\mathcal{L}_v(\mathrm{d}u) = \mathrm{d}(\mathcal{L}_v u) = \mathrm{d}(\partial_v u)$ (with $\partial_v u$ the derivative of u in direction v), as well as the induction hypothesis applied to the $(k-1)$-form β, we calculate

$$\begin{aligned}\mathcal{L}_v(\mathrm{d}u \wedge \beta) &= (\mathcal{L}_v \mathrm{d}u) \wedge \beta + \mathrm{d}u \wedge (\mathcal{L}_v\beta)\\ &= \mathrm{d}(\partial_v u) \wedge \beta + \mathrm{d}u \wedge \big(\mathrm{d}(\iota_v\beta) + \iota_v(\mathrm{d}\beta)\big).\end{aligned} \tag{3.11}$$

On the other hand, by Lemma 3.10, Lemma 3.11, and (3.9) (and where we consider d to bind stronger than ι_v),

$$\begin{aligned}\mathrm{d}(\iota_v(\mathrm{d}u \wedge \beta)) &= \mathrm{d}\big((\iota_v \mathrm{d}u) \wedge \beta - \mathrm{d}u \wedge (\iota_v\beta)\big)\\ &= \mathrm{d}\big((\partial_v u)\,\beta - \mathrm{d}u \wedge (\iota_v\beta)\big)\\ &= \mathrm{d}(\partial_v u) \wedge \beta + (\partial_v u)\,\mathrm{d}\beta + \mathrm{d}u \wedge \mathrm{d}(\iota_v\beta).\end{aligned}$$

Moreover, by similar arguments, we obtain

$$\begin{aligned}\iota_v(\mathrm{d}(\mathrm{d}u \wedge \beta)) &= -\iota_v(\mathrm{d}u \wedge \mathrm{d}\beta)\\ &= -(\iota_v \mathrm{d}u) \wedge \mathrm{d}\beta + \mathrm{d}u \wedge (\iota_v\,\mathrm{d}\beta)\\ &= -(\partial_v u)\,\mathrm{d}\beta + \mathrm{d}u \wedge (\iota_v \mathrm{d}\beta).\end{aligned}$$

Combining these computations,

$$\mathrm{d}(\iota_v(\mathrm{d}u \wedge \beta)) + \iota_v(\mathrm{d}(\mathrm{d}u \wedge \beta)) = \mathrm{d}(\partial_v u) \wedge \beta + \mathrm{d}u \wedge \mathrm{d}(\iota_v\beta) + \mathrm{d}u \wedge (\iota_v\,\mathrm{d}\beta),$$

which agrees with (3.11). This completes the inductive step and hence the proof. □

3.6 Hodge duality

In the vector space $\mathbb{R}^N$, equipped with the usual Euclidean inner product, we will now consider an operation that gives the orthogonal (oriented) $(n-k)$-plane defined by a given k-vector.

For a k-vector $\eta \in \bigwedge_k \mathbb{R}^N$, we define the **Hodge dual** $\star\eta \in \bigwedge_{N-k} \mathbb{R}^N$ as the unique vector satisfying, with $\mathrm{E} := \mathrm{e}_1 \wedge \cdots \wedge \mathrm{e}_N$,

$$\xi \wedge \star\eta = (\xi, \eta)\,\mathrm{E}$$

for all $\xi \in \bigwedge_k \mathbb{R}^N$. The existence and uniqueness of $\star\eta$ here follows from the fact that the map

$$(\xi, \zeta) \in \left(\bigwedge_k \mathbb{R}^N\right) \times \left(\bigwedge_{N-k} \mathbb{R}^N\right) \mapsto f(\xi, \zeta) \in \mathbb{R},$$

defined by $\xi \wedge \zeta = f(\xi, \zeta)\,\mathrm{E}$ is non-degenerate (note that the only N-vectors are $\pm\mathrm{E}$), meaning that $f(\xi, \cdot)$ and $f(\cdot, \zeta)$ are non-zero linear maps for any ξ, ζ (this is elementary to see). On the other hand, the map $\xi \mapsto (\xi, \eta)$ defines a linear functional on $\bigwedge_k \mathbb{R}^N$ for any fixed $\eta \in \bigwedge_k \mathbb{R}^N$. Hence, by a version of the Riesz representation theorem for bilinear forms, there is a unique element $\zeta_\eta \in \bigwedge_{N-k} \mathbb{R}^N$ such $f(\xi, \zeta_\eta) = (\xi, \eta)$ for all $\xi \in \bigwedge_k \mathbb{R}^N$. Then set $\star\eta := \zeta_\eta$.

For k-covectors $\alpha \in \bigwedge^k \mathbb{R}^N$ we may use the same formula to define $\star\alpha \in \bigwedge^{N-k} \mathbb{R}^N$, with $\mathrm{e}_1 \wedge \cdots \wedge \mathrm{e}_n$ replaced by $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$. All of the following properties also hold for the Hodge star applies to k-covectors.

In the special case $n = 3$ we also have the following geometric interpretation of the Hodge dual of a 2-vector: For $\eta \in \bigwedge_2 \mathbb{R}^3$, its dual $\star\eta$ is an (oriented) normal vector to any 2-dimensional hyperplane with orientation η. The orientation is given by the right-hand rule. Note that no similar representation of a 2-plane by a normal vector holds in $\mathbb{R}^4$ since the Hodge dual of a 2-vector in $\mathbb{R}^4$ is again a 2-vector!

The following result collects some properties of the Hodge star:

Lemma 3.20. *The following rules hold:*

(i) For any $k \in \{0, \ldots, N\}$ it holds that

$$\star(\mathrm{e}_1 \wedge \cdots \wedge \mathrm{e}_k) = \mathrm{e}_{k+1} \wedge \cdots \wedge \mathrm{e}_N$$

and

$$\star(\mathrm{e}_{k+1} \wedge \cdots \wedge \mathrm{e}_N) = (-1)^{k(N-k)}(\mathrm{e}_1 \wedge \cdots \wedge \mathrm{e}_k).$$

(ii) The Hodge star is an isometry, that is, $(\star\xi, \star\eta) = (\xi, \eta)$ for all $\xi, \eta \in \bigwedge_k \mathbb{R}^N$. In particular, $|\star\xi| = |\xi|$.

(iii) The inverse of the Hodge star is given as

$$\star^{-1} = (-1)^{k(N-k)} \star.$$

Proof. Ad (i). Since the Hodge dual is unique, the first formula follows directly from the definition.

For the second formula, we set $\mathrm{e}_{\langle k \rangle} := \mathrm{e}_1 \wedge \cdots \wedge \mathrm{e}_k$ and $\eta := \mathrm{e}_{k+1} \wedge \cdots \wedge \mathrm{e}_N$. Then, by counting how many times the sign switches as we exchange vectors,

$$\eta \wedge \mathrm{e}_{\langle k \rangle} = (-1)^{k(N-k)}\, \mathrm{e}_{\langle k \rangle} \wedge \eta = (-1)^{k(N-k)}\, \mathrm{E}.$$

In particular,

$$\eta \wedge \left[(-1)^{k(N-k)} \mathrm{e}_{\langle k \rangle}\right] = \mathrm{E}.$$

This means that $\star\eta = (-1)^{k(N-k)} \mathrm{e}_{\langle k \rangle}$ because any $(N-k)$-vector ξ can be extended by using the canonical basis of $\bigwedge_k \mathbb{R}^N$ as $(\xi, \eta)\, \eta$ plus simple $(N-k)$-vectors that have one (or more) vectors in common with ξ and hence wedge to zero with it. Thus,

$$\xi \wedge \left[(-1)^{k(N-k)} \mathrm{e}_{\langle k \rangle}\right] = (\xi, \eta)\, \mathrm{E}$$

for all $\xi \in \bigwedge_k \mathbb{R}^N$, whereby the claim holds.

Ad (ii). We have

$$(\star\xi, \star\eta)\, \mathrm{E} = (-1)^{k(N-k)} \star\xi \wedge \eta = \eta \wedge \star\xi = (\eta, \xi)\, \mathrm{E},$$

from which the claim already follows.

Ad (iii). For any $\xi, \eta \in \bigwedge_k \mathbb{R}^N$ we compute, using (ii),

$$(\xi, \eta)\, \mathrm{E} = (\star\xi, \star\eta)\, \mathrm{E} = \star\xi \wedge \star\star\eta = (-1)^{k(N-k)} \star\star\eta \wedge \star\xi = (-1)^{k(N-k)} (\star\star\eta, \xi)\, \mathrm{E}.$$

Since ξ is arbitrary, this implies the claim. □

In $\mathbb{R}^3$, the wedge product has a close relationship with the vector cross product, namely:

Lemma 3.21. *For $a, b \in \bigwedge_1 \mathbb{R}^3 \cong \mathbb{R}^3$ the identities*

$$\star(a \times b) = a \wedge b, \qquad \star(a \wedge b) = a \times b$$

hold, where "$\times$" denotes the classical vector cross product in $\mathbb{R}^3$.

Proof. For any $v \in \mathbb{R}^3 \sim \bigwedge_1 \mathbb{R}^3$, the triple product $v \cdot (a \times b)$ is equal to the determinant $\det(v, a, b)$ of matrix with columns v, a, b, and so

$$v \wedge \star(a \times b) = v \cdot (a \times b)\, \mathrm{e}_1 \wedge \mathrm{e}_2 \wedge \mathrm{e}_3 = \det(v, a, b)\, \mathrm{e}_1 \wedge \mathrm{e}_2 \wedge \mathrm{e}_3 = v \wedge (a \wedge b).$$

Hence, the first identity follows. The second identity follows by applying $\star$ on both sides and using $\star^{-1} = \star$ via Lemma 3.20 (iii) for $N = 3$ and $k = 1$). □

The transformation rules for the Hodge star in $\mathbb{R}^N$ under linear transformations are given in the following lemma:

Lemma 3.22. *Let $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ be an invertible linear map, which is identified with its matrix representation $L \in \mathbb{R}^{3\times3}$. For $\eta \in \bigwedge_2 \mathbb{R}^3$, and $v \in \mathbb{R}^3$, the transformation rules*

$$\star(L\eta) = (\det L)\, L^{-\mathsf{T}}(\star\eta) \tag{3.12}$$

and

$$\star(Lv) = (\det L)\, L^{-\mathsf{T}}(\star v)$$

hold.

Proof. Concerning the first claim, for every $\xi \in \bigwedge_2 \mathbb{R}^3$ it holds that

$$\xi \wedge \star(L\eta) = (\xi, L\eta)\, \mathrm{E} = (L^{\mathsf{T}}\xi, \eta)\, \mathrm{E} = (L^{\mathsf{T}}\xi) \wedge \star\eta = \xi \wedge [(\det L)\, L^{-\mathsf{T}}(\star\eta)],$$

where in the last equality we used that

$$L^{-\mathsf{T}}\zeta = (\det L)^{-1}\zeta$$

for any $\zeta \in \bigwedge_3 \mathbb{R}^3$ (all of which are of the form $\lambda\mathrm{E}$ for $\lambda \in \mathbb{R}$). Thus, (3.12) follows.

For the second claim one argues in a similar way, using also that $\star^{-1} = \star$. □

More geometrically, in three dimensions, we will denote by $[\![v \wedge w]\!]$ the hyperplane (through the origin) with orientation and weighting given by $v \wedge w \in \bigwedge_2 \mathbb{R}^3$. Its (weighted) normal is denoted by

$$\star[\![v \wedge w]\!] := \star(v \wedge w) = v \times w.$$

One advantage of this description is that it makes it easy to transform planes between different frames (an operation we will have to perform many times in the sequel): If the transformation matrix for vectors is $R \in \mathrm{GL}^+(3)$, then vectors transform via the relation $v \mapsto Rv$. Applying this to the wedge product $v \wedge w$ gives $R(v \wedge w) = Gv \wedge Gw$, so the transformed plane is $R_*[\![v \wedge w]\!] := [\![Rv \wedge Rw]\!]$. We may compute from the definition that for $\eta := v \wedge w$ the Hodge dual $\star R\eta$ satisfies, for every $\xi \in \bigwedge_2 \mathbb{R}^3$,

$$\xi \wedge \star R\eta = (\xi, R\eta)\,\mathrm{E} = (R^{\mathsf{T}}\xi, \eta)\,\mathrm{E} = (R^{\mathsf{T}}\xi) \wedge \star\eta = \xi \wedge [(\det R)\, R^{-\mathsf{T}} \star\eta],$$

where in the last equality we used that $R^{-\mathsf{T}}\zeta = (\det R)^{-1}\zeta$ for any $\zeta \in \bigwedge_3 \mathbb{R}^3$ (all of which are of the form $\lambda\mathrm{E}$ for $\lambda \in \mathbb{R}$). For the weighted normal to the transformed plane $R_*[\![v \wedge w]\!]$ we get

$$\star(R_*[\![v \wedge w]\!]) = (\det R)\, R^{-\mathsf{T}} \star[\![v \wedge w]\!] = (\det R)\, R^{-\mathsf{T}}(v \times w)$$

Note that these formulas express the transformation rules for normals to planes.

Chapter 4

Currents

Currents are generalizations of orientable surfaces, "density fields" of surfaces, as well as much more pathological structures. Technically, they are defined as the dual objects to (smooth, compactly supported) differential forms by abstract pairing, which one may think of as the "integration" of the differential form over the current. This is in complete analogy to the theory of distributions, which are defined dual to smooth and compactly supported functions. We also discuss boundaries, mass as well as various operations on currents.

4.1 Currents via duality

We first need to specify the topological (not just vector space) dual to $\mathcal{D}^k(\Omega)$ ($\Omega \subset \mathbb{R}^d$ an open set), which requires the following definition: A sequence $(\omega_j)_j \subset \mathcal{D}^k(\Omega)$ is said to **converge to zero in** $\mathcal{D}^k(\Omega)$, in symbols "$\omega_j \to 0$ in $\mathcal{D}^k(\Omega)$", if there is a compact set $K \subset \Omega$ such that $\operatorname{supp} \omega_j \subset K$ for all $j \in \mathbb{N}$ and $\|\partial^\alpha \omega_j\|_\infty \to 0$ as $j \to \infty$ for all multi-indices $\alpha \in \mathbb{N}_0^d$. We say that $(\omega_j)_j \subset \mathcal{D}^k(\Omega)$ **converges to** $\omega \in \mathcal{D}^k(\Omega)$, in symbols $\omega_j \to \omega$ in $\mathcal{D}^k(\Omega)$, if $\omega_j - \omega \to 0$ in $\mathcal{D}^k(\Omega)$. It is an exercise to see that this convergence is not induced by a metric.

A k**-current** on an open set $\Omega \subset \mathbb{R}^d$ is a linear map $T \colon \mathcal{D}^k(\Omega) \to \mathbb{R}$ with the property that for every compact set $K \subset \Omega$ there is a constant $C = C(K) > 0$ and an integer $N = N(K) \in \mathbb{N}_0$ such that

$$\big|\langle T, \omega \rangle\big| \le C \sum_{|\alpha| \le N} \|\partial^\alpha \omega\|_\infty \tag{4.1}$$

for all $\omega \in \mathcal{D}^k(\Omega)$ with $\operatorname{supp} \omega \subset K$. Here, $\langle T, \omega \rangle = T(\omega)$ is the application of the linear map T on ω and $\partial^\alpha \omega$ is the α-derivative of ω when considered as a function with values in the linear space $\bigwedge^k \mathbb{R}^d$. We collect all k-currents on Ω into the set $\mathcal{D}_k(\Omega)$, which is a vector space over $\mathbb{R}$ under addition and scalar multiplication.

Currents can also be characterized as those linear maps on $\mathcal{D}^k(\Omega)$ that are continuous with respect to convergence (to zero) in $\mathcal{D}^k(\Omega)$:

Lemma 4.1. *Let $T\colon \mathcal{D}^k(\Omega) \to \mathbb{C}$ be a linear map. Then, the following are equivalent:*

(i) $T \in \mathcal{D}_k(\Omega)$;

(ii) for every sequence $\omega_j \to 0$ in $\mathcal{D}^k(\Omega)$ it holds that $\langle T, \omega_j\rangle \to 0$;

(iii) for every sequence $\omega_j \to \omega$ in $\mathcal{D}^k(\Omega)$ it holds that $\langle T, \omega_j\rangle \to \langle T, \omega\rangle$.

Proof. Since the equivalence between (ii) and (iii) is obvious, we only need to show that (i) implies (ii) and that (ii) implies (i).

If $T \in \mathcal{D}_k(\Omega)$ and $\omega_j \to 0$ in $\mathcal{D}^k(\Omega)$, then by (4.1) there is an $N \in \mathbb{N}_0$ and a $C > 0$ such that

$$\big|\langle T, \omega_j\rangle\big| \le C \sum_{|\alpha|\le N} \|\partial^\alpha \omega_j\|_\infty \to 0$$

as $j \to \infty$, where we have used the definition of the convergence $\omega_j \to 0$ in $\mathcal{D}^k(\Omega)$.

For the other direction, let $T\colon \mathcal{D}^k(\Omega) \to \mathbb{C}$ be linear and such that for every sequence $\omega_j \to 0$ in $\mathcal{D}^k(\Omega)$ it holds that $\langle T, \omega_j\rangle \to 0$. Arguing by contradiction, we assume that $T \notin \mathcal{D}_k(\Omega)$. So, there exists a compact set $K \subset \Omega$ with the property that for all $N \in \mathbb{N}_0$ there is a $\omega_N \in \mathcal{D}^k(\Omega) \setminus \{0\}$ with $\operatorname{supp} \omega_N \subset K$ and

$$\big|\langle T, \omega_N\rangle\big| \ge N \sum_{|\alpha|\le N} \|\partial^\alpha \omega_N\|_\infty .$$

Set

$$\psi_N := \frac{\omega_N}{N \sum_{|\alpha|\le N} \|\partial^\alpha \omega_N\|_\infty},$$

so that

$$\big|\langle T, \psi_N\rangle\big| \ge 1. \tag{4.2}$$

On the other hand, for every multi-index $\beta \in \mathbb{N}_0^d$,

$$\|\partial^\beta \psi_N\|_\infty = \frac{1}{N} \cdot \frac{\|\partial^\beta \omega_N\|_\infty}{\sum_{|\alpha|\le N} \|\partial^\alpha \omega_N\|_\infty} \to 0,$$

hence $\psi_N \to 0$ in $\mathcal{D}^k(\Omega)$. By our assumption on T, we have $|\langle T, \psi_N\rangle| \to 0$, contradicting (4.2). $\square$

For $T \in \mathcal{D}_k(\Omega)$ we define the **support** as

$$\operatorname{supp} T := \bigcap \bigl\{ F \subset \Omega \text{ relatively closed } : \langle T, \omega \rangle = 0 \text{ for all } \omega \in \mathcal{D}^k(\Omega \setminus F) \bigr\}$$

The support of T is always a relatively closed subset of Ω.

Example 4.2. Let R be a k-box. Then, we can associate a k-current $[\![R]\!] \in \mathcal{D}_k(\mathbb{R}^k)$ with R by setting

$$\bigl\langle [\![R]\!], \omega \bigr\rangle := \int_R \omega, \qquad \omega \in \mathcal{D}^k(\mathbb{R}^k).$$

The support of $[\![R]\!]$ is $\operatorname{supp}[\![R]\!] = R$. For the oppositely-oriented box, we define

$$[\![\ominus R]\!] := -[\![R]\!],$$

which is consistent with the earlier definition of "$\int_{\ominus R}$".

Example 4.3. We can immediately extend the previous definition to chains of k-boxes by setting, for k-boxes R_1, R_2 in $\mathbb{R}^k$,

$$[\![R_1 \oplus R_2]\!] := [\![R_1]\!] + [\![R_2]\!], \qquad [\![R_1 \ominus R_2]\!] := [\![R_1]\!] - [\![R_2]\!],$$

and so on.

Example 4.4. Also k-dimensional oriented manifolds M that are embedded in $\mathbb{R}^d$ can be considered as currents $[\![M]\!] \in \mathcal{D}_k(\mathbb{R}^d)$ by setting

$$\bigl\langle [\![M]\!], \omega \bigr\rangle := \int_M \omega, \qquad \omega \in \mathcal{D}^k(\mathbb{R}^d).$$

For a k-current $[\![R]\!]$ or $[\![F_* R]\!]$ as defined before and $\lambda \in \mathbb{R}$, we say that the k-current $\lambda[\![R]\!]$ has **multiplicity** λ.

Example 4.5. Let $Q := (-1/2, 1/2) \subset \mathbb{R}^2$ be the unit square in $\mathbb{R}^2$. One can check easily that $T_1 := \mathrm{e}_1 \mathcal{L}^2 \llcorner Q$ and $T_2 := \mathrm{e}_1 \delta_0$ are both 1-currents in Q, i.e., $T_1, T_2 \in \mathcal{D}_1(Q)$. This shows that the (Hausdorff-)dimension of the support ($\operatorname{supp} T_1 = Q$, $\operatorname{supp} T_2 = \{0\}$) does not have to agree with the "geometric" dimension k. Later, when we will restrict to special classes of currents, we will restore this correspondence.

4.2 Boundaries

Define the **boundary** of a k-current $T \in \mathcal{D}_k(\mathbb{R}^d)$, where now $k \geq 1$, as the $(k-1)$-current $\partial T \in \mathcal{D}_{k-1}(\mathbb{R}^d)$ determined via

$$\bigl\langle \partial T, \omega \bigr\rangle := \bigl\langle T, \mathrm{d}\omega \bigr\rangle, \qquad \omega \in \mathcal{D}^{k-1}(\mathbb{R}^d).$$

For a 0-current $T \in \mathcal{D}_0(\mathbb{R}^d)$, we formally set $\partial T := 0$. By Proposition 3.11, we have that $\partial^2 T = \partial(\partial T) = 0$.

Example 4.6. In the same way as in Examples 4.2, 4.3, one may consider the $(k-1)$-current $[\![Q]\!]$ associated to a $(k-1)$-box Q in $\mathbb{R}^k$. In particular, for a k-box R, $[\![R_i^\pm]\!]$ can be defined in such a way and we obtain

$$[\![\partial R]\!] = \left[\!\!\left[\bigoplus_{i=1}^{d} \ominus^{i-1}(R_i^+ \ominus R_i^-)\right]\!\!\right] = \sum_{i=1}^{d}(-1)^{i-1}([\![R_i^+]\!] - [\![R_i^-]\!]).$$

An important observation is now that, by Stokes's Theorem 3.15, this in fact agrees with the boundary of $[\![R]\!]$ in the sense of currents: For any $\omega \in \mathcal{D}^{k-1}(\mathbb{R}^k)$ we have

$$\langle \partial[\![R]\!], \omega \rangle = \langle [\![R]\!], \mathrm{d}\omega \rangle = \int_R \mathrm{d}\omega = \int_{\partial R} \omega = \langle [\![\partial R]\!], \omega \rangle.$$

Hence, $\partial[\![R]\!] = [\![\partial R]\!]$. In this way, we see that the boundary operation for currents has the correct geometric meaning, at least for boxes.

Example 4.7. Let R be a k-box in $\mathbb{R}^k$ and let $F\colon U \to \mathbb{R}^d$ be smooth and proper, where $U \supset R$ is open. Then, also F_*R corresponds to a current $[\![F_*R]\!] \in \mathcal{D}_k(\mathbb{R}^d)$ given by

$$\langle [\![F_*R]\!], \omega \rangle := \int_{F_*R} \omega = \int_R F^*\omega, \qquad \omega \in \mathcal{D}^k(\mathbb{R}^d).$$

Once again we obtain from Stokes's theorem that

$$\partial[\![F_*R]\!] = [\![\partial(F_*R)]\!] = [\![F_*(\partial R)]\!].$$

Example 4.8. For a map $g \in \mathrm{L}^1(\mathbb{R}^d)$, we can define the d-current $T = g\,\mathrm{e}_1 \wedge \cdots \wedge \mathrm{e}_d\,\mathcal{L}^d$ via

$$\langle T, \omega \rangle := \int_{\mathbb{R}^d} \langle \mathrm{e}_1 \wedge \cdots \wedge \mathrm{e}_d, \omega(x) \rangle\, g(x)\,\mathrm{d}x, \qquad \omega \in \mathcal{D}^d(\mathbb{R}^d).$$

In order to compute the boundary ∂T of this top-dimensional current T, let $\omega \in \mathcal{D}^{d-1}(\mathbb{R}^d)$ be of the form

$$\omega = \sum_{i=1}^{d} \omega_i\,\widehat{\mathrm{d}x^i}, \qquad \omega_i \in \mathrm{C}_c^\infty(\mathbb{R}^d).$$

By definition, and a similar computation as the one after (3.2) in the proof of Stokes's theorem,

$$\begin{aligned}
\langle \partial T, \omega \rangle &= \langle T, \mathrm{d}\omega \rangle \\
&= \left\langle T, \sum_i (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d \right\rangle \\
&= \sum_i (-1)^{i-1} \int_{\mathbb{R}^d} \frac{\partial \omega_i}{\partial x^i} \cdot g\,\mathrm{d}x.
\end{aligned}$$

If we further assume that g is continuously differentiable with integrable partial derivatives, then we may integrate by parts to arrive at

$$\langle \partial T, \omega \rangle = \sum_i (-1)^i \int_{\mathbb{R}^d} \omega_i \cdot \frac{\partial g}{\partial x^i} \,\mathrm{d}x.$$

Example 4.9. In a similar way to the previous example, one may associate to a map $G = (G^1, \ldots, G^d) \in \mathrm{L}^1(\mathbb{R}^d; \mathbb{R}^d)$ a 1-current

$$T := \sum_i G^i \,\mathrm{e}_i,$$

that is, for $\omega = \sum_i \omega_i \,\mathrm{d}x^i \in \mathcal{D}^1(\mathbb{R}^d)$,

$$\langle T, \omega \rangle = \sum_i \int_{\mathbb{R}^d} G^i \cdot \omega_i \,\mathrm{d}x.$$

For its boundary we compute, if $G \in \mathrm{C}^1(\mathbb{R}^d; \mathbb{R}^d)$, as follows: For $\omega \in \mathcal{D}^0(\mathbb{R}^d)$ we have $\mathrm{d}\omega = \sum_i \frac{\partial \omega}{\partial x^i} \,\mathrm{d}x^i$ and hence

$$\langle \partial T, \omega \rangle = \langle T, \mathrm{d}\omega \rangle = \sum_i \int_{\mathbb{R}^d} G^i \cdot \frac{\partial \omega}{\partial x^i} \,\mathrm{d}x = - \int_{\mathbb{R}^d} \operatorname{div} G \cdot \omega \,\mathrm{d}x.$$

Thus,

$$\partial T = - \operatorname{div} G.$$

Note however, that the boundary of T is defined even if G is only integrable (and not necessarily differentiable). In this case, we may consider $-\partial T$ as a "weak divergence" of G.

Example 4.10. Recall the definition of the Cantor set C_∞ in Example 1.35 as well as the definition of the sets C_j used in the construction of C_∞. Set for $t \in \mathbb{R}$,

$$f_j(t) := \int_0^t \left(\frac{3}{2}\right)^j \mathbb{1}_{C_j}(s) \,\mathrm{d}s,$$

and define the linear and continuous functional $L_j \colon \mathrm{C}^1_0(\mathbb{R}) \to \mathbb{R}$ via

$$\langle L_j, \varphi \rangle := \int_0^1 f_j'(t)\varphi(t) \,\mathrm{d}t, \qquad \varphi \in \mathrm{C}^1_0(\mathbb{R}).$$

It is not difficult to see that the sequence $(f_j)_j$ is Cauchy in $\mathrm{C}(\mathbb{R})$ and thus converges to a continuous and monotone map $f_C \in \mathrm{C}(\mathbb{R})$, the **Cantor function**. Since, for $\varphi \in \mathrm{C}_0(\mathbb{R})$,

$$|\langle L_j, \varphi \rangle| \le \|\varphi\|_\infty \int_0^1 f_j'(t) \,\mathrm{d}t = \|\varphi\|_\infty f_j(1) = \|\varphi\|_\infty,$$

the L_j are uniformly bounded in the dual space to $\mathrm{C}_0(\mathbb{R})$. Thus, we have $L_j \overset{*}{\rightharpoonup} L$ in $\mathrm{C}_0(\mathbb{R})^*$ for some functional L. By the Riesz Representation Theorem 1.4, there exists a measure $\mu_C \in \mathcal{M}(\mathbb{R})$ with

$$\langle L, \varphi \rangle = \int \varphi(t) \, \mathrm{d}\mu_C(t).$$

In fact, using the self-similarity property of the Cantor set C_∞, one can show (exercise) that

$$\mu_C = \frac{1}{\mathcal{H}^\gamma(C_\infty)} \mathcal{H}^\gamma \llcorner C_\infty, \qquad \gamma = \frac{\ln 2}{\ln 3}.$$

On the other hand, using an integration by parts,

$$\langle L_j, \varphi \rangle = -\int_0^1 f_j(t)\varphi'(t) \, \mathrm{d}t \to -\int_0^1 f_C(t)\varphi'(t) \, \mathrm{d}t.$$

for any $\varphi \in \mathrm{C}_c^\infty(\mathbb{R})$. Thus, for such φ,

$$\int_0^1 f_C(t)\varphi'(t) \, \mathrm{d}t = -\int \varphi(t) \, \mathrm{d}\mu_C(t).$$

If we consider the 1-current $T_C := f_C \, \mathrm{e}_1 \, \mathcal{L}^1 \in \mathcal{D}_1(\mathbb{R})$ we obtain from Example 4.8 that

$$\partial T_C = -\mu_C \in \mathcal{D}_0(\mathbb{R}).$$

In particular, even though a current may be carried by a rectifiable set (here, $[0, \infty)$), this does not imply that also the current's boundary is carried by a rectifiable set (here, the Cantor set C_∞).

The boundary also satisfies a Leibniz formula:

Lemma 4.11. *For any $T \in \mathcal{D}_k(\mathbb{R}^d)$, for any $\omega \in \mathcal{D}^l(\mathbb{R}^d)$, the Leibniz rule holds in the form*

$$\partial(T \llcorner \omega) = (-1)^l \big[(\partial T) \llcorner \omega - T \llcorner \mathrm{d}\omega\big].$$

In particular, for any $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$, the Leibniz rule holds in the form

$$\partial(fT) = f\partial T - T \llcorner \mathrm{d}f.$$

Proof. Let $\eta \in \mathcal{D}^k(\mathbb{R}^d)$ be a test form. By the Leibniz rule for exterior forms (Lemma 3.10),

$$(-1)^l \omega \wedge \mathrm{d}\eta = \mathrm{d}(\omega \wedge \eta) - \mathrm{d}\omega \wedge \eta.$$

Therefore, using also the definition of ∂T and of restriction,

$$\begin{aligned} (-1)^l \langle T, \omega \wedge \mathrm{d}\eta \rangle &= \big\langle T, \mathrm{d}(\omega \wedge \eta) \big\rangle - \langle T, \mathrm{d}\omega \wedge \eta \rangle \\ &= \big\langle (\partial T) \llcorner \omega, \eta \big\rangle - \langle T \llcorner \mathrm{d}\omega, \eta \rangle. \end{aligned}$$

So,

$$\big\langle \partial(T \llcorner \omega), \eta \big\rangle = \big\langle T, \omega \wedge \mathrm{d}\eta \big\rangle = (-1)^l \big\langle (\partial T) \llcorner \omega - T \llcorner \mathrm{d}\omega,\, \eta \big\rangle.$$

Since η was arbitrary, this gives the sought formula.

For the special case $\omega = f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ (a 0-form, so $l = 0$), the specialized formula follows since $T \llcorner f = fT$ and $(\partial T) \llcorner f = f\partial T$. □

4.3 Mass and Radon–Nikodym decomposition

The **mass** of a current $T \in \mathcal{D}_k(\Omega)$ in an open set $U \subset \Omega$ is defined to be

$$\mathbf{M}_U(T) := \sup\big\{ \big\langle T, \omega \big\rangle \;:\; \omega \in \mathcal{D}^k(U),\ \|\omega\| \le 1 \big\}.$$

We simply write $\mathbf{M}(T)$ for $\mathbf{M}_\Omega(T)$.

Proposition 4.12. *If $T \in \mathcal{D}_k(\Omega)$ has finite mass, i.e., $\mathbf{M}(T) < \infty$, then T is representable by integration: There is a measure $\|T\| \in \mathcal{M}^+(\Omega)$ and a $\|T\|$-measurable map $\vec{T} \colon \Omega \to \bigwedge_k \mathbb{R}^d$ such that "$T = \vec{T}\,\|T\|$", that is,*

$$\big\langle T, \omega \big\rangle = \int_{\mathbb{R}^d} \big\langle \vec{T}(x), \omega(x) \big\rangle \,\mathrm{d}\|T\|(x), \qquad \omega \in \mathcal{D}^k(\Omega).$$

Moreover, $\|\vec{T}(x)\| = 1$ for $\|T\|$-almost every x, and $\mathbf{M}(T) = \|T\|(\Omega)$.

The measure $\|T\|$ is called the **total variation measure** of T and $\vec{T}$ is called the **orienting map**. The resulting decomposition $T = \vec{T}\,\|T\|$, if it exists, is often called the **Radon–Nikodym decomposition** of a current.

Proof. Consider the 0-current $T^{i_1,\ldots,i_k} \in \mathcal{D}_0(\Omega)$, with a set of indices $1 \le i_1 < \cdots < i_k \le d$, given by

$$\big\langle T^{i_1,\ldots,i_k}, \varphi \big\rangle := \langle T, \varphi\,\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \rangle, \qquad \varphi \in \mathcal{D}^0(\Omega) \cong \mathrm{C}_c^\infty(\Omega).$$

As $\mathbf{M}(T) < \infty$, we have that

$$\big| \big\langle T^{i_1,\ldots,i_k}, \varphi \big\rangle \big| \le \mathbf{M}(T) \|\varphi\|_\infty.$$

Hence, $T^{i_1,\ldots,i_k}$ lies in the dual space to $\mathrm{C}_c^\infty(\Omega)$ and then also in the dual space to $\mathrm{C}_0(\Omega)$ since the bound above is uniform in the support of φ and only depends on $\|\varphi\|_\infty$. By the Riesz Representation Theorem 1.4 there is a measure $\mu^{i_1,\ldots,i_k} \in \mathcal{M}^+(\Omega)$ with

$$\big\langle T^{i_1,\ldots,i_k}, \varphi \big\rangle = \int_\Omega \varphi \,\mathrm{d}\mu^{i_1,\ldots,i_k}, \qquad \varphi \in \mathrm{C}_c^\infty(\Omega).$$

Then, decomposing an arbitrary differential form $\omega \in \mathcal{D}^k(\Omega)$ as

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le d} \varphi_{i_1,\ldots,i_k} \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k},$$

it follows that

$$T = \sum_{1 \le i_1 < \cdots < i_k \le d} \mathrm{e}_{i_1} \wedge \cdots \wedge \mathrm{e}_{i_k} \, \mu^{i_1,\ldots,i_k}.$$

Setting

$$\mu := \sum_{1 \le i_1 < \cdots < i_k \le d} \mu^{i_1,\ldots,i_k} \in \mathcal{M}^+(\Omega)$$

we see from the Radon–Nikodym Theorem 1.14 that $T = V\mu$ with a μ-measurable map $V \colon \Omega \to \bigwedge_k \mathbb{R}^d \setminus \{0\}$. With $\vec{T} := \|V\|^{-1}V$ and $\|T\| := \|V\|\mu$ we conclude that $T = \vec{T}\,\|T\|$.

It is straightforward to see that $\mathbf{M}(T) \le \|T\|(\Omega)$. The other direction follows by choosing $\omega(x)$ such that approximately $\langle \vec{T}(x), \omega(x) \rangle = 1$ $\|T\|$-almost everywhere and a smoothing argument (see below); we leave the details as an exercise. □

Example 4.13. If $T_1 \in \mathcal{D}_{k_1}(\mathbb{R}^{d_1})$, $T_2 \in \mathcal{D}_{k_2}(\mathbb{R}^{d_2})$ with $\mathbf{M}(T_1), \mathbf{M}(T_2) < \infty$. Then,

$$\mathbf{M}(T_1 \times T_2) \le \mathbf{M}(T_1) \cdot \mathbf{M}(T_2).$$

As convergence for currents we use the weak* convergence, i.e., we say that a sequence $(T_j)_j \subset \mathcal{D}_k(\mathbb{R}^d)$ **converges weakly*** to $T \in \mathcal{D}_k(\mathbb{R}^d)$, in symbols "$T_j \overset{*}{\rightharpoonup} T$", if

$$\langle T_j, \omega \rangle \to \langle T, \omega \rangle \qquad \text{for all } \omega \in \mathcal{D}^k(\mathbb{R}^d).$$

Note that it follows directly from the definition that if $T_j \overset{*}{\rightharpoonup} T$, then also $\partial T_j \overset{*}{\rightharpoonup} \partial T$.

Example 4.14. In Example 4.10 we have already shown that

$$T_j := f_j \, \mathrm{e}_1 \, \mathcal{L}^1 \overset{*}{\rightharpoonup} T_C := f_C \, \mathrm{e}_1 \, \mathcal{L}^1$$

weakly* in $\mathcal{D}_1(\mathbb{R})$.

Example 4.15. Consider the currents $T_j \in \mathcal{D}_1(\mathbb{R}^2)$ given as

$$T_j := \mathrm{e}_1 \mathcal{H}^1 \llcorner (R_j \cup Q_j),$$

where

$$R_j := \bigcup_{i=0}^{j-1} \Big(\frac{2i}{2j}, \frac{2i+1}{2j} \Big) \times \{0\}, \qquad Q_j := \bigcup_{i=0}^{j-1} \Big(\frac{2i+1}{2j}, \frac{2i+2}{2j} \Big) \times \{1\}.$$

Then, for any $\omega \in \mathcal{D}^1(\mathbb{R}^2)$, we have (using an argument with Riemann sums, the details of which are an exercise)

$$\begin{aligned}\langle T_j, \omega \rangle &= \sum_{i=0}^{j-1} \int_{\left(\frac{2i}{2j}, \frac{2i+1}{2j}\right)} \langle \mathrm{e}_1, \omega(x,0) \rangle \,\mathrm{d}x + \sum_{i=0}^{j-1} \int_{\left(\frac{2i+1}{2j}, \frac{2i+2}{2j}\right)} \langle \mathrm{e}_1, \omega(x,1) \rangle \,\mathrm{d}x \\ &\to \frac{1}{2} \int_0^1 \langle \mathrm{e}_1, \omega(x,0) \rangle + \langle \mathrm{e}_1, \omega(x,1) \rangle \,\mathrm{d}x.\end{aligned}$$

Thus, $T_j \overset{*}{\rightharpoonup} \frac{1}{2}\mathrm{e}_1\mathcal{H}^1 \llcorner [(0,1) \times \{0,1\}]$.

The condition $\mathbf{M}(T) + \mathbf{M}(\partial T) < \infty$ appears so often in the theory of currents that it has a name: One says that $T \in \mathcal{D}_k(\mathbb{R}^d)$ is a **normal current** if

$$\mathbf{M}(T) + \mathbf{M}(\partial T) < \infty.$$

In this case, both T and ∂T are representable by integration as in Proposition 4.12. We collect all normal currents into the set ($k \in \mathbb{N} \cup \{0\}$):

$$\mathrm{N}_k(\mathbb{R}^d) := \big\{ T \in \mathcal{D}_k(\mathbb{R}^d) \;:\; \mathbf{M}(T) + \mathbf{M}(\partial T) < \infty \big\}. \tag{4.3}$$

The following compactness result holds for normal currents:

Proposition 4.16. *Let $(T_j) \subset \mathrm{N}_k(\mathbb{R}^d)$ with $\operatorname{supp} T_j \subset K$ for a fixed compact set $K \subset \mathbb{R}^d$ and*

$$\sup_{j \in \mathbb{N}} \big(\mathbf{M}(T_j) + \mathbf{M}(\partial T_j)\big) < \infty.$$

Then, there is a subsequence (not relabeled) with $T_j \overset{*}{\rightharpoonup} T$.

Proof. This follows directly from Theorem 1.19. □

Example 4.17. Assume that $T \in \mathrm{N}_1(\mathbb{R})$. Since $\mathbf{M}(\partial T) < \infty$, we know from Proposition 4.12 that there is a measure $\mu \in \mathcal{M}(\mathbb{R};\mathbb{R})$ such that for all $f \in \mathcal{D}^0(\mathbb{R}) \simeq \mathrm{C}_c^\infty(\mathbb{R})$ it holds that

$$\langle \partial T, f \rangle = \int f \,\mathrm{d}\mu.$$

For the 1-current

$$\widetilde{T} := \varrho\, \mathrm{e}_0\, \mathcal{L}^1, \qquad \varrho(t) := -\mu((-\infty, t)), \quad t \in \mathbb{R}$$

we obtain

$$\langle \partial \widetilde{T}, f \rangle = \langle \widetilde{T}, f' \,\mathrm{d}t \rangle = \int f'(t)\varrho(t) \,\mathrm{d}t = \int f(t) \,\mathrm{d}\mu(t) = \langle \partial T, f \rangle.$$

Thus, by the Constancy Theorem 4.23, $T = \widetilde{T}$ (first, $T = \widetilde{T} + \lambda \,\mathrm{d}t\, \mathcal{L}^1$, but then $\lambda = 0$ because the mass must be finite) and we have concluded that $T \ll \mathcal{L}^1$.

We define the **flat norm** of $T \in \mathrm{N}_k(\mathbb{R}^d)$ as

$$\mathbf{F}(T) := \inf\big\{\, \mathbf{M}(T - \partial S) + \mathbf{M}(S) \;:\; S \in \mathrm{N}_{k+1}(\mathbb{R}^d) \,\big\}, \tag{4.4}$$

where we adopt the convention that $\inf \emptyset := \infty$. Clearly,

$$\mathbf{F} \le \mathbf{M}$$

and $\mathbf{F}$ is a norm on $\mathrm{N}_k(K)$ for any compact $K \subset \mathbb{R}^d$, as can be checked easily. It turns out that the infimum in (4.4) is attained, which is immediate from Proposition 4.16 in conjunction with Reshetnyak's Lower Semicontinuity Theorem 1.22.

We also define the **homogeneous flat norm** of $T \in \mathrm{N}_k(\mathbb{R}^d)$ as follows:

$$\mathbf{F}^\circ(T) := \inf\big\{\, \mathbf{M}(S) \;:\; S \in \mathrm{N}_{k+1}(\mathbb{R}^d),\ \partial S = T \,\big\}.$$

Evidently, $\mathbf{F}^\circ(T) = \infty$ if $T \neq \partial S$ for all $S \in \mathrm{N}_{k+1}(\mathbb{R}^d)$ (that is, T is not a boundary). Moreover,

$$\mathbf{F} \le \mathbf{F}^\circ,$$

but the reverse inequality does not always hold:

Example 4.18. For

$$T := \delta_2 - \delta_{-2} \in \mathrm{N}_0([-2,2])$$

we have $\mathbf{F}_{[-2,2]}(T) \le \mathbf{M}(T) = 2$, but one can show using the Constancy Theorem 4.23 that

$$S := [\![(-2,2)]\!] \in \mathrm{N}_1([-2,2])$$

is the unique normal 1-current with $\partial S = T$. So, $\mathbf{F}^\circ(T) = 4$.

Given a compact set $K \subset \mathbb{R}^d$, one also defines $\mathbf{F}_K$ and $\mathbf{F}^\circ_K$, where the fill current S is restricted to have support in K and we only consider the masses of the currents restricted to K.

4.4 Operations on currents

Let $T_1 \in \mathcal{D}_{k_1}(\mathbb{R}^{d_1})$ and $T_2 \in \mathcal{D}_{k_2}(\mathbb{R}^{d_2})$. Then, the **product current** of T_1, T_2 is the current $T_1 \times T_2 \in \mathcal{D}_{k_1+k_2}(\mathbb{R}^{d_1+d_2})$ defined for simple forms $\omega \in \mathcal{D}^{k_1+k_2}(\mathbb{R}^{d_1+d_2})$ via

$$\big\langle T_1 \times T_2, \omega \big\rangle := \big\langle T_1, \langle T_2, \varphi\, \omega_2\rangle\, \omega_1 \big\rangle$$

if $\omega = \varphi\,\omega_1 \wedge \omega_2$ with $\varphi = \varphi(x,y) = \varphi(x_1,\ldots,x_{d_1},y_1,\ldots,y_{d_2}) \in \mathrm{C}_c^\infty(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$ and $\omega_1 = \mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_{k_1}}$, $\omega_2 = \mathrm{d}y^{j_1}\wedge\cdots\wedge\mathrm{d}y^{j_{k_2}}$ for some indices $1 \le i_1 < \cdots < i_{k_1} \le d_1$, $1 \le j_1 < \cdots < j_{k_2} \le d_2$; and

$$\langle T_1 \times T_2, \omega\rangle := 0$$

if $\omega = \varphi\,\omega_1 \wedge \omega_2$ with $\omega_1 \in \mathcal{D}^{l_1}(\mathbb{R}^{d_1})$, $\omega_2 \in \mathcal{D}^{l_2}(\mathbb{R}^{d_2})$ but $l_1 \neq k_1$, $l_2 \neq k_2$; and for non-simple forms defined by linear extension of the above rules.

For its boundary we obtain the following formula:

Lemma 4.19. *Let $T_1 \in \mathcal{D}_{k_1}(\mathbb{R}^{d_1})$ and $T_2 \in \mathcal{D}_{k_2}(\mathbb{R}^{d_2})$. Then,*

$$\partial(T_1\times T_2) = \partial T_1 \times T_2 + (-1)^{k_1} T_1 \times \partial T_2.$$

Proof. The proof is a direct computation (exercise). □

Example 4.20. For $T_1 = [\![(0,1)]\!]$, i.e., the canonical current associated to the interval $(0,1)$ (with orientation $+1$ and multiplicity 1), and $T_2 = T \in \mathcal{D}_k(\mathbb{R}^d)$, we have that $[\![(0,1)]\!] \times T \in \mathcal{D}_{k+1}(\mathbb{R}^d)$ and, via Lemma 4.19,

$$\partial([\![(0,1)]\!]\times T) = \delta_1 \times T - \delta_0 \times T - [\![(0,1)]\!] \times \partial T,$$

where δ_z as usual denotes the Dirac point mass at z, here understood as a 0-dimensional current.

Let $\varrho \in \mathrm{C}_c^\infty(B_1;[0,\infty))$ with $\varrho(x) = \varrho(y)$ whenever $|x| = |y|$ $(x,y \in B_1)$ and $\int_{\mathbb{R}^d}\varrho\,\mathrm{d}x = 1$; we call such a ϱ a **(radial) mollifier**. Defining $\varrho_\delta(x) := \delta^{-d}\varrho(x/\delta)$, recall that the **mollification** of $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d;W)$, where W is a finite-dimensional normed vector space, is defined as

$$(\varrho_\delta * f)(x) := \int_{\mathbb{R}^d}\varrho_\delta(x-y)f(y)\,\mathrm{d}y.$$

Then, $\varrho_\delta * f \in \mathrm{C}^\infty(\mathbb{R}^d)$ and $\varrho_\delta * f \to f$ in $\mathrm{L}^1_{\mathrm{loc}}$; analogous statements hold for L^p-spaces, $1 \le p < \infty$. Moreover,

$$\frac{\partial}{\partial x^i}(\varrho_\delta * f) = \frac{\partial\varrho_\delta}{\partial x^i} * f$$

for $i = 1,\ldots,d$ and analogously for higher derivatives. Finally, if $\varphi \in \mathrm{C}(\mathbb{R}^d)$, then $\varrho_\delta*\varphi \to \varphi$ uniformly on every compact subset of $\mathbb{R}^d$. Finally,

$$\frac{\partial}{\partial x^i}(\varrho_\delta * \varphi) = \varrho_\delta * \frac{\partial\varphi}{\partial x^i}$$

for $i = 1,\ldots,d$ and analogously for higher derivatives.

Now, for $T \in \mathcal{D}_k(\mathbb{R}^d)$ the **mollification** $T * \varrho_\delta \in \mathcal{D}_k(\mathbb{R}^d)$ is given via

$$\langle T * \varrho_\delta, \omega \rangle := \langle T, \varrho_\delta * \omega \rangle, \qquad \omega \in \mathcal{D}^k(\mathbb{R}^d).$$

Here, $\varrho_\delta * \omega$ is defined as above with $W = \bigwedge^k \mathbb{R}^d$ (normed by the comass norm).

Lemma 4.21. *For $T \in \mathcal{D}_k(\mathbb{R}^d)$ it holds that $T * \varrho_\delta \in \mathcal{D}_k(\mathbb{R}^d)$ and $T * \varrho_\delta$ is a smooth k-current, that is,*

$$T * \varrho_\delta = \sum_{1 \le i_1 < \cdots < i_k \le d} v_\delta^{i_1,\ldots,i_k} \, (\mathrm{e}_{i_1} \wedge \cdots \wedge \mathrm{e}_{i_k}) \, \mathcal{L}^d$$

with $v_\delta^{i_1,\ldots,i_k} \in \mathrm{C}^\infty(\mathbb{R}^d)$; in fact, $v_\delta^{i_1,\ldots,i_k}(x) = \langle T, \varrho_\delta(x - \bullet) \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \rangle$. Moreover, $T * \varrho_\delta \overset{*}{\rightharpoonup} T$ as $\delta \to 0$.

Proof. It is a standard property of the convolution that for $\omega \in \mathcal{D}^k(\Omega)$ it holds that

$$\|\partial^\alpha(\varrho_\delta * \omega)\|_\infty = \|\varrho_\delta * (\partial^\alpha \omega)\|_\infty \le \|\varrho_\delta\|_{\mathrm{L}^1} \cdot \|\partial^\alpha \omega\|_\infty = \|\partial^\alpha \omega\|_\infty.$$

The fact that $T * \varrho_\delta \in \mathcal{D}_k(\mathbb{R}^d)$ is then easy to see using the definition of currents.

To show that $T * \varrho_\delta$ is given by integration, it suffices to consider the case where

$$T = \langle T, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \rangle \, \mathrm{e}_{i_1} \wedge \cdots \wedge \mathrm{e}_{i_k}$$

for a set of indices $1 \le i_1 < \cdots < i_k \le d$. The general case then follows by a linear decomposition of T into all these parts and arguing separately. We will show that in this case, $T = v_\delta \, \mathrm{e}^I \, \mathcal{L}^d$ with

$$v_\delta(x) := \langle T, \varrho_\delta(x - \bullet) \, \mathrm{d}x^I \rangle, \qquad x \in \mathbb{R}^d,$$

where we have employed the abbreviations $\mathrm{e}^I := \mathrm{e}_{i_1} \wedge \cdots \wedge \mathrm{e}_{i_k}$ and $\mathrm{d}x^I := \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$.

We may first observe that v_δ is smooth: For $i \in \{1, \ldots, d\}$ we have

$$\frac{v_\delta(x + h\mathrm{e}_i) - v_\delta(x)}{h} = \left\langle T, \frac{\varrho_\delta(x + h\mathrm{e}_i - \bullet) - \varrho_\delta(x - \bullet)}{h} \, \mathrm{d}x^I \right\rangle \to \langle T, \partial_i \varrho_\delta(x - \bullet) \, \mathrm{d}x^I \rangle$$

since the difference quotient converges to the derivative in $\mathcal{D}^k(\Omega)$ (this is an easy exercise). Iterating this argument for higher derivatives, we obtain that v is smooth.

To show that $\widetilde{T}_\delta := v_\delta \, \mathrm{e}^I \, \mathcal{L}^d$ agrees with $T * \varrho_\delta$, for any $\omega \in \mathcal{D}^k(\Omega)$ we may write $\langle \widetilde{T}_\delta, \omega \rangle$ as a Riemann sum,

$$\begin{aligned}
\langle \widetilde{T}_\delta, \omega \rangle &= \int v_\delta(x) \, \langle \mathrm{e}^I, \omega(x) \rangle \, \mathrm{d}x \\
&= \lim_{\varepsilon \downarrow 0} \sum_{z \in \mathbb{Z}^d} \varepsilon^d v_\delta(\varepsilon z) \, \langle \mathrm{e}^I, \omega(\varepsilon z) \rangle \\
&= \lim_{\varepsilon \downarrow 0} \sum_{z \in \mathbb{Z}^d} \varepsilon^d \langle T, \varrho_\delta(\varepsilon z - \bullet) \, \mathrm{d}x^I \rangle \cdot \langle \mathrm{e}^I, \omega(\varepsilon z) \rangle \\
&= \lim_{\varepsilon \downarrow 0} \sum_{z \in \mathbb{Z}^d} \varepsilon^d \varrho_\delta(\varepsilon z - \bullet) \, \big\langle \langle T, \mathrm{d}x^I \rangle \, \mathrm{e}^I, \omega(\varepsilon z) \big\rangle \\
&= \lim_{\varepsilon \downarrow 0} \sum_{z \in \mathbb{Z}^d} \varepsilon^d \varrho_\delta(\varepsilon z - \bullet) \, \langle T, \omega(\varepsilon z) \rangle \\
&= \lim_{\varepsilon \downarrow 0} \Big\langle T, \sum_{z \in \mathbb{Z}^d} \varepsilon^d \varrho_\delta(\varepsilon z - \bullet) \, \omega(\varepsilon z) \Big\rangle
\end{aligned}$$

since only finitely many terms in the sum are nonzero. Observe that

$$h_\varepsilon := \sum_{z \in \mathbb{Z}^d} \varepsilon^d \varrho_\delta(\varepsilon z - \bullet) \omega(\varepsilon z)$$

is a Riemann sum for the k-form $\varrho_\delta * \omega$. Thus, by the smoothness of ω and ϱ_δ, we have $h_\varepsilon \to \varrho_\delta * \omega$ in $\mathcal{D}^k(\Omega)$, meaning that the difference converges to zero (this is an exercise). We can now invoke the continuity property of T (see Lemma 4.1) to see that

$$\langle \widetilde{T}_\delta, \omega \rangle = \langle T, \varrho_\delta * \omega \rangle = \langle T * \varrho_\delta, \omega \rangle.$$

We have thus indeed verified that $T * \varrho_\delta = v_\delta \, \mathrm{e}^I \, \mathcal{L}^d$.

To see that $T * \varrho_\delta \overset{*}{\rightharpoonup} T$ as $\delta \to 0$, it suffices to observe

$$\langle T * \varrho_\delta, \omega \rangle = \langle T, \varrho_\delta * \omega \rangle \to \langle T, \omega \rangle$$

for all $\omega \in \mathcal{D}^k(\mathbb{R}^d)$ by standard properties of the mollification, namely that $\partial^\alpha(\varrho_\delta * \omega) = \varrho_\delta * (\partial^\alpha \omega) \to \partial^\alpha \omega$ uniformly for any (multiple) partial derivative ∂^α. □

Lemma 4.22. *For $T \in \mathcal{D}_k(\mathbb{R}^d)$ it holds that $\partial(T * \varrho_\delta) = (\partial T) * \varrho_\delta$.*

Proof. For $k = 0$ there is nothing to show. For $k \geq 1$, we note the following formula (exercise) for $\omega \in \mathcal{D}^k(\mathbb{R}^d)$:

$$\langle \partial T, \omega \rangle = \sum_{i=1}^d \Big\langle T, \frac{\partial}{\partial x^i} \big[\mathrm{d}x^i \wedge \omega \big] \Big\rangle.$$

Then, by the properties of the convolution,

$$\begin{aligned}\big\langle \partial(T * \varrho_\delta), \omega \big\rangle &= \sum_{i=1}^{d} \Big\langle T * \varrho_\delta, \frac{\partial}{\partial x^i}\big[\mathrm{d}x^i \wedge \omega\big] \Big\rangle \\ &= \sum_{i=1}^{d} \Big\langle T, \frac{\partial}{\partial x^i}\big[\mathrm{d}x^i \wedge (\varrho_\delta * \omega)\big] \Big\rangle \\ &= \big\langle \partial T, \varrho_\delta * \omega \big\rangle \\ &= \big\langle (\partial T) * \varrho_\delta, \omega \big\rangle\end{aligned}$$

and the claim has been proved. □

The boundary of a current has another important use besides its geometric interpretation: It turns out that boundaryless currents are given by integration against a *constant* function on a hyperplane; in this context also consider the formula for the boundary in Example 4.8.

Theorem 4.23 (Constancy theorem). *Let $T \in \mathcal{D}_k(\mathbb{R}^d)$ satisfy $\partial T = 0$ and $\operatorname{supp} T \subset H$ with H a k-dimensional (affine) hyperplane in $\mathbb{R}^d$. Then,*

$$T = \lambda\, [\![H]\!] = \lambda\, (v_1 \wedge \cdots \wedge v_k)\, \mathcal{H}^k \llcorner H$$

for any orthonormal basis $\{v_i\}$ of H and $\lambda \in \mathbb{R}$. More generally, if only $\partial T = 0$ in an open and connected set $U \subset \mathbb{R}^d$, then $T = \lambda\, [\![H]\!]$ on U (that is, when tested with forms supported in U). In particular, if $T \in \mathcal{D}_d(\mathbb{R}^d)$ with $\partial T = 0$, then $T = \lambda\, [\![\mathbb{R}^d]\!] = \lambda\, (\mathrm{e}_1 \wedge \cdots \wedge \mathrm{e}_d)\, \mathcal{L}^d$ for some $\lambda \in \mathbb{R}$.

Proof. We consider the approximation $T_\delta := T * \varrho_\delta$ with a family $(\varrho_\delta)_{\delta>0}$ of (radial) mollifiers. By Lemma 4.21, we have that T_δ is smooth, and from Lemma 4.22, we infer that $\partial T_\delta = (\partial T) * \varrho_\delta = 0$. Then, considering T, T_δ to be currents on $H \cong \mathbb{R}^k$, we also have that this restriction is boundaryless (exercise). From Example 4.8 we infer that all partial derivatives of the density v_δ (see Lemma 4.21, considered now on $\mathbb{R}^k$, so there is only one density) vanish. Thus, the densities v_δ of T_δ must all be a constant for all $\delta > 0$. Letting $\delta \to 0$, we get that also T is given by integration against a constant density, as required.

The additional statement has the same proof once we notice that $\partial T_\delta = (\partial T) * \varrho_\delta = 0$ in $U' \Subset U$ for $\delta > 0$ sufficiently small. □

Finally, the following lemma relates the boundary operator to the curl via Hodge duality; it will be used many times in the sequel.

Lemma 4.24. *For $T \in \mathcal{D}_2(\mathbb{R}^3)$ it holds that $\partial T = \operatorname{curl}(\star T)$.*

Proof. It can be checked (for instance by working with a basis) that the Hodge dual for a k-covector α satisfies

$$\star\alpha = (\star\alpha^\sharp)^\flat$$

with the musical isomorphisms $\flat\colon \bigwedge_k V \to \bigwedge^k V$ ("lowering an index") and $\sharp\colon \bigwedge^k V \to \bigwedge_k V$ ("raising an index"), which are determined by

$$\langle \xi, \alpha \rangle = (\xi^\flat, \alpha) = (\xi, \alpha^\sharp), \qquad \xi \in \textstyle\bigwedge_k V,\ \alpha \in \bigwedge^k V.$$

Then, let $S \in \mathcal{D}_2(\mathbb{R}^3)$ and compute, using the symmetry of the curl operator, the classical formula

$$(\mathrm{curl}\, \omega^\sharp)^\flat = \star\mathrm{d}\omega$$

for any $\omega \in \mathcal{D}^1(\mathbb{R}^3)$, which is proved in the following lemma, as well as the duality rule $\langle \star\xi, \star\alpha \rangle = \langle \xi, \alpha \rangle$ (which is analogous to the isometry property of the Hodge star, see Lemma 3.20 (ii)), as follows:

$$\big\langle \mathrm{curl}(\star S), \omega \big\rangle = \big\langle \star S, (\mathrm{curl}\, \omega^\sharp)^\flat \big\rangle = \big\langle \star S, \star\mathrm{d}\omega \big\rangle = \big\langle S, \mathrm{d}\omega \big\rangle = \big\langle \partial S, \omega \big\rangle,$$

which proves the claim. □

Lemma 4.25. *It holds that* $(\mathrm{curl}\, \omega^\sharp)^\flat = \star\mathrm{d}\omega$.

Proof. Write $\omega = \omega_1 \mathrm{d}x^1 + \omega_2 \mathrm{d}x^2 + \omega_3 \mathrm{d}x^3$. Then, for cyclic indices $i\colon 1 \to 2 \to 3 \to 1$,

$$\mathrm{d}\omega = \sum_{i=1}^{3} (\omega_{i+1} \mathrm{e}_i - \omega_i \mathrm{e}_{i+1})\, \mathrm{d}x^i \wedge \mathrm{d}x^{i+1}.$$

Since $\star(\mathrm{d}x^i \wedge \mathrm{d}x^{i+1}) = \mathrm{d}x^{i+2}$,

$$\star\mathrm{d}\omega = \sum_{i=1}^{3} (\omega_{i+1} \mathrm{e}_i - \omega_i \mathrm{e}_{i+1})\, \mathrm{d}x^{i+2}.$$

On the other hand, $\omega^\sharp = \omega_1 \mathrm{e}_1 + \omega_2 \mathrm{e}_2 + \omega_3 \mathrm{e}_3$, and by the definition of curl,

$$\mathrm{curl}(\omega^\sharp) = \sum_{i=1}^{3} (\omega_{i+1} \mathrm{e}_i - \omega_i \mathrm{e}_{i+1})\, \mathrm{e}_{i+2}.$$

Lowering the index thus shows the sought equality. □

4.5 Pushforwards

Let $T \in \mathcal{D}_k(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is open, and let $\theta\colon \Omega \to \Omega'$ be a smooth and proper map between Ω and another open set $\Omega' \subset \mathbb{R}^m$ (recall the definition of properness from just before Lemma 3.12). Then, the **pushforward** $\theta_* T \in \mathcal{D}_k(\Omega')$ of T under θ (this is also denoted by "$\theta_\# T$" in the literature, but this can lead to confusion with the measure-theoretic pushforward) is defined via

$$\langle \theta_* T, \omega \rangle := \langle T, \theta^* \omega \rangle, \qquad \omega \in \mathcal{D}^k(\Omega'),$$

where $\theta^*\omega \in \mathcal{D}^k(\mathbb{R}^d)$ denotes the pullback of the k-form ω.

More generally, we could only require that $\theta|_{\operatorname{supp} T}$ is proper, i.e., $\theta^{-1}(K) \cap \operatorname{supp} T$ is compact in Ω for every compact set $K \subset \Omega'$. Then, in the definition of $\theta_* T$ above we need to instead apply T to $\eta \cdot \theta^*\omega$ with a cut-off function $\eta \in \mathrm{C}_c^\infty(\Omega; [0,1])$ with $\eta \equiv 1$ on $\operatorname{supp} T \cap \operatorname{supp} \theta^*\omega$, which is compact by the properness assumption and contained in the open set $\mathbb{R}^d$, hence such an η exists.

Lemma 4.26. *Under the above assumptions, $\partial(\theta_* T) = \theta_*(\partial T)$.*

Proof. This is a direct consequence of Lemma 3.12. □

Lemma 4.27. *Let $T \in \mathcal{D}_k(\Omega)$ with $\mathbf{M}(T) < \infty$ and let $\theta\colon \Omega \to \Omega'$ (with $\Omega' \subset \mathbb{R}^m$ open) be a smooth and proper map. Then, $\theta_* T$ is representable by integration,*

$$\langle \theta_* T, \omega \rangle = \int \langle \mathrm{D}\theta(x)[\vec{T}(x)], \omega(\theta(x)) \rangle \, \mathrm{d}\|T\|(x) \tag{4.5}$$

for all $\omega \in \mathcal{D}^k(\Omega')$ (here, $\mathrm{D}\theta(x) = \bigwedge^k \mathrm{D}\theta(x)\colon \bigwedge^k \mathbb{R}^d \to \bigwedge^k \mathbb{R}^m$ is the differential of θ at x, extended to k-vectors).

Proof. This is just the definition of the pullback $\theta^*\omega$ written out for currents representable by integration, which our T is by Proposition 4.12. □

We now consider homotopies, that is, smooth deformations of the ambient space. A homotopy induces a corresponding deformation of currents, for which we have the following **homotopy formula**:

Lemma 4.28. *Let H be a smooth* ***homotopy****, that is, $H \in \mathrm{C}^\infty([0,1] \times \Omega; \mathbb{R}^m)$, where $\Omega \subset \mathbb{R}^d$ is open, and let $T \in \mathcal{D}_k(\Omega)$. Then, with $f := H(0, \cdot)$ and $g := H(1, \cdot)$, we have*

$$g_* T - f_* T = \partial[H_*(\llbracket (0,1) \rrbracket \times T)] + H_*(\llbracket (0,1) \rrbracket \times \partial T).$$

Proof. We compute, using Lemma 4.26 and Lemma 4.19 that

$$\partial[H_*(\llbracket (0,1) \rrbracket \times T)] = H_* \partial(\llbracket (0,1) \rrbracket \times T) = H_*(\delta_1 \times T - \delta_0 \times T - \llbracket (0,1) \rrbracket \times \partial T),$$

where δ_z as usual denotes the Dirac point mass at z, here understood as a 0-dimensional current. Realizing that $H_*(\delta_0 \times T) = f_*T$ and $H_*(\delta_1 \times T) = g_*T$ and rearranging, the result follows. $\square$

Example 4.29. Let $T \in \mathcal{D}_k(\Omega)$ with $\mathbf{M}(T) < \infty$. An **affine homotopy** (in $\Omega \subset \mathbb{R}^d$ open) is a map $H \in \mathrm{C}^\infty([0,1] \times \Omega; \mathbb{R}^m)$ which can be written with $f, g \colon \Omega \to \mathbb{R}^m$ as

$$H(t,x) = (1-t)f(x) + tg(x), \qquad (t,x) \in [0,1] \times \Omega.$$

A computation shows, where we denote by e_0 the unit vector in the "time" direction,

$$\mathrm{D}H(t,x)[\mathrm{e}_0 \wedge \vec{T}(x)] = (g(x) - f(x)) \wedge ((1-t)\mathrm{D}f(x) + t\mathrm{D}g(x))[\vec{T}(x)].$$

Thus, if $\mathbf{M}(T) < \infty$, using Lemma 4.27 we have for $\omega \in \mathcal{D}^{k+1}(\mathbb{R}^{1+m})$ that

$$\begin{aligned}
&\big\langle H_*(\llbracket(0,1)\rrbracket \times T), \omega \big\rangle \\
&\qquad = \int_0^1 \int \big\langle \mathrm{D}H[\mathrm{e}_0 \wedge \vec{T}], \omega(H(t,\bullet)) \big\rangle \;\mathrm{d}\|T\| \;\mathrm{d}t \\
&\qquad = \int_0^1 \int \big\langle (g-f) \wedge ((1-t)\mathrm{D}f + t\mathrm{D}g)[\vec{T}], \omega \circ H \big\rangle \;\mathrm{d}\|T\| \;\mathrm{d}t.
\end{aligned}$$

Since

$$\begin{aligned}
\big|\mathrm{D}H[\mathrm{e}_0 \wedge \vec{T}])\big| &= \big|(g-f) \wedge ((1-t)\mathrm{D}f + t\mathrm{D}g)[\vec{T}]\big| \\
&\le \|g-f\|_\infty \cdot (|\mathrm{D}f|^k + |\mathrm{D}g|^k),
\end{aligned}$$

we obtain the estimate (see also Example 4.13)

$$\begin{aligned}
\mathbf{M}\big(H_*(\llbracket(0,1)\rrbracket \times T)\big) &\le \|g-f\|_\infty \int |\mathrm{D}f|^k + |\mathrm{D}g|^k \;\mathrm{d}\|T\| \\
&\le \|g-f\|_\infty \cdot (\|\mathrm{D}f\|_\infty^k + \|\mathrm{D}g\|_\infty^k) \cdot \mathbf{M}(T). \qquad (4.6)
\end{aligned}$$

In particular, $H_*(\llbracket(0,1)\rrbracket \times T)$ is a $(k+1)$-current on all of $\mathbb{R}^m$.

Since Lipschitz continuity is the natural notion of regularity in the Geometric Measure Theory, we would like to extend the definition of pushforwards to this class of maps. This is complicated by the fact that a-priori the differential of a Lipschitz map, which by Rademacher's Theorem 1.18 exists $\mathcal{L}^1$-almost everywhere, may however not be defined on the carrier set of a current. It turns out, though, that at least for normal currents we can use the homotopy formula to extend the definition of pushforwards to maps with mere Lipschitz regularity:

Proposition 4.30. *Let $T \in \mathrm{N}_k(\Omega)$ be a normal current and let $\theta\colon \Omega \to \Omega'$ (with $\Omega' \subset \mathbb{R}^m$ open) be a Lipschitz and proper map. Then, the* ***pushforward*** *$\theta_* T \in \mathrm{N}_k(\Omega')$ is defined as*

$$\big\langle \theta_* T, \omega \big\rangle := \lim_{\delta \to 0} \big\langle (\varrho_\delta * \theta)_* T, \omega \big\rangle, \qquad \omega \in \mathcal{D}^k(\Omega'),$$

where $(\varrho_\delta)_{\delta>0}$ is any family of (radial) mollifiers. This expression is well-defined and does not depend on the choice of mollifier. Moreover,

$$\mathbf{M}(\theta_* T) \le \|\mathrm{D}\theta\|_\infty^k \cdot \mathbf{M}(T), \qquad \mathbf{M}(\partial(\theta_* T)) \le \|\mathrm{D}\theta\|_\infty^{k-1} \cdot \mathbf{M}(\partial T).$$

Proof. To show the well-definedness of the above limit, let $\theta^\delta := \varrho_\delta * \theta$ and for $\delta, \eta > 0$ define the affine homotopy

$$H^{\delta,\eta}(t,x) := (1-t)\theta^\delta(x) + t\theta^\eta(x).$$

Then, by the homotopy formula,

$$\big\langle \theta^\eta_* T, \omega \big\rangle - \big\langle \theta^\delta_* T, \omega \big\rangle = \big\langle H^{\delta,\eta}_*([\![(0,1)]\!] \times T), \mathrm{d}\omega \big\rangle + \big\langle H^{\delta,\eta}_*([\![(0,1)]\!] \times \partial T), \omega \big\rangle.$$

Applying (4.6), we arrive at

$$\big|\big\langle \theta^\eta_* T, \omega \big\rangle - \big\langle \theta^\delta_* T, \omega \big\rangle\big| \le \|\theta^\eta - \theta^\delta\|_\infty \cdot 2L^k \cdot (\mathbf{M}(T) + \mathbf{M}(\partial T)) \cdot (\|\omega\|_\infty + \|\mathrm{d}\omega\|_\infty),$$

where we have denoted by L (assumed to be at least 1) the Lipschitz constant of θ, which also constitutes a bound on the Lipschitz constant of $\theta^\delta, \theta^\eta$. Thus, $\langle \theta^\eta_* T, \omega \rangle$ is a Cauchy sequence as $\delta \to 0$ and the well-definedness of $\langle \theta_* T, \omega \rangle$ is established. The independence from the choice of mollifier is obtained from the fact that the difference of the mollified θ's must go to zero uniformly since both of these mollifications converge to the (continuous) θ. The mass bounds also follows by a limit passage from the corresponding bound for θ^δ. In particular, $\theta_* T$ is normal. □

It turns out that a representation formula for the pushforward as in (4.5) holds also for normal currents:

Theorem 4.31 (Alberti–Marchese [3]). *Let $T \in \mathrm{N}_k(\Omega)$ and let $\theta\colon \Omega \to \Omega'$ be a Lipschitz, proper (on $\operatorname{supp} T$) and injective map. Then, $\mathrm{D}^T\theta(x)$, that is, the differential of θ restricted to the space $\operatorname{span} \vec{T}$ exists for $\|T\|$-almost every $x \in \Omega$, and*

$$\theta_* T = \theta_*[\vec{T}]\, \theta_\# \mu$$

with

$$\theta_*[\vec{T}](y) := \mathrm{D}\theta(\theta^{-1}(y))[\vec{T}(\theta^{-1}(y))]$$

for $\|T\|$-almost every $y \in \operatorname{img} \theta$.

Proof. The key ingredient to the proof is that $\mathrm{D}^T\theta(x)$ indeed exists for $\|T\|$-almost every $x \in \mathbb{R}^d$. This deep fact, whose proof goes beyond the present work, was first established by Alberti–Marchese in Theorem 5.10 of [3]. It uses the normality of T in an essential way. This result is essentially a measure-theoretic analogue of Proposition 2.20 (which would be enough if we additionally assumed that T is integral, as defined in Chapter 5 below) and intuitively says that Lipschitz maps have much better tangential differentiability properties than may be expected.

One can furthermore prove, see Corollary 8.3 in [3] that there exists a sequence $(\theta_j)_j \subset \mathrm{C}^\infty(\mathbb{R}^d;\Omega')$ such that $\theta_j \to \theta$ uniformly, $\mathrm{Lip}(\theta_j) \to \mathrm{Lip}(\theta)$ and $\mathrm{D}^T\theta_j \to \mathrm{D}^T\theta$ in the operator norm (this is a technical result, which is not very hard to establish). Note that here $\mathrm{D}^T\theta_j$ is the restriction of the (existing!) differential $\mathrm{D}\theta_j$, whereas $\mathrm{D}^T\theta$ is not defined outside $\operatorname{span}\vec{T}$.

Finally, we use that

$$\big\langle \theta_* T, \omega \big\rangle = \lim_{\delta \to 0} \big\langle (\theta_j)_* T, \omega \big\rangle, \qquad \omega \in \mathcal{D}^k(\Omega'),$$

which follows by the same argument as in the proof of (4.30). Expanding the right-hand side as in (4.5) and passing to the limit via the dominated convergence theorem (here we use the normality), we see that

$$\big\langle \theta_* T, \omega \big\rangle = \int \big\langle \mathrm{D}^T\theta_j[\vec{T}], \omega \circ \theta \big\rangle \,\mathrm{d}\|T\| \to \int \big\langle \mathrm{D}^T\theta[\vec{T}], \omega \circ \theta \big\rangle \,\mathrm{d}\|T\|.$$

Combining, we arrive at the claimed representation of $\theta_* T$. $\square$

Chapter 5

Integral Currents

As we have seen, the class of general currents includes some highly pathological specimens, at least when thinking of currents as being "geometric" objects. It was a decisive breakthrough when Federer and Fleming in 1960 found a subclass of currents with a clear geometric meaning that is closed under weak* convergence. This class consists of the so-called *integral* currents, which are given by integration over oriented $\mathcal{H}^k$-rectifiable subsets with integer multiplicity and such that the boundary has the same property. This chapter will discuss integral currents in some detail.

5.1 Integer-multiplicity rectifiable and integral currents

A (local) Borel measure $T \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d; \bigwedge_k \mathbb{R}^d) \subset \mathcal{D}_k(\mathbb{R}^d)$ (cf. Proposition 4.12 in this context) is called an **integer-multiplicity rectifiable k-current (IMR k-current)** if it is of the form

$$T = m\vec{T}\,\mathcal{H}^k \llcorner R,$$

that is,

$$\langle T, \omega \rangle = \int_R \langle \vec{T}(x), \omega(x) \rangle \, m(x) \, \mathrm{d}\mathcal{H}^k(x), \qquad \omega \in \mathcal{D}^k(\mathbb{R}^d),$$

where

(i) $R \subset \mathbb{R}^d$ is a countably $\mathcal{H}^k$-rectifiable set;

(ii) $\vec{T} \colon R \to \bigwedge_k \mathbb{R}^d$ is an $\mathcal{H}^k$-measurable map such that for $\mathcal{H}^k$-almost every $x \in R$ the k-vector $\vec{T}(x)$ is a simple unit k-vector spanning the approximate tangent space $\mathrm{T}_x R$;

(iii) $m \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{H}^k \llcorner R; \mathbb{N})$;

One calls $\vec{T}$ the **orientation map** of T and m the **multiplicity**. By Theorem 1.14 and Proposition 4.12 the **total variation measure** of T can then be seen to be

$$\|T\| = m\,\mathcal{H}^k \llcorner R \in \mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}^d),$$

so that $T = \vec{T}\|T\|$ is the Radon–Nikodym decomposition of T. The **(global) mass** of T is

$$\mathbf{M}(T) = \|T\|(\mathbb{R}^d) = \int_R m(x)\;\mathrm{d}\mathcal{H}^k(x).$$

The members of the following sets are called **integral k-currents** ($k \in \mathbb{N} \cup \{0\}$):

$$\begin{aligned}\mathrm{I}_k(\mathbb{R}^d) := \bigl\{\, T \text{ IMR } k\text{-current on } \mathbb{R}^d \;:\; &\partial T \text{ IMR } (k-1)\text{-current and} \\ &\mathbf{M}(T) + \mathbf{M}(\partial T) < \infty \,\bigr\}.\end{aligned}$$

We quote without proof the following fundamental theorem, showing that the condition on ∂T to be integer-multiplicity rectifiable is in fact redundant under a mass bound:

Theorem 5.1 (Boundary rectifiability). *Let $T \in \mathcal{D}_k(\mathbb{R}^d)$ be integer-multiplicity rectifiable such that $\mathbf{M}(T) + \mathbf{M}(\partial T) < \infty$. Then, also ∂T is integer-multiplicity rectifiable and thus $T \in \mathrm{I}_k(\mathbb{R}^d)$.*

For integral currents we can obtain a mode detailed understanding of products and push-forwards: Let $T_1 = m_1\,\vec{T}_1\,\mathcal{H}^{k_1} \llcorner R_1 \in \mathrm{I}_{k_1}(\mathbb{R}^{d_1})$ and $T_2 = m_2\,\vec{T}_2\,\mathcal{H}^{k_2} \llcorner R_2 \in \mathrm{I}_{k_2}(\mathbb{R}^{d_2})$ with the property that the product set $R_1 \times R_2$ is countably $\mathcal{H}^{k_1+k_2}$-rectifiable (in particular, $\mathcal{H}^{k_1+k_2}$-measurable, which is not automatic). Then, the product current of T_1, T_2 is

$$T_1 \times T_2 = m_1 m_2\,(\vec{T}_1 \wedge \vec{T}_2)\,\mathcal{H}^{k_1+k_2} \llcorner (R_1 \times R_2) \in \mathrm{I}_{k_1+k_2}(\mathbb{R}^{d_1+d_2}).$$

The fact that $T_1 \times T_2 \in \mathrm{I}_{k_1+k_2}(\mathbb{R}^{d_1+d_2})$ follows from straightforward mass estimates (cf. Example 4.13) and Lemma 4.19.

Proposition 5.2. *Let $T = m\,\vec{T}\,\mathcal{H}^k \llcorner R \in \mathrm{I}_k(\mathbb{R}^d)$ and let $\theta\colon \mathbb{R}^d \to \mathbb{R}^m$ be a smooth and proper map (on $\operatorname{supp} T$). Then, $\theta_* T \in \mathrm{I}_k(\mathbb{R}^m)$ and*

$$\begin{aligned}\langle \theta_* T, \omega\rangle &= \int_R \langle \mathrm{D}^R\theta(x)[\vec{T}(x)], \omega(\theta(x))\rangle\, m(x)\;\mathrm{d}\mathcal{H}^k(x) \\ &= \int_{\theta(R)} \Biggl\langle \sum_{x \in R^* \cap \theta^{-1}(y)} m(x) \cdot \frac{\mathrm{D}^R\theta(x)[\vec{T}(x)]}{|\mathrm{D}^R\theta(x)[\vec{T}(x)]|}, \omega(y) \Biggr\rangle\;\mathrm{d}\mathcal{H}^k(y)\end{aligned}$$

for all $\omega \in \mathcal{D}^k(\mathbb{R}^m)$. Here, R^* is the subset of all $x \in R$ for which $\mathrm{T}_x R$, $\mathrm{D}^R\theta(x)$ exist and $\mathrm{D}^R\theta(x)$ is of rank k.

Proof. The first equality is a rewriting of the result of Lemma 4.27 for integral currents.

Observe that $\mathrm{T}_x R$, $\mathrm{D}^R\theta(x)$ exist for $\mathcal{H}^k$-almost every $x \in R$ by Proposition 2.23. For $\mathcal{H}^k$-almost every $y \in \theta(R)$ and every $x \in R^* \cap \theta^{-1}(y)$ we observe that

$$\mathrm{D}^R\theta(x)[\vec{T}(x)] = \pm w_1 \wedge \cdots w_k$$

for an orthonormal basis $\{w_1, \ldots, w_k\}$ of $\mathrm{T}_y\,\theta(R)$, and that

$$\mathbf{J}_k\theta(x) = |\mathrm{D}^R\theta(x)[\vec{T}(x)]|.$$

Moreover, the set where $\mathrm{D}^R\theta(x)$ is not of rank k (hence $\mathbf{J}_k\theta(x) = 0$) can be neglected by virtue of Lemma 2.3. The second equality in the statement of the proposition follows from the area formula in its rectifiable version, see Theorem 2.24.

It remains to show the integrality of $\theta_* T$. Proposition 2.17 implies that $\theta(R^*)$ is $\mathcal{H}^k$-rectifiable (note here that R^* is $\mathcal{H}^k$-rectifiable as a subset of an $\mathcal{H}^k$-rectifiable set and we have a finite mass bound) and by Lemma 2.3, $\mathcal{H}^k(\theta(R) \setminus \theta(R^*)) = 0$. Thus, $\theta_* T$ is a rectifiable current and it only remains to show the integrality. At $y \in \theta(R^*)$ such that $\mathrm{T}_y\,\theta(R)$ exists, and for every $x \in R^* \cap \theta^{-1}(y)$ we have either

$$\mathrm{D}^R\theta(x)[\vec{T}(x)] = w_1 \wedge \cdots w_k \qquad \text{or} \qquad \mathrm{D}^R\theta(x)[\vec{T}(x)] = -w_1 \wedge \cdots w_k.$$

Since m is integer-valued, we thus obtain that

$$\sum_{x \in R^* \cap \theta^{-1}(y)} m(x) \cdot \frac{\mathrm{D}^R\theta(x)[\vec{T}(x)]}{|\mathrm{D}^R\theta(x)[\vec{T}(x)]|}$$

is an *integer* multiple of $w_1 \wedge \cdots w_k$. Hence, $\theta_* T$ is integral. □

5.2 Deformation theorem

The following is one of the most important theorems in the theory of currents with numerous applications:

Theorem 5.3 (Deformation). *Let $T \in \mathrm{I}_k(\mathbb{R}^d)$. Then, for all $\varrho > 0$ there exist $R \in \mathrm{I}_{1+k}(\mathbb{R}^d)$, $S \in \mathrm{I}_k(\mathbb{R}^d)$, and a **polyhedral k-current***

$$P = \sum_{F \in \mathcal{F}_k(\varrho)} p_F [\![F]\!] \in \mathrm{I}_k(\mathbb{R}^d),$$

where $[\![F]\!]$ is the integral current associated to an oriented k-face $F \in \mathcal{F}_k(\varrho)$ of one of the cubes $\varrho z + [0, \varrho]^d$ (with unit multiplicity and a fixed choice of orientation), $z \in \mathbb{Z}^d$, and $p_F \in \mathbb{Z}$, such that

$$T = P + \partial R + S$$

and

$$\begin{aligned}\mathbf{M}(P) &\le C(\mathbf{M}(T) + \varrho\mathbf{M}(\partial T)),\\ \mathbf{M}(\partial P) &\le C\mathbf{M}(\partial T),\\ \mathbf{M}(R) &\le C\varrho\mathbf{M}(T),\\ \mathbf{M}(S) &\le C\varrho\mathbf{M}(\partial T).\end{aligned}$$

Here, the constant $C = C(d,k) > 0$ depends only on the dimensions.

Proof. Step 1. We start by observing that it suffices to prove the deformation theorem for $\varrho = 1$; the general case is reduced to $\varrho = 1$ by scaling. Indeed, setting $r^\alpha(x) := \alpha x$ for $\alpha > 0$ and $x \in \mathbb{R}^d$, we may apply the result in the version for $\varrho = 1$ to $\widetilde{T} := r^{1/\varrho}_* T$ and in this way obtain $\widetilde{P}, \widetilde{R}, \widetilde{S}$ as in the statement of the theorem for $\varrho = 1$. Then, set $P := r^\varrho_* \widetilde{P}$, $R := r^\varrho_* \widetilde{R}$, $S := r^\varrho_* \widetilde{S}$. These P, S satisfy the conclusion of the theorem for our ϱ since P, S have the same dimension k and R has dimension $k+1$, whereby

$$\mathbf{M}(R) = \varrho^{k+1}\mathbf{M}(\widetilde{R}) \le C\varrho^{k+1}\mathbf{M}(\widetilde{T}) = C\varrho\mathbf{M}(T).$$

by the area formula. The other scalings follow in a similar way. So, in the following let $\varrho = 1$.

Step 2. Define for $n = 0, 1, \ldots, d$ the n-skeleton of the cubical decomposition as

$$Y_n := \bigcup_{F \in \mathcal{F}_n(1)} F$$

and the shifted n-skeleton as

$$Z_n := (\tfrac{1}{2}, \ldots, \tfrac{1}{2}) + Y_n.$$

Denote by $\psi_n \colon Y_{d-n} \setminus Z_n \to Y_{d-n-1}$ the radial projection of a point x in a face $F \in \mathcal{F}_{d-n}(1)$, except the center of F, onto the boundary of F, i.e., the union of F's $(d-n-1)$-faces. More precisely $\psi_n(x)$, for $x \in F$ not the center of F, is the intersection of the ray originating at the center of F with the boundary of F. In the definition of ψ_n we have in fact artificially excluded all of Z_n (not just the mid-points), but this does not matter in the following. Then define

$$\psi := \psi_{d-k-1} \circ \cdots \circ \psi_0 \colon \mathbb{R}^d \setminus Z_{d-k-1} \to Y_k.$$

Note that the map ψ_{d-k-1} limits the domain of definition of ψ to the complement of Z_{d-k-1}.

An elementary, but uninspiring, proof shows that we may bound the gradient of ψ as follows:

$$1 \le |\nabla\psi(x)| \le C(\operatorname{dist}(x, Z_{d-k-1}))^{-1}, \qquad x \in \mathbb{R}^d \setminus Z_{d-k-1} \tag{5.1}$$

for a constant $C = C(d,k) > 0$. The interested reader can find the details in Lemmas 7.7.3, 7.7.4 of [23].

We now show that

$$\int_{[-\frac{1}{2},\frac{1}{2}]^d} |\nabla\psi(x)|^k \,\mathrm{d}x < \infty. \tag{5.2}$$

Using (5.1), the splitting $x = (x', x'') \in \mathbb{R}^{k+1} \times \mathbb{R}^{d-k-1}$, and Fubini's theorem, we have

$$\begin{aligned}\int_{[-\frac{1}{2},\frac{1}{2}]^d} |\nabla\psi(x)|^k \,\mathrm{d}x &\le C \int_{[-\frac{1}{2},\frac{1}{2}]^d} \frac{1}{\operatorname{dist}(x, Z_{d-k-1})^k} \,\mathrm{d}x \\ &\le C \int_{[-\frac{1}{2},\frac{1}{2}]^d} |x'|^{-k} \,\mathrm{d}x \\ &\le C \int_{[-\frac{1}{2},\frac{1}{2}]^{k+1}} |x'|^{-k} \,\mathrm{d}x' \\ &< \infty\end{aligned}$$

since the $|x'|^{-k}$ is integrable in compact sets of $\mathbb{R}^{k+1}$ (one can for instance use spherical coordinates to see this explicitly).

Step 3. Our next claim is that there is a translation $a \in [-\frac{1}{2}, \frac{1}{2}]^d$ with the property that

$$\int_{\mathbb{R}^d} |\nabla\psi|^k \,\mathrm{d}\|\mathbf{t}^a_* T\| \le C\mathbf{M}(T), \qquad \int_{\mathbb{R}^d} |\nabla\psi|^{k-1} \,\mathrm{d}\|\mathbf{t}^a_* \partial T\| \le C\mathbf{M}(\partial T), \tag{5.3}$$

where $\mathbf{t}^a(x) := x + a$ and $C = C(d,k) > 0$ is a dimensional constant.

We first observe that from the construction of ψ and (5.2) it follows that

$$\int_{[-\frac{1}{2},\frac{1}{2}]^d} |\nabla\psi(x+a)|^k \,\mathrm{d}a = \int_{[-\frac{1}{2},\frac{1}{2}]^d} |\nabla\psi(a)|^k \,\mathrm{d}a =: \alpha$$

for almost every $x \in \mathbb{R}^d$. Thus, by Fubini's theorem,

$$\begin{aligned}&\int_{[-\frac{1}{2},\frac{1}{2}]^d} \int_{\mathbb{R}^d} |\nabla\psi(x)|^k \,\mathrm{d}\|\mathbf{t}^a_* T\|(x) \,\mathrm{d}a \\ &\qquad = \int_{\mathbb{R}^d} \int_{[-\frac{1}{2},\frac{1}{2}]^d} |\nabla\psi(x+a)|^k \,\mathrm{d}a \,\mathrm{d}\|T\|(x) \\ &\qquad = \alpha\mathbf{M}(T).\end{aligned}$$

Thus, there exists a set of a's of measure at least $2/3$ such that for the a's from this set it holds that

$$\int_{\mathbb{R}^d} |\nabla\psi(x)|^k \,\mathrm{d}\|\mathbf{t}^a_* T\|(x) \le 3\alpha\mathbf{M}(T).$$

Otherwise, the set where the above fails would have measure larger than $1/3$, in contradiction with the previous equality. A similar argument applies to the other integral in the claim. Hence, the claim is established the intersection of the two exceptional sets, whose measure is at least $1/3$.

Step 4. Fix an $a \in [-\frac{1}{2}, \frac{1}{2}]^d$ with the property (5.3) and set

$$Q := \psi_* \mathbf{t}^a_* T.$$

For the affine homotopy (see Example 4.29) $H^1(t,x) := (1-t)x + t\mathbf{t}^a(x)$ we may apply the homotopy formula from Lemma 4.28 to obtain

$$T = \mathbf{t}^a_* T - \partial[H^1_*([\![(0,1)]\!] \times T)] - H^1_*([\![(0,1)]\!] \times \partial T).$$

Further, with the affine homotopy $H^2(t,x) := (1-t)x + t\psi(x)$ we write

$$\mathbf{t}^a_* T = Q - \partial[H^2_*([\![(0,1)]\!] \times \mathbf{t}^a_* T)] - H^2_*([\![(0,1)]\!] \times \partial \mathbf{t}^a_* T).$$

Thus,

$$T - Q = \partial R + \widetilde{S}$$

with

$$\begin{aligned} R &:= -H^1_*([\![(0,1)]\!] \times T)] - H^2_*([\![(0,1)]\!] \times \mathbf{t}^a_* T), \\ \widetilde{S} &:= -H^1_*([\![(0,1)]\!] \times \partial T) - H^2_*([\![(0,1)]\!] \times \partial \mathbf{t}^a_* T). \end{aligned}$$

Via Proposition 5.2 we know that R and S' are integral currents of dimension $k+1$ and k, respectively.

We can further estimate, using (4.6), (5.3), and Lemma 4.26 that

$$\begin{aligned} \mathbf{M}(R) &\le C\bigg(|a|\mathbf{M}(T) + \int |\nabla\psi|^k \, \mathrm{d}\|\mathbf{t}^a T\|\bigg) \le C\mathbf{M}(T), \\ \mathbf{M}(\widetilde{S}) &\le C\bigg(|a|\mathbf{M}(\partial T) + \int |\nabla\psi|^{k-1} \, \mathrm{d}\|\partial \mathbf{t}^a T\|\bigg) \le C\mathbf{M}(\partial T), \end{aligned}$$

with a dimensional constant $C = C(d,k) > 0$.

Step 5. Our Q is supported in Y_k by construction. Let us first suppose in addition that our current Q has boundary ∂Q concentrated on Y_{k-1}; this is the case for instance if T was boundaryless to begin with. Let $F \subset \mathcal{F}_k(1)$ be a face of the k-skeleton of the cubical decomposition of $\mathbb{R}^d$, and let $\mathring{F}$ be its relative interior (when F is considered embedded in a k-plane). Then, since $Q \llcorner \mathring{F}$ is boundaryless, by the Constancy Theorem 4.23 we deduce

$$Q \llcorner \mathring{F} = p_F [\![F]\!]$$

for some $p_F \in \mathbb{Z}$ (recall that Q is integral). Thus, with

$$P := Q = \sum_{F \in \mathcal{F}_k(1)} p_F [\![F]\!]$$

and $S := \widetilde{S}$ we have arrived at the sought decomposition

$$T = P + \partial R + S.$$

Similar estimates as in Step 4 give that for $P = Q = \psi_* \mathbf{t}^a_* T$ we have

$$\mathbf{M}(P) \leq C\mathbf{M}(T).$$

The estimates for R and $S = \widetilde{S}$ have already been shown and the proof is finished in the case that Q has boundary ∂Q concentrated on Y_{k-1}.

Step 6. If ∂Q is not concentrated on Y_{k-1}, we cannot conclude that Q is polyhedral (it is an exercise to find a counterexample). However, we may apply Steps 1–5 to ∂Q, for which $\partial(\partial Q) = 0$. In this way, we obtain a polyhedral $(k-1)$-current P' supported on Y_{k-1}, and $R' \in \mathrm{I}_k(\mathbb{R}^d)$ (the corresponding S' vanishes since ∂Q is boundaryless) such that

$$\partial Q = P' + \partial R'.$$

Set

$$P := Q - R', \qquad S := \widetilde{S} + R'.$$

Since this P satisfies the assumption at the beginning of Step 5, we conclude as before that it is polyhedral. Then,

$$T = Q + \partial R + \widetilde{S} = P + \partial R + S.$$

Because $\mathbf{M}(R') \leq C\mathbf{M}(\partial Q) \leq C\mathbf{M}(\partial T)$ (using (4.6) and Lemma 4.26), we have

$$\mathbf{M}(P) \leq C(\mathbf{M}(T) + \mathbf{M}(\partial T)), \qquad \mathbf{M}(S) \leq C\mathbf{M}(\partial T).$$

Thus, also in this case we have shown the desired claim. □

Remark 5.4. A slightly more involved proof in the non-boundaryless case (due to Leon Simon) shows that in fact one may sharpen the bound on P to $\mathbf{M}(P) \leq C\mathbf{M}(T)$. This improvement, however, needs a better choice of P and a Poincaré-inequality for functions of bounded variation; see Section 7.7 in [23] for the details.

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A nice geometric consequence of the deformation theorem is the following.

Theorem 5.5 (Isoperimetric inequality). *Let $T \in \mathrm{I}_k(\mathbb{R}^d)$, $k \geq 1$, with $\partial T = 0$. Then, there exists a current $R \in \mathrm{I}_{1+k}(\mathbb{R}^d)$ such that*

$$T = \partial R$$

and

$$\mathbf{M}(R) \leq C\mathbf{M}(T)^{(k+1)/k}$$

Here, the constant $C = C(d,k) > 0$ depends only on the dimensions.

Proof. If $k = d$, then by the Constancy Theorem 4.23, $T = \alpha[\![\mathbb{R}^d]\!] = 0$, and so the result is trivially true.

Assuming that $1 \leq k < d$ and $T \neq 0$, we let $T = P + \partial R$ as in the deformation theorem with

$$\varrho := [2\overline{C}\mathbf{M}(T)]^{1/k},$$

where $\overline{C} > 0$ is the constant from said theorem. By a simple scaling argument, $\mathbf{M}(P) = N(\varrho)\varrho^k$ for some nonnegative integer $N(\varrho)$. From the estimates in the deformation theorem we have $\mathbf{M}(P) \leq \overline{C}\mathbf{M}(T)$ and thus

$$N(\varrho) \cdot 2\overline{C}\mathbf{M}(T) = \mathbf{M}(P) \leq \overline{C}\mathbf{M}(T).$$

So, $2N(\varrho) \leq 1$, whereby $N(\varrho) = 0$, and hence $P = 0$. This immediately yields all the claimed statements. □

5.3 Compactness and flat convergence

The fundamental theorem about convergence of integral currents is the following:

Theorem 5.6 (Federer–Fleming compactness theorem). *Let $(T_j)_j \subset \mathrm{I}_k(\mathbb{R}^d)$ with*

$$\sup_{j \in \mathbb{N}} \big(\mathbf{M}(T_j) + \mathbf{M}(\partial T_j)\big) < \infty.$$

Then, there is a subsequence (not relabeled) of $(T_j)_j$ and a $T \in \mathrm{I}_k(\mathbb{R}^d)$ such that $T_j \overset{*}{\rightharpoonup} T$. Moreover,

$$\mathbf{M}(T) \leq \liminf_{j \to \infty} \mathbf{M}(T_j),$$
$$\mathbf{M}(\partial T) \leq \liminf_{j \to \infty} \mathbf{M}(\partial T_j).$$

We will not give a proof of this theorem here and only consider a few comments: First, it is not too hard to see from abstract results in Functional Analysis, namely weak* compactness results like the Banach–Alaoglu theorem, that there always is a weakly* convergent subsequence. The difficult part is to show that the limit current is integral (hence, the above theorem is sometimes referred to as the *closure theorem* for integral currents). There are several strategies to achieve this, but all rely fundamentally on a number of further tools, most prominently a form of slicing procedure for currents (either by level sets of a Lipschitz map or just on balls) as well as a rectifiability theorem which relates the rectifiability of slices to the rectifiability of the whole current. The slicing part requires a number of technical preparations, which are somewhat lengthy, but not particularly hard. The rectifiability is a delicate matter, and builds on the rectifiability theorems surveyed in Section 2.6. The interested reader can find a complete and readable proof in Chapter 8 of [23]. Another proof based on slicing is in [33].

Example 5.7. In Example 4.15 we saw a sequence of integral currents that converge to a non-integral current (rectifiable, but with multiplicity $1/2$). It can be checked that for this sequence the boundary masses are not uniformly bounded, so the Federer–Fleming compactness theorem does not apply.

The Federer–Fleming compactness theorem has a number of important consequences, including the following solution to (one possible version) of **Plateau's Problem**:

Theorem 5.8. *Given $R \in \mathrm{I}_k(\mathbb{R}^d)$, there exists a* ***mass-minimizing*** *current $T \in \mathrm{I}_k(\mathbb{R}^d)$ with the same boundary values as R, that is,*

$$\mathbf{M}(T) = \inf \left\{ \mathbf{M}(\widehat{T}) \;:\; \widehat{T} \in \mathrm{I}_k(\mathbb{R}^d),\ \partial T = \partial R \right\}$$

and $\operatorname{supp} T \subset \overline{B}_N$ *with N so large that* $\operatorname{supp} R \subset B_N$.

Before we come to the proof of the theorem, we need a fact about pushforwards:

Proposition 5.9. *Let $T \in \mathrm{I}_k(\mathbb{R}^d)$ and let $\theta \colon \mathbb{R}^d \to \mathbb{R}^m$ be a Lipschitz and proper map (on an open set containing* $\operatorname{supp} T$*). Then, $\theta_* T \in \mathrm{I}_k(\mathbb{R}^m)$.*

Proof. From (the proof of) Proposition 4.30 we know that $\theta_* T = \text{w*-}\lim_{\delta \to 0} \theta^\delta_* T$, where $(\theta^\delta)_{\delta > 0}$ is a smooth approximation of θ. From Proposition 5.2 we know that $\theta^\delta_* T$ is integral. Since $\limsup_{\delta \to 0} (\mathbf{M}(\theta^\delta_* T) + \mathbf{M}(\partial \theta^\delta_* T)) < \infty$, the Federer–Fleming Compactness Theorem 5.6 yields that also $\theta_* T$ is integral. □

Proof of Theorem 5.8. Set

$$m_* := \inf \left\{ \mathbf{M}(\widehat{T}) \;:\; \widehat{T} \in \mathrm{I}_k(\mathbb{R}^d),\ \partial T = \partial R \right\} \in [0, \infty).$$

We employ the *Direct Method* of the Calculus of Variations: Let $(T_j)_j \subset \mathrm{I}_k(\mathbb{R}^d)$ with $\partial T_j = \partial R$ and such that $\mathbf{M}(T_j) \to m_*$. Let $N \in \mathbb{N}$ be so large that $\operatorname{supp} R \subset B_N$ and denote by $r\colon \mathbb{R}^d \to \overline{B}_N$ the retraction of $\mathbb{R}^d$ onto the nearest point in $\overline{B}_N$. This map r is well-defined by the convexity of $\overline{B}_N$, has Lipschitz constant 1, and $r_*(\partial R) = \partial R$. Then, using the estimate (4.6),

$$\mathbf{M}(r_* T_j) \le \mathbf{M}(T_j)$$

and also, using Lemma 4.26, $\partial(r_* T_j) = r_*(\partial T_j) = r_*(\partial R) = \partial R$. Hence, replacing T_j by $r_* T_j$, which is integral by Proposition 5.9 below, we may assume that $\operatorname{supp} T_j \subset \overline{B}_N$.

Now apply the Federer–Fleming Compactness Theorem 5.6 to obtain the existence of $T_* \in \mathrm{I}_k(\mathbb{R}^d)$ with

$$m_* \le \mathbf{M}(T_*) \le \liminf_{j\to\infty} \mathbf{M}(T_j) = m_*.$$

Hence, this T_* constitutes a solution to the minimization problem. □

5.4 Flat convergence

Recall from (4.4) the definition of the flat norm, now on the whole space:

$$\mathbf{F}(T) := \inf\bigl\{\, \mathbf{M}(T - \partial S) + \mathbf{M}(S) \;:\; S \in \mathrm{N}_{k+1}(\mathbb{R}^d) \,\bigr\}.$$

Lemma 5.10. *Let $T_j, T \in \mathrm{N}_k(\mathbb{R}^d)$. Then, for every $\omega \in \mathcal{D}^k(\mathbb{R}^d)$,*

$$\bigl\langle T_j - T, \omega \bigr\rangle \le \mathbf{F}(T_j - T)(\|\omega\|_\infty + \|\mathrm{d}\omega\|_\infty).$$

Consequently, $\mathbf{F}(T_j - T) \to 0$ implies $T_j \overset{*}{\rightharpoonup} T$ as currents.

Proof. Set $R_j := T_j - T$. Take an arbitrary $S_j \in \mathrm{N}_{k+1}(\mathbb{R}^d)$. For $\omega \in \mathcal{D}^k(\mathbb{R}^d)$, then

$$\begin{aligned}
\bigl|\bigl\langle R_j, \omega \bigr\rangle\bigr| &\le \bigl|\bigl\langle R_j - \partial S_j, \omega \bigr\rangle\bigr| + \bigl|\bigl\langle \partial S_j, \omega \bigr\rangle\bigr| \\
&= \bigl|\bigl\langle R_j - \partial S_j, \omega \bigr\rangle\bigr| + \bigl|\bigl\langle S_j, \mathrm{d}\omega \bigr\rangle\bigr| \\
&\le \bigl(\mathbf{M}(R_j - \partial S_j) + \mathbf{M}(S_j)\bigr)(\|\omega\|_\infty + \|\mathrm{d}\omega\|_\infty).
\end{aligned}$$

Taking the infimum over $S_j \in \mathrm{N}_{k+1}(\mathbb{R}^d)$ gives the sought inequality. The convergence assertion is then clear. □

In fact, even the following strengthening of this lemma is true:

Theorem 5.11 (Federer–Fleming flat-norm duality). *For every $T \in \mathrm{N}_k(\mathbb{R}^d)$,*

$$\mathbf{F}(T) = \sup\bigl\{\, \bigl\langle T, \omega \bigr\rangle \;:\; \omega \in \mathcal{D}^k(\mathbb{R}^d),\ \|\omega\|_\infty \le 1,\ \|\mathrm{d}\omega\|_\infty \le 1 \,\bigr\}.$$

The proof of this result can be found in Section 4.1.12 of [19].

For an *integral* $T \in \mathrm{I}_k(\mathbb{R}^d)$, the **integral flat norm** is given by

$$\mathbf{FI}(T) := \inf \Big\{ \mathbf{M}(R) + \mathbf{M}(S) \ : \ R \in \mathrm{I}_{1+k}(\mathbb{R}^d),\, S \in \mathrm{I}_k(\mathbb{R}^d) \text{ with } T = \partial R + S \Big\}.$$

Then, one can consider the **flat convergence** $\mathbf{FI}(T_j - T) \to 0$ for a sequence $(T_j)_j \subset \mathrm{I}_k(\mathbb{R}^d)$ and $T \in \mathrm{I}_k(\mathbb{R}^d)$. Notice that

$$\mathbf{F}(T) \le \mathbf{FI}(T)$$

for all $T \in \mathrm{I}_k(\mathbb{R}^d)$.

Under a uniform normality assumption this notion of convergence agrees with the weak* one:

Proposition 5.12. *Let $(T_j)_j \subset \mathrm{I}_k(\mathbb{R}^d)$ with $\operatorname{supp} T_j \subset K$ for a fixed compact set $K \subset \mathbb{R}^d$ and*

$$\sup_{j \in \mathbb{N}} \big(\mathbf{M}(T_j) + \mathbf{M}(\partial T_j)\big) < \infty.$$

Then, $T_j \overset{*}{\rightharpoonup} T$ for some $T \in \mathrm{I}_k(\mathbb{R}^d)$ if and only if $\mathbf{FI}(T_j - T) \to 0$. If $\partial T_j = 0$ for all j, then we may replace $\mathbf{FI}$ by $\mathbf{FI}^\circ$.

Proof. For the first direction, let us observe that for $\omega \in \mathcal{D}^k(\mathbb{R}^d)$ it holds that for any $R \in \mathrm{I}_{1+k}(\mathbb{R}^d)$, $S \in \mathrm{I}_k(\mathbb{R}^d)$ with $T_j - T = \partial R + S$ we have

$$\begin{aligned} \big|\langle T_j - T, \omega \rangle\big| &= \big|\langle \partial R + S, \omega \rangle\big| \\ &\le \big|\langle R, \mathrm{d}\omega \rangle\big| + \big|\langle S, \omega \rangle\big| \\ &\le (\mathbf{M}(R) + \mathbf{M}(S)) \cdot \max\big\{\|\omega\|_\infty, \|\mathrm{d}\omega\|_\infty\big\}. \end{aligned}$$

Taking the supremum over all such R, S, we obtain

$$\big|\langle T_j - T, \omega \rangle\big| \le \mathbf{FI}(T_j - T) \cdot \max\big\{\|\omega\|_\infty, \|\mathrm{d}\omega\|_\infty\big\}.$$

Hence, if $\mathbf{FI}(T_j - T) \to 0$, then also $T_j \overset{*}{\rightharpoonup} T$.

For the other direction, assume $T_j \overset{*}{\rightharpoonup} T$ in $\mathrm{I}_k(\mathbb{R}^d)$ with $M := \sup_{j \in \mathbb{N}} (\mathbf{M}(T_j) + \mathbf{M}(\partial T_j)) < \infty$. We need to show that

$$\mathbf{FI}(T_j - T) \to 0. \tag{5.4}$$

The first step is to observe that for all $N \in \mathbb{N}$ sufficiently large there exists a finite collection $\mathbf{P}_N \subset \mathrm{I}_k(\mathbb{R}^d)$ such that for all $\widehat{T} \in \mathrm{I}_k(\mathbb{R}^d)$ with $\operatorname{supp} \widehat{T} \subset K$ and $\mathbf{M}(\widehat{T}) + \mathbf{M}(\partial \widehat{T}) \le M$ it holds that

$$\mathbf{FI}(\widehat{T}, P) < 2^{-N} \text{ for some } P \in \mathbf{P}_N, \tag{5.5}$$

where the lower bound for N depend only on the dimensions. Indeed, for $\mathbf{P}_N$ we take the collection of all polyhedral k-currents P that can possibly satisfy the conclusion of the Deformation Theorem 5.3 for $\varrho := 2^{-N}/(C_{d,k}M)$ with $C_{d,k}$ the constant from the deformation theorem, for a $\widehat{T}$ as above (which implies a mass bound on P and hence a bound on the polyhedral coefficients). The set of these P's is clearly finite.

Returning to our sequence $(T_j)_j$, for every $N \in \mathbb{N}$ sufficiently large we find via (5.5) a $P \in \mathbf{P}_N$ such that $\mathbf{Fl}(T_j, P) < 2^{-N}$ for infinitely many j's. Applying this argument repeatedly and selecting a subsequence at every step (such that the constraint holds for all elements of that subsequence), we may find a diagonal subsequence, still denoted by $(T_j)_j$, such that $\mathbf{Fl}(T_l, P_j) < 2^{-(j+1)}$ for all $l \geq j$ and a $P_j \in \bigcup_N \mathbf{P}_N \subset \mathrm{I}_k(\mathbb{R}^d)$. Then, via the triangle inequality,

$$\mathbf{Fl}(T_j, T_{j+1}) < 2^{-j}.$$

Hence, there exist $R_j \in \mathrm{I}_{1+k}(\mathbb{R}^d)$, $S_j \in \mathrm{I}_k(\mathbb{R}^d)$ with

$$T_{j+1} - T_j = \partial R_j + S_j, \qquad \mathbf{M}(R_j) + \mathbf{M}(S_j) < 2^{-j}.$$

For

$$R_j^m := \sum_{l=0}^{m-1} R_{j+l}, \qquad S_j^m := \sum_{l=0}^{m-1} S_{j+l}$$

it holds that

$$T_{j+m} - T_j = \partial R_j^m + S_j^m$$

and

$$\mathbf{M}(R_j^m) + \mathbf{M}(S_j^m) = \sum_{l=0}^{m-1} \big(\mathbf{M}(R_{j+l}) + \mathbf{M}(S_{j+l})\big) \leq 2^{-j+1}.$$

We now pass to the limit $m \to \infty$. Via the Federer–Fleming Compactness Theorem 5.6 this yields $\overline{R}_j \in \mathrm{I}_{1+k}(\mathbb{R}^d)$ and $\overline{S}_j \in \mathrm{I}_k(\mathbb{R}^d)$ with

$$\partial \overline{R}_j + \overline{S}_j = \Big(\operatorname*{w^*\text{-}lim}_{m\to\infty} T_{j+m}\Big) - T_j = T - T_j$$

and

$$\mathbf{M}(\overline{R}_j) + \mathbf{M}(\overline{S}_j) = \sum_{l=0}^{m-1} \big(\mathbf{M}(R_{j+l}) + \mathbf{M}(S_{j+l})\big) \leq 2^{-j+1}.$$

Our $\overline{R}_j, \overline{S}_j$ are admissible in the definition of the flat norm for $T_j - T$. So,

$$\mathbf{Fl}(T_j - T) \leq \mathbf{M}(\overline{R}_j) + \mathbf{M}(\overline{S}_j) \to 0 \qquad \text{as } j \to \infty.$$

In this way we can find for every subsequence of the original sequence $(T_j)_j$ (before taking the repeated subsequences above) a further subsequence that converges in the flat sense to T. Hence, also $\mathbf{FI}(T_j - T) \to 0$ for the original sequence, proving our claim (5.4).

The proof in the case of boundaryless currents is the same, with only minor adjustments. □

The deformation theorem can also be written in the following approximative form, which allows to reduce many questions for general (integral) currents to the corresponding question for (integral) polyhedral chains.

Theorem 5.13 (Polyhedral approximation). *Let $T \in \mathrm{I}_k(\mathbb{R}^d)$. Then, there exists a sequence $(P_j)_j$ of polyhedral k-currents, i.e.,*

$$P_j = \sum_{F \in \mathcal{F}_k(\varrho)} p_F^{(j)} [\![F]\!] \in \mathrm{I}_k(\mathbb{R}^d),$$

where $p_F^{(j)} \in \mathbb{Z}$, such that $P_j \overset{*}{\rightharpoonup} T$.

Proof. From the Deformation Theorem 5.3, applied with $\varrho = 1/j$, we obtain a sequence P_j with $\mathbf{FI}(P_j - T) \to 0$ as $j \to \infty$. By Proposition 5.12, then also $P_j \overset{*}{\rightharpoonup} T$. □

Remark 5.14. One can prove a stronger version of the above theorem in which also $\mathbf{M}(P_j) \to \mathbf{M}(P)$, see [19, Theorem 4.2.24].

We can now also prove the Boundary Rectifiability Theorem 5.1.

Proof of Theorem 5.1. The Polyhedral Approximation Theorem 5.13 entails that there is a sequence $(P_j)_j$ of polyhedral k-currents with $P_j \overset{*}{\rightharpoonup} P$. Then also $\partial P_j \overset{*}{\rightharpoonup} \partial T$ (this just means we test with $\mathrm{d}\omega$ instead of ω in the convergence). Since all the ∂P_j are clearly integral, so is ∂T by the Federer–Fleming Theorem 5.6. □

Finally, for $T \in \mathrm{I}_k(\mathbb{R}^d)$ with $\partial T = 0$, we also define the **homogeneous integral flat norm** of T as follows:

$$\mathbf{FI}^\circ(T) := \inf\bigl\{ \mathbf{M}(R) \;:\; R \in \mathrm{I}_{1+k}(\mathbb{R}^d),\ \partial R = T \bigr\}.$$

For general it is not known if $\mathbf{FI}^\circ$ and $\mathbf{F}^\circ$ are comparable, besides the dimensions in the following result:

Proposition 5.15. *Let $T \in \mathrm{I}_k(\mathbb{R}^d)$ with $\partial T = 0$. If $k \in \{0, d-2, d-1, d\}$, then*

$$\mathbf{F}^\circ(T) = \mathbf{FI}^\circ(T).$$

Proof. The case $k = d$ is trivial, since the only boundaryless normal d-current in $\mathbb{R}^d$ is the zero current. In the case $k = d - 1$, one can show that there exists an integral current S such that $\partial S = T$ by the isoperimetric inequality, see Theorem 5.5. Such a current S is also unique (in the class of normal currents with boundary T): Suppose $R_1, R_2 \in \mathrm{N}_d(\mathbb{R}^d)$ are such that $\partial R_1 = \partial R_2 = T$, Then, $R := R_1 - R_2$ must then be a boundaryless normal current. By the Constancy Theorem 4.23, R is a fixed multiple of $\mathcal{L}^d$, which must vanish since both R_1, R_2 have finite mass.

For a proof of Proposition 5.15 for the case $k = d - 2$ we refer the reader to, e.g., [20] or [15, Remark 5], which also contains the proof of the case $k = 0$. $\square$

Part III

Space-Time Theory

Chapter 6

Space-Time Integral Currents

The basic idea of the space-time approach to transport of geometric structures is to consider evolutions of integral k-currents to be identified with the space-time $(1+k)$-current "traced out" by the moving k-current. This turns out to yield a satisfying mathematical theory, as we will see in the present and the following chapters. In particular, we discuss how to slice space-time currents in time and specialize various results for currents to that class. Finally, we discuss how one can think of an integral space-time $(1+k)$-current as "connecting" the two k-currents at its "ends" and how this gives a way to metrize the weak*-convergence of integral currents.

6.1 Time-slicing

We will work in the (Galilean) **space-time** $\mathbb{R}^{1+d} \cong \mathbb{R} \times \mathbb{R}^d$, where the first component takes the role of "time" and the remaining components take the role of "space". The unit vectors in $\mathbb{R}^{1+d}$ are denoted by $\mathrm{e}_0, \mathrm{e}_1, \dots, \mathrm{e}_d$ with e_0 the "time" unit vector (pointing in the positive direction).

It will be convenient to write the **time projection** as

$$\mathbf{t} \colon \mathbb{R}^{1+d} \to \mathbb{R}, \qquad \mathbf{t}(t, x) := t,$$

and the **space projection**

$$\mathbf{p} \colon \mathbb{R}^{1+d} \to \mathbb{R}^d, \qquad \mathbf{p}(t, x) := x.$$

By convention, we also denote the linear extensions of these projections to multi-vectors by the same symbols. to simply lengthy expressions, we usually write just $\mathbf{t}v$ and $\mathbf{p}v$ for the application of $\mathbf{t}, \mathbf{p}$ to a vector $v \in \mathbb{R}^{1+d}$.

Given a (normal or integral) **space-time current**

$$S \in \mathrm{N}_{1+k}(\mathbb{R}^{1+d}),$$

it is a fundamental operation to determine the "slice" of S at a time t, that is, the trace of S with the plane

$$\{\mathbf{t} = t\} := \bigl\{ (t,x) \in \mathbb{R}^{1+d} \;:\; x \in \mathbb{R}^d \bigr\}.$$

Of course we need to take into account orientations and also deal with singularities.

Proposition 6.1. *Let $S \in \mathrm{N}_{1+k}(\mathbb{R}^{1+d})$. For $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, the* ***time-slice****, defined by*

$$\bigl\langle S|_t, \omega \bigr\rangle := \lim_{r \downarrow 0} \frac{1}{2r} \bigl\langle S \llcorner \mathbf{t}^*(\mathbb{1}_{[t-r,t+r]}\, \mathrm{d}t), \omega \bigr\rangle, \qquad \omega \in \mathcal{D}^k(\mathbb{R}^{1+d}),$$

exists as a normal current, that is,

$$S|_t \in \mathrm{N}_k(\mathbb{R}^{1+d}),$$

and the ***boundary formula***

$$(\partial S)|_t = -\partial (S|_t) \tag{6.1}$$

holds. Moreover, for every bounded Borel function $g\colon \mathbb{R} \to \mathbb{R}$,

$$\int S|_t\, g(t)\, \mathrm{d}t = S \llcorner (g \circ \mathbf{t}\, \mathrm{d}t). \tag{6.2}$$

Technically, g merely Borel is too rough to make the right-hand side of (6.2) well-defined, but since S is representable by integration as a normal current (see Proposition 4.12), we can also make sense of this for bounded Borel functions g as

$$\bigl\langle S \llcorner (g \circ \mathbf{t}\, \mathrm{d}t), \omega \bigr\rangle := \int \bigl\langle \vec{S}, (g \circ \mathbf{t}\, \mathrm{d}t) \wedge \omega \bigr\rangle \, \mathrm{d}\|S\|$$

for $\omega \in \mathcal{D}^k(\mathbb{R}^{1+d})$.

Proof. Let $\omega \in \mathcal{D}^k(\mathbb{R}^{1+d})$ and $g\colon \mathbb{R} \to [0,\infty)$ be a bounded Borel function. Then, we may compute as follows:

$$\begin{aligned} \bigl\langle S \llcorner \mathbf{t}^*(g\, \mathrm{d}t), \omega \bigr\rangle &= \bigl\langle S, \mathbf{t}^*(g\, \mathrm{d}t) \wedge \omega \bigr\rangle \\ &= (-1)^k \bigl\langle S, \omega \wedge \mathbf{t}^*(g\, \mathrm{d}t) \bigr\rangle \\ &= (-1)^k \bigl\langle \mathbf{t}_*(S \llcorner \omega), g\, \mathrm{d}t \bigr\rangle. \end{aligned} \tag{6.3}$$

The same remark as above regarding the interpretation of these duality products between measures and rough functions applies.

In particular, for $g := \mathbb{1}_{[t-r,t+r]}$ in (6.3) with $t \in \mathbb{R}$, $r > 0$ we obtain

$$\bigl\langle S \llcorner \mathbf{t}^*(\mathbb{1}_{[t-r,t+r]}\, \mathrm{d}t), \omega \bigr\rangle = (-1)^k \bigl\langle \mathbf{t}_*(S \llcorner \omega), \mathbb{1}_{[t-r,t+r]}\, \mathrm{d}t \bigr\rangle. \tag{6.4}$$

Since $\mathbf{t}_*(S \llcorner \omega) \in \mathrm{N}_1(\mathbb{R})$ (see Proposition 4.30), we can apply the argument of Example 4.17 to see that there is an $\mathcal{L}^1$-integrable $\varrho_\omega \colon \mathbb{R} \to \mathbb{R}$ with

$$\mathbf{t}_*(S \llcorner \omega) = \varrho_\omega \mathrm{e}_0 \, \mathcal{L}^1. \tag{6.5}$$

So, by Lebesgue's Differentiation Theorem 1.16,

$$\frac{1}{2r}\big\langle \mathbf{t}_*(S \llcorner \omega), \mathbb{1}_{[t-r,t+r]} \,\mathrm{d}t \big\rangle = \frac{1}{2r}\int_{t-r}^{t+r} \varrho_\omega(s)\,\mathrm{d}s \to \varrho_\omega(t) \qquad \text{as } r \to 0, \tag{6.6}$$

for $\mathcal{L}^1$-almost every $t \in \mathbb{R}$.

Now take a countable set $\{\omega_j\}_j \subset \mathcal{D}^k(\mathbb{R}^{1+d})$ that is $\|\bullet\|_\infty$-dense in $\mathcal{D}^k(\mathbb{R}^{1+d})$. By the above reasoning, in particular combining (6.4) with (6.6), there is a set $N \subset \mathbb{R}$ with $\mathcal{L}^1(N) = 0$ such that

$$\big\langle S|_t, \omega_j \big\rangle := (-1)^k \varrho_{\omega_j}(t) = \lim_{r \downarrow 0} \frac{1}{2r}\big\langle S \llcorner \mathbf{t}^*(\mathbb{1}_{[t-r,t+r]}\,\mathrm{d}t), \omega_j \big\rangle \tag{6.7}$$

for $t \in \mathbb{R} \setminus N$ and $j \in \mathbb{N}$. Without loss of generality, we may furthermore require that for $t \in \mathbb{R} \setminus N$ it holds that

$$\Theta^{1*}(\mathbf{t}_\#\|S\|, t) := \limsup_{r \downarrow 0} \frac{\mathbf{t}_\#\|S\|([t-r,t+r])}{2r} < \infty,$$

where $\Theta^{1*}(\mathbf{t}_\#\|S\|, t)$ is the upper 1-density of the measure $\mathbf{t}_\#\|S\|$, cf. the definition in (1.4). Indeed, this follows by the Besicovitch Derivation Theorem 1.13 (with $\nu := \mathbf{t}_\#\|S\|$ and $\mu := \mathcal{L}^1$).

Fix $t \in \mathbb{R} \setminus N$. For any $\omega \in \mathcal{D}^k(\mathbb{R}^{1+d})$ and $r > 0$ we have the estimate

$$\frac{1}{2r}\big|\big\langle S \llcorner \mathbf{t}^*(\mathbb{1}_{[t-r,t+r]}\,\mathrm{d}t), \omega \big\rangle\big| \le \|\omega\|_\infty \frac{\mathbf{t}_\#\|S\|([t-r,t+r])}{2r}. \tag{6.8}$$

For any $\varepsilon > 0$, there is a $j \in \mathbb{N}$ with $\|\omega - \omega_j\|_\infty \le \varepsilon$ and hence, applying (6.8) to $\omega - \omega_j$ and combining this with (6.7),

$$\begin{aligned} &\left|\limsup_{r \downarrow 0} \frac{1}{2r}\big\langle S \llcorner \mathbf{t}^*(\mathbb{1}_{[t-r,t+r]}\,\mathrm{d}t), \omega \big\rangle - \liminf_{r \downarrow 0} \frac{1}{2r}\big\langle S \llcorner \mathbf{t}^*(\mathbb{1}_{[t-r,t+r]}\,\mathrm{d}t), \omega \big\rangle\right| \\ &\qquad \le 2\varepsilon \cdot \Theta^{1*}(\mathbf{t}_\#\|S\|, t). \end{aligned}$$

As $\varepsilon > 0$ was arbitrary, we conclude that

$$\big\langle S|_t, \omega \big\rangle = \lim_{r \downarrow 0} \frac{1}{2r}\big\langle S \llcorner \mathbf{t}^*(\mathbb{1}_{[t-r,t+r]}\,\mathrm{d}t), \omega \big\rangle \tag{6.9}$$

exists for every $\omega \in \mathcal{D}^k(\mathbb{R}^{1+d})$. Moreover, from (6.8) we conclude that

$$\big|\big\langle S|_t, \omega \big\rangle\big| \le \|\omega\|_\infty \cdot \Theta^{1*}(\mathbf{t}_\#\|S\|, t),$$

whereby $S|_t$ is a k-current with finite mass, that is, $\mathbf{M}(S|_t) \le \Theta^{1*}(\mathbf{t}_\#\|S\|, t)$.

For $\eta \in \mathcal{D}^{k-1}(\mathbb{R}^{1+d})$, write (6.3) for ∂S to observe

$$\big\langle \partial S \llcorner \mathbf{t}^*(g\ \mathrm{d}t), \eta \big\rangle = (-1)^{k-1} \big\langle \mathbf{t}_*(\partial S \llcorner \eta), g\ \mathrm{d}t \big\rangle.$$

The conclusion of Lemma 4.11 reads as

$$\partial(S \llcorner \eta) = (-1)^{k-1}\big[\partial S \llcorner \eta - S \llcorner \mathrm{d}\eta\big]$$

Taking the pushforward under $\mathbf{t}$ and noticing that $\mathbf{t}_*(S \llcorner \eta) = 0$ (as $S \llcorner \eta \in \mathcal{D}_2(\mathbb{R}^{1+d})$ and there are no non-zero 2-currents in $\mathbb{R}$), we have

$$\mathbf{t}_*(\partial S \llcorner \eta) = \mathbf{t}_*(S \llcorner \mathrm{d}\eta).$$

Combining, and also applying the original (6.3), we arrive at

$$\begin{aligned} \big\langle \partial S \llcorner \mathbf{t}^*(g\ \mathrm{d}t), \eta \big\rangle &= (-1)^{k-1} \big\langle \mathbf{t}_*(S \llcorner \mathrm{d}\eta), g\ \mathrm{d}t \big\rangle \\ &= -\big\langle S \llcorner \mathbf{t}^*(g\ \mathrm{d}t), \mathrm{d}\eta \big\rangle \\ &= -\big\langle \partial(S \llcorner \mathbf{t}^*(g\ \mathrm{d}t)), \eta \big\rangle \end{aligned}$$

Specializing to $g := \mathbb{1}_{[t-r,t+r]}$ for $t \in \mathbb{R}$ and $r > 0$,

$$\frac{1}{2r} \partial S \llcorner \mathbf{t}^*(\mathbb{1}_{[t-r,t+r]}\ \mathrm{d}t) = -\frac{1}{2r} \partial(S \llcorner \mathbf{t}^*(\mathbb{1}_{[t-r,t+r]}\ \mathrm{d}t)). \tag{6.10}$$

For any $t \in \mathbb{R}$ such that $S|_t$ exists, we can then repeat the above arguments to let $r \downarrow 0$ in (6.10). In this way we deduce that $(\partial S)|_t$ exists and the boundary formula (6.1) holds. Moreover, $\mathbf{M}(\partial(S|_t)) \le \Theta^{1*}(\mathbf{t}_\#\|\partial S\|, t)$, which we may assume to be finite for all but a null set of t's, as before. We have thus shown that

$$S|_t \in \mathrm{N}_k(\{t\} \times \mathbb{R}^d) \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in \mathbb{R}.$$

It remains to prove (6.2). For this, fix $\omega \in \mathcal{D}^k(\mathbb{R}^{1+d})$ and $\psi \in \mathrm{C}_c^\infty(\mathbb{R})$. Using the representation $\mathbf{t}_*(S \llcorner \omega) = \varrho_\omega \mathrm{e}_0\, \mathcal{L}^1$ from above together with (6.9), (6.5), we compute

$$\begin{aligned} \left\langle \int S|_t\, \psi(t)\ \mathrm{d}t, \omega \right\rangle &= \int \big\langle S|_t, \omega \big\rangle\, \psi(t)\ \mathrm{d}t \\ &= (-1)^k \int \varrho_\omega(t)\, \psi(t)\ \mathrm{d}t \\ &= (-1)^k \big\langle \mathbf{t}_*(S \llcorner \omega), \psi\, \mathrm{d}t \big\rangle. \end{aligned}$$

Using $g := \psi \circ \mathbf{t}$ in (6.3), we find

$$(-1)^k \big\langle \mathbf{t}_*(S \llcorner \omega), \psi\, \mathrm{d}t \big\rangle = \big\langle S \llcorner (\psi \circ \mathbf{t})\, \mathrm{d}t, \omega \big\rangle.$$

Since $\omega \in \mathcal{D}^k(\mathbb{R}^{1+d})$ was arbitrary, (6.2) holds for $g = \psi$. The case for general bounded Borel functions g then follows by approximation and the dominated convergence theorem. □

Proposition 6.2. *Let $S \in \mathrm{N}_{1+k}(\mathbb{R}^{1+d})$ and let $g \colon \mathbb{R} \colon \mathbb{R}$ be a bounded Borel function. Then, the following* ***coarea formula*** *holds:*

$$\int g(t)\,\mathbf{M}(S|_t)\;\mathrm{d}t = \int (g \circ \mathbf{t})\,\|\vec{S} \llcorner \mathrm{d}t\|\;\mathrm{d}\|S\|$$

To prove this, we first need an approximation lemma:

Lemma 6.3. *Let $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ and let $\xi \colon \mathbb{R}^d \to \bigwedge_k \mathbb{R}^d$ with*

$$\|\xi(x)\| \le M \qquad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d$$

for an $M > 0$. Then there is a Borel set $G \subset \mathbb{R}^d$ with $\mu(\mathbb{R}^d \setminus G) < \varepsilon$ and a smooth k-covector field $\omega \colon \mathbb{R}^d \to \bigwedge^k \mathbb{R}^d$ with $\|\omega\|_\infty \le 1$ and

$$\big\langle \xi(x), \omega(x) \big\rangle \ge \|\xi(x)\| - \varepsilon \qquad \text{for } x \in G.$$

Proof. Take a countable dense subset $\{\alpha_n\}_n \subset \{\, \alpha \in \bigwedge^k \mathbb{R}^d \;:\; \|\alpha\| \le 1 \,\}$ and let for $x \in \mathbb{R}^d$,

$$\widetilde{\omega}(x) := \alpha_{j(x)}, \qquad \text{where} \qquad j(x) := \min\big\{\, n \in \mathbb{N} \;:\; \big\langle \xi(x), \alpha_n \big\rangle \ge \|\xi(x)\| - \varepsilon \,\big\}.$$

This is a measurable map satisfying $\|\widetilde{\omega}\|_\infty \le 1$ and

$$\big\langle \xi(x), \widetilde{\omega}(x) \big\rangle > \|\xi(x)\| - \varepsilon$$

for all $x \in \mathbb{R}^d$. Now uniformly approximate $\widetilde{\omega}$ by a sequence of smooth k-covector fields. The existence of the set G and ω with the desired properties then follows from Egorov's theorem applied to that approximating sequence. □

Proof of Proposition 6.2. Let $t \in \mathbb{R}$ be such that $S|_t$ exists. Since $\mathcal{D}^k(\mathbb{R}^{1+d})$ is separable, there exists a countable dense sequence $\{\omega_n\}_n \subset \{\, \omega \in \mathcal{D}^k(\mathbb{R}^{1+d}) \;:\; \|\omega\|_\infty \le 1 \,\}$ such that

$$\mathbf{M}(S|_t) = \sup_{n \in \mathbb{N}} \big\langle S|_t, \omega_n \big\rangle. \tag{6.11}$$

Thus, $t \mapsto \mathbf{M}(S|_t)$ is measurable. Estimating in a similar way to (6.8),

$$\frac{1}{2r}\big|\big\langle S \llcorner \mathbf{t}^*(\mathbb{1}_{[t-r,t+r]}\;\mathrm{d}t), \omega \big\rangle\big| \le \frac{1}{2r}\int_{[t-r,t+r]\times\mathbb{R}^d} |\omega|\;\mathrm{d}\|S \llcorner \mathrm{d}t\|.$$

Letting $r \downarrow 0$, we thus obtain from (6.11) that

$$\mathbf{M}(S|_t) \le \Theta^{1*}(\mathbf{t}_\#\|S \llcorner \mathrm{d}t\|, t).$$

By the Besicovitch Differentiation Theorem, this gives

$$\mathbf{M}(S|_t)\,\mathrm{d}t \le \mathbf{t}_\# \|S \llcorner \mathrm{d}t\|,$$

as measures on $\mathbb{R}$. Hence,

$$\int g(t)\,\mathbf{M}(S|_t)\;\mathrm{d}t \le \int g(t)\;\mathrm{d}\mathbf{t}_\# \|S \llcorner \mathrm{d}t\| = \int g \circ \mathbf{t}\;\mathrm{d}\|S \llcorner \mathrm{d}t\|. \tag{6.12}$$

For the opposite inequality, fix $\varepsilon > 0$. From Lemma 6.3 we obtain a Borel set $G \subset \mathbb{R}^{1+d}$ with $\|S \llcorner \mathrm{d}t\|(\mathbb{R}^{1+d} \setminus G) \le \varepsilon$ and a smooth maps $\omega\colon \mathbb{R}^{1+d} \to \bigwedge^k \mathbb{R}^{1+d}$ with $\|\omega\|_\infty \le 1$ such that

$$1 - \varepsilon \le \bigl\langle \overrightarrow{S \llcorner \mathrm{d}t}(t,x), \omega(t,x) \bigr\rangle \qquad \text{for } (t,x) \in G.$$

Since $\operatorname{supp} S$ is compact, we may without loss of generality assume that ω has compact support, so $\omega \in \mathcal{D}^k(\mathbb{R}^{1+d})$. So, also using (6.2),

$$\begin{aligned}
\int g \circ \mathbf{t}\;\mathrm{d}\|S \llcorner \mathrm{d}t\| &\le \bigl\langle S \llcorner (g \circ \mathbf{t}\;\mathrm{d}t), \omega \bigr\rangle + \varepsilon \|g\|_\infty (1 + \mathbf{M}(S \llcorner \mathrm{d}t)) \\
&= \int g(t)\,\bigl\langle S|_t, \omega \bigr\rangle\;\mathrm{d}t + \varepsilon \|g\|_\infty (1 + \mathbf{M}(S \llcorner \mathrm{d}t)) \\
&\le \int g(t)\,\mathbf{M}(S|_t)\;\mathrm{d}t + \varepsilon \|g\|_\infty (1 + \mathbf{M}(S \llcorner \mathrm{d}t)).
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we obtain the opposite inequality to (6.12) and hence the claimed identity. $\square$

Proposition 6.4. *Let $S \in \mathrm{N}_{1+k}(\mathbb{R}^{1+d})$ and $t \in \mathbb{R}$ be such that*

$$\mathbf{t}_\#(\|S\| + \|\partial S\|)(\{t\}) = 0,$$

which holds outside of a countable set. Then $S|_t$ exists in the sense of Proposition 6.1, and it satisfies the ***cylinder formula***

$$S|_t = \partial(S \llcorner \{\mathbf{t} < t\}) - (\partial S) \llcorner \{\mathbf{t} < t\}. \tag{6.13}$$

Proof. Step 1. Write for any $t \in \mathbb{R}$,

$$H_t^+ := \{\mathbf{t} > t\}, \qquad H_t^- := \{\mathbf{t} < t\}, \qquad H_t := \{\mathbf{t} = t\},$$

and set

$$S|_t^+ := \partial S \llcorner H_t^+ - \partial(S \llcorner H_t^+), \qquad S|_t^- := \partial(S \llcorner H_t^-) - \partial S \llcorner H_t^-,$$

both of which are well-defined elements of $\mathcal{D}_k(\mathbb{R}^{1+d})$ for every $t \in \mathbb{R}$, since restriction by a Borel set is allowed for currents representable by integration (see Proposition 4.12).

For a t as in the statement of the proposition, it holds that $\|S\|(H_t) = \|\partial S\|(H_t) = 0$ by hypothesis. This is indeed satisfied outside of a countable set, since $\mathbf{t}_\# \|S\|$ and $\mathbf{t}_\# \|\partial S\|$ are finite Borel measures on $\mathbb{R}$. Then, $S \llcorner H_t = 0$, $\partial S \llcorner H_t = 0$, whereby

$$S = S \llcorner H_t^+ + S \llcorner H_t^-, \qquad \partial S = \partial S \llcorner H_t^+ + \partial S \llcorner H_t^-.$$

Substituting these identities into the definition of $S|_t^+$,

$$\begin{aligned} S|_t^+ &= \big(\partial S - \partial S \llcorner H_t^-\big) - \partial\big(S - S \llcorner H_t^-\big) \\ &= \partial(S \llcorner H_t^-) - \partial S \llcorner H_t^- \\ &= S|_t^-. \end{aligned}$$

Step 2. It remains to identify $S|_t^- = S|_t^+$ with the limit defining $S|_t$ in Proposition 6.1. Then, since $H_t^- = \{\mathbf{t} < t\}$, the cylinder formula (6.13) will be established.

For $r > 0$ define $g_r \colon \mathbb{R} \to [0, 1]$ by

$$g_r(s) := \begin{cases} 1 & s \le t - r, \\ \dfrac{t-s}{r} & t - r \le s \le t, \\ 0 & s \ge t, \end{cases}$$

and $h_r \colon \mathbb{R} \to [0, 1]$ by

$$h_r(s) := \begin{cases} 0 & s \le t, \\ \dfrac{s-t}{r} & t \le s \le t + r, \\ 1 & s \ge t + r. \end{cases}$$

Both are continuous, piecewise-linear, and $[0, 1]$-valued, with

$$g_r' = -\frac{1}{r}\,\mathbb{1}_{[t-r,t]}, \qquad (h_r)' = \frac{1}{r}\,\mathbb{1}_{[t,t+r]},$$

and as $r \downarrow 0$,

$$g_r \to \mathbb{1}_{(-\infty,t)}, \qquad h_r \to \mathbb{1}_{(t,\infty)}$$

pointwise almost everywhere (in fact for every $s \neq t$) on $\mathbb{R}$.

Next, we will show the Leibniz rule

$$\partial\big(S \llcorner (g_r \circ \mathbf{t})\big) = \partial S \llcorner (g_r \circ \mathbf{t}) - S \llcorner \mathbf{t}^*\big(g_r'\,\mathrm{d}t\big). \tag{6.14}$$

Indeed, the function g_r is Lipschitz (it has corners at $s = t - r$ and $s = t$), so Lemma 4.11 does not apply to it directly. We obtain the Leibniz formula (6.14) by approximation with a family $(\varrho_\delta)_{\delta>0}$ of standard mollifiers with $\operatorname{supp} \varrho_\delta \subset [-\delta, \delta]$. Set

$$g_r^\delta := \varrho_\delta * g_r \in \mathrm{C}^\infty(\mathbb{R}).$$

Then $g_r^\delta \to g_r$ uniformly on $\mathbb{R}$ as $\delta \downarrow 0$ and $(g_r^\delta)' = \varrho_\delta * g_r' \to g_r'$ in $\mathrm{L}^1(\mathbb{R})$ as $\delta \downarrow 0$, where $g_r' = -\frac{1}{r}\mathbb{1}_{[t-r,t]}$ is the almost everywhere defined derivative of g_r.

Applying Lemma 4.11 to the smooth function $f_\delta := g_r^\delta \circ \mathbf{t}$,

$$\partial\big(S \llcorner (g_r^\delta \circ \mathbf{t})\big) = \partial S \llcorner (g_r^\delta \circ \mathbf{t}) - S \llcorner \mathbf{t}^*\big((g_r^\delta)' \circ \mathbf{t}\,\mathrm{d}t\big).$$

As $\delta \downarrow 0$, we have that $g_r^\delta \circ \mathbf{t} \to g_r \circ \mathbf{t}$ uniformly and hence $S \llcorner (g_r^\delta \circ \mathbf{t}) \to S \llcorner (g_r \circ \mathbf{t})$ weakly* by dominated convergence. Moreover, ∂ is weakly* continuous and thus the left-hand side converges weakly* to $\partial(S \llcorner (g_r \circ \mathbf{t}))$. Similarly, the first term on the right-hand side converges to $\partial S \llcorner (g_r \circ \mathbf{t})$.

The second term on the right-hand side converges to $S \llcorner \mathbf{t}^*(g_r'\,\mathrm{d}t)$ because $(g_r^\delta)' \circ \mathbf{t} \to g_r' \circ \mathbf{t}$ in $\mathrm{L}^1(\mathbf{t}_\# \|S\|)$ by dominated convergence since the only points where the convergence does not hold pointwise are $t - r$ and t, but without loss of generality we may require $\mathbf{t}_\# \|S\|(\{t - r, t\}) = 0$ (for t this is by assumption and for $t - r$ we can arrange it since $\mathbf{t}_\# \|S\|$ is a finite measure).

Step 3. Letting $r \downarrow 0$ (along a sequence of r's satisfying the additional constraint in the preceding paragraph) in (6.14) and using $g_r \circ \mathbf{t} \to \mathbb{1}_{(-\infty,t)}$ pointwise, we obtain via the dominated convergence theorem and the weak* continuity of ∂ that

$$\frac{1}{r} S \llcorner \mathbf{t}^*(\mathbb{1}_{[t-r,t]}\,\mathrm{d}t) \overset{*}{\rightharpoonup} \partial(S \llcorner H_t^-) - \partial S \llcorner H_t^- = S|_t^-,$$

and an analogous reasoning for h_r gives

$$\frac{1}{r} S \llcorner \mathbf{t}^*(\mathbb{1}_{[t,t+r]}\,\mathrm{d}t) \overset{*}{\rightharpoonup} \partial S \llcorner H_t^+ - \partial(S \llcorner H_t^+) = S|_t^+.$$

Since $S \llcorner H_t = 0$, we have $S \llcorner \mathbf{t}^*(\mathbb{1}_{\{t\}}\,\mathrm{d}t) = 0$, so for every $r > 0$,

$$S \llcorner \mathbf{t}^*(\mathbb{1}_{[t-r,t+r]}\,\mathrm{d}t) = S \llcorner \mathbf{t}^*(\mathbb{1}_{[t-r,t]}\,\mathrm{d}t) + S \llcorner \mathbf{t}^*(\mathbb{1}_{[t,t+r]}\,\mathrm{d}t).$$

Dividing by $2r$ and letting $r \downarrow 0$, the two terms on the right converge weakly* to $\frac{1}{2}S|_t^-$ and $\frac{1}{2}S|_t^+$ respectively, so

$$\frac{1}{2r} S \llcorner \mathbf{t}^*(\mathbb{1}_{[t-r,t+r]}\,\mathrm{d}t) \overset{*}{\rightharpoonup} \frac{1}{2}\big(S|_t^- + S|_t^+\big) = S|_t^-,$$

using that $S|_t^+ = S|_t^-$ shown in Step 1. The left-hand side is exactly the difference quotient whose limit as $r \downarrow 0$ defines $S|_t$ in Proposition 6.1, so $S|_t$ exists and equals $S|_t^-$. □

For integral currents, also the slices are integral, as summarized in the following proposition. For this, recall that for a (countably) $\mathcal{H}^{1+k}$-rectifiable set $R \subset \mathbb{R}^{1+d}$ the tangential differential

$$\mathrm{D}^R\mathbf{t}(x)\colon\ \mathrm{T}_{(t,x)}\,R \to \mathbb{R}$$

of $\mathbf{t}$ at $x \in R$, which was first encountered in Proposition 2.23, is the restriction of the differential

$$\mathrm{D}\mathbf{t}(t,x)\colon \mathbb{R}^{1+d} \to \mathbb{R}, \qquad \mathrm{D}\mathbf{t}(t,x)[s,z] = s,$$

to the approximate tangent space $\mathrm{T}_{(t,x)}\,R$ to R at x. Moreover, this differential (a liner map) can be represented by the **tangential gradient** ∇^R, which is the projection of the gradient

$$\nabla\mathbf{t} = \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix} \in \mathbb{R}^{1\times(1+d)}$$

onto $\mathrm{T}_{(t,x)}\,R$. That is, if $P_{(t,x)} \in \mathbb{R}^{(1+d)\times(1+d)}$ is the orthogonal projection onto $\mathrm{T}_{(t,x)}\,R$, then

$$\nabla^R\mathbf{t}(t,x) = \nabla\mathbf{t}\,P_{(t,x)}^{\mathsf{T}} \in \mathbb{R}^{1\times(1+d)}.$$

Indeed, observe that for $v \in \mathrm{T}_{(t,x)}$,

$$\mathrm{D}^R\mathbf{t}[v] = \big(\nabla\mathbf{t}, v\big) = \big(\nabla\mathbf{t}, P_{(t,x)}v\big) = \big(\nabla\mathbf{t}P_{(t,x)}^{\mathsf{T}}, v\big),$$

from which the above formula follows. Unlike $\nabla\mathbf{t}$, which is the same at every point and equal to $\mathrm{e}_0^{\mathsf{T}}$, the tangential gradient $\nabla^R\mathbf{t}$ depends on (t,x). Note also that $\nabla^R\mathbf{t} = 0$ if and only if $\mathrm{T}_{(t,x)}\,R$ is contained in $\{0\}\times\mathbb{R}^d$, i.e., $\mathrm{T}_{(t,x)}\,R$ is "space-like".

Proposition 6.5. *Let $S = \vec{S}\,m\,\mathcal{H}^{k+1}\llcorner R \in \mathrm{I}_{1+k}(\mathbb{R}^{1+d})$ be an integral space-time current. Then, for $\mathcal{L}^1$-almost every $t \in \mathbb{R}$, the following assertions hold:*

(i) The set

$$R|_t := R \cap \mathbf{t}^{-1}(\{t\})$$

is countably $\mathcal{H}^k$-rectifiable.

(ii) For $\mathcal{H}^k$-almost every $z \in R|_t$ with $\nabla^R\mathbf{t}(z) \neq 0$, the approximate tangent spaces T_zR and $\mathrm{T}_zR|_t$ exist, $\mathrm{T}_zR|_t = \ker \mathrm{D}^R\mathbf{t}(z)$ in T_zR, and

$$\mathrm{T}_zR = \mathrm{T}_zR|_t \oplus \operatorname{span}\{\xi(z)\}, \qquad \xi(z) := \frac{\nabla^R\mathbf{t}(z)}{|\nabla^R\mathbf{t}(z)|} \perp \mathrm{T}_zR|_t.$$

Moreover, $\xi(z)$ is a simple unit (1-)vector.

(iii) Define

$$m_+(z) := \begin{cases} m(z), & \text{if } \nabla^R \mathbf{t}(z) \neq 0, \\ 0, & \text{if } \nabla^R \mathbf{t}(z) = 0, \end{cases}$$

and

$$\xi^*(z) := \frac{\mathrm{D}^R \mathbf{t}(z)}{|\mathrm{D}^R \mathbf{t}(z)|}$$

wherever $\mathrm{D}^R \mathbf{t}(z) \neq 0$*, and choose* ξ^* *arbitrarily on the remaining set. With*

$$\vec{S}|_t(z) := \vec{S}(z) \llcorner \xi^*(z) \in \textstyle\bigwedge_k \mathrm{T}_z R|_t,$$

where $\llcorner$ *denotes contraction of a multivector by a covector, the time-slice satisfies*

$$S|_t = \vec{S}|_t \, m_+ \, \mathcal{H}^k \llcorner R|_t \in \mathrm{I}_k(\mathbb{R}^{1+d}).$$

Proof. Ad (i). The claim is precisely the rectifiability conclusion in the rectifiable version of the coarea formula, Theorem 2.25, applied to $f := \mathbf{t}$, that is, the slices $R|_t$ are countably $\mathcal{H}^k$-rectifiable after discarding a null set of t's.

Ad (ii). Take a $t \in \mathbb{R}$ such that the conclusion (i) holds. Then, by the rectifiability the approximate tangent space $\mathrm{T}_z R|_t$ exists for $\mathcal{H}^k$-almost every $z \in R|_t$ by Proposition 2.20.

Fix a point $z \in R|_t$ at which $\mathrm{T}_z R$ exists, $\mathbf{t}$ is approximately differentiable along R, and $\nabla^R \mathbf{t}(z) \neq 0$. From the rectifiable coarea formula, Theorem 2.25,

$$\int_{\mathbb{R}} \mathcal{H}^k\big(R|_t \cap \{\nabla^R \mathbf{t} = 0\}\big) \, \mathrm{d}t = \int_{R \cap \{\nabla^R \mathbf{t} = 0\}} |\nabla^R \mathbf{t}| \, \mathrm{d}\mathcal{H}^{k+1} = 0.$$

This implies the $\mathcal{H}^k$-negligibility of the critical part $\{\nabla^R \mathbf{t} = 0\}$ on almost every slice. Thus, points z as above exhaust almost every slice up to a $\mathcal{H}^k$-null set and so it suffices to show the claim for them.

Choose a countable rectifiable decomposition

$$R \subset N \cup \bigcup_{j=1}^{\infty} h_j(\mathbb{R}^{k+1}), \qquad \mathcal{H}^{k+1}(N) = 0,$$

where each $h_j \colon \mathbb{R}^{k+1} \to \mathbb{R}^d$ is injective and Lipschitz (see Remark 2.12). Refining the pieces if necessary, we may assume that, for $\mathcal{H}^{k+1}$-almost every point $z \in h_j(\mathbb{R}^{k+1})$ under consideration, if $z = h_j(y)$, then h_j is differentiable at y (cf. Proposition 2.23), $\mathbf{J}_{k+1} h_j(y) > 0$, and

$$\mathrm{T}_z \, R = \nabla h_j(y)[\mathbb{R}^{k+1}]. \tag{6.15}$$

Set

$$g_j := \mathbf{t} \circ h_j \colon \mathbb{R}^{k+1} \to \mathbb{R}.$$

At the corresponding good points y, the map g_j is differentiable and

$$\nabla g_j(y) = \mathrm{D}^R \mathbf{t}(z) \circ \nabla h_j(y), \qquad z = h_j(y). \tag{6.16}$$

Indeed, this is exactly the chain rule for the tangential differential, using (6.15). Since $\nabla h_j(y)$ maps $\mathbb{R}^{k+1}$ onto $\mathrm{T}_z\, R$ and $\mathrm{D}^R\mathbf{t}(z) \neq 0$ on $\mathrm{T}_z\, R$, we have

$$\nabla g_j(y) \neq 0.$$

We will use the following elementary consequence of the coarea formula: If $g \colon \mathbb{R}^{k+1} \to \mathbb{R}$ is Lipschitz, then for $\mathcal{L}^1$-almost every level s and for $\mathcal{H}^k$-almost every point $y \in K \cap g^{-1}(s)$ at which g is differentiable and $\nabla g(y) \neq 0$,

$$\mathrm{T}_y\big(K \cap g^{-1}(s)\big) = \ker \nabla g(y). \tag{6.17}$$

Applying (6.17) to each g_j and intersecting the resulting full-measure sets of levels over j, we may also assume that, for the fixed t above and for $\mathcal{H}^k$-almost every $y \in \mathbb{R}^{k+1} \cap g_j^{-1}(t)$ with $\nabla g_j(y) \neq 0$,

$$\mathrm{T}_y\big(\mathbb{R}^{k+1} \cap g_j^{-1}(t)\big) = \ker \nabla g_j(y).$$

Now let $z = h_j(y)$ be such a point. Since h_j is differentiable at y and has rank $k+1$, Proposition 2.20, applied to the rectifiable set $\mathbb{R}^{k+1} \cap g_j^{-1}(t)$ and then pushed forward by h_j, gives

$$\mathrm{T}_z(R|_t) = \nabla h_j(y)\big[\mathrm{T}_y(\mathbb{R}^{k+1} \cap g_j^{-1}(t))\big] = \nabla h_j(y)[\ker \nabla g_j(y)].$$

Using (6.16), and writing $A := \nabla h_j(y)$ and $L := \mathrm{D}^R\mathbf{t}(z)$, we obtain

$$A[\ker \nabla g_j(y)] = A[\ker(L \circ A)] = \big\{\, v \in A[\mathbb{R}^{k+1}] \;:\; L[v] = 0 \,\big\}.$$

By (6.15), $A[\mathbb{R}^{k+1}] = \mathrm{T}_z\, R$. Therefore,

$$\mathrm{T}_z(R|_t) = \ker \mathrm{D}^R\mathbf{t}(z) \qquad \text{inside } \mathrm{T}_z\, R.$$

Finally, by definition of the tangential gradient,

$$\mathrm{D}^R\mathbf{t}(z)[v] = \nabla^R\mathbf{t}(z)v \qquad \text{for every } v \in \mathrm{T}_z\, R.$$

Thus

$$\mathrm{T}_z(R|_t) = \big\{\, v \in \mathrm{T}_z\, R \;:\; \nabla^R\mathbf{t}(z)v = 0 \,\big\}.$$

Since $\nabla^R \mathbf{t}(z) \neq 0$, the vector

$$\xi(z) := \frac{\nabla^R \mathbf{t}(z)}{|\nabla^R \mathbf{t}(z)|}$$

is a unit vector in $\mathrm{T}_z\, R$, and the previous identity yields

$$\mathrm{T}_z\, R = \operatorname{span}\big\{\mathrm{T}_z(R|_t), \xi(z)\big\}, \qquad \xi(z) \perp \mathrm{T}_z(R|_t),$$

so assertion (ii) follows.

Ad (iii). Let $\omega \in \mathcal{D}^k(\mathbb{R}^d)$. By the definition of the slice, see Proposition 6.1,

$$\begin{aligned}
\big\langle S|_t, \omega \big\rangle &= \lim_{r \downarrow 0} \frac{1}{2r} \big\langle S \llcorner \mathbf{t}^*(\mathbb{1}_{[t-r,t+r]}\, \mathrm{d}t), \omega \big\rangle \\
&= \lim_{r \downarrow 0} \frac{1}{2r} \int_{[t-r,t+r]\times\mathbb{R}^d} \big\langle \vec{S}, \mathrm{d}t \wedge \omega \big\rangle \, \mathrm{d}\|S\| \\
&= \lim_{r \downarrow 0} \frac{1}{2r} \int_{R \cap \{t-r<\mathbf{t}<t+r\}} m \,\big\langle \vec{S}, \mathrm{D}\mathbf{t} \wedge \omega \big\rangle \, \mathrm{d}\mathcal{H}^{k+1}.
\end{aligned}$$

Only the tangential part $\mathrm{D}^R\mathbf{t}$ contributes to the pairing with $\vec{S}(z)$. At points where $\mathrm{D}^R\mathbf{t} \neq 0$, set

$$\xi^* := \frac{\mathrm{D}^R\mathbf{t}}{|\mathrm{D}^R\mathbf{t}|}, \qquad \vec{S}|_{\mathbf{t}(z)} := \vec{S} \llcorner \xi^*.$$

Then,

$$\big\langle \vec{S}(z), \mathrm{D}\mathbf{t}(z) \wedge \omega(z) \big\rangle = |\nabla^R \mathbf{t}(z)| \, \big\langle \vec{S}|_{\mathbf{t}(z)}(z), \omega(z) \big\rangle.$$

Using $m_+ = 0$ on $\{\nabla^R \mathbf{t} = 0\}$ and applying the rectifiable coarea formula, Theorem 2.25,

$$\frac{1}{2r} \int_{R \cap \{t-r<\mathbf{t}<t+r\}} m \,\big\langle \vec{S}, \mathrm{D}\mathbf{t} \wedge \omega \big\rangle \, \mathrm{d}\mathcal{H}^{k+1} = \frac{1}{2r} \int_{t-r}^{t+r} \int_{R|_s} m_+ \,\big\langle \vec{S}|_s, \omega \big\rangle \, \mathrm{d}\mathcal{H}^k \, \mathrm{d}s.$$

The inner integral is locally integrable in s, so the Lebesgue Differentiation Theorem 1.16 gives for almost every t that

$$\big\langle S|_t, \omega \big\rangle = \int_{R|_t} m_+ \,\big\langle \vec{S}|_t, \omega \big\rangle \, \mathrm{d}\mathcal{H}^k.$$

Therefore,

$$S|_t = \vec{S}|_t \, m_+ \, \mathcal{H}^k \llcorner R|_t.$$

This current is rectifiable with integer multiplicity by (i), (ii), and its boundary has finite mass for almost every t by the slice-boundary identity (6.1). Hence $S|_t \in \mathrm{I}_k(\mathbb{R}^{1+d})$. $\square$

6.2 Space-time currents

The fundamental idea of the space-time approach is to consider a **space-time integral current**

$$S \in \mathrm{I}_{1+k}(\mathbb{R}^{1+d})$$

and think of it as describing the motion of its slices $S|_t$, wherever these are defined. Based on this intuition we can then study the motions's velocity, traversed area, variation, etc., all of which we will treat in detail in the following.

However, the intuition of "moving slices" is somewhat flawed since there may be **jumps**, that is, times $t \in \mathbb{R}$ with

$$\|S\|(\{0\} \times \mathbb{R}^d) > 0.$$

In general, jump points cannot be avoided since they may occur as the limit of faster and faster transient motions, and we will need to develop a theory that is able to deal with jumps and also with more general fractal-like phenomena, like singularities that are "smeared out in time and space". Much of the technical challenges associated with the theory of space-time currents are related to these kinds of phenomena.

Remark 6.6. While from an applied one may be tempted to view the aforementioned singular phenomena as "pathologies", there are an inescapable consequence of continuum mechanics modeling: Unless there is a mechanism that imposes a lower bound on the smallest observable structures (like in gradient plasticity), jumps and more complex fractal-like structures may indeed occur in (weak) limits. This is the price one pays for having a mathematically closed theory.

In the following we will repeatedly make use of the measure $\mathrm{d}t \otimes \|S|_t\|$, defined via

$$\big(\mathrm{d}t \otimes \|S|_t\|\big)(A) := \int \|S|_t\|(A) \; \mathrm{d}t$$

for any Borel measurable set $A \subset \mathbb{R}^{1+d}$. Note, however, that as a consequence of the above discussion, $\mathrm{d}t \otimes \|S|_t\|$ does not tell the whole story since it misses any part of S not seen by the slices $S|_t$.

Remark 6.7. Technically, the notation "$\mathrm{d}t \otimes \|S|_t\|$" is slightly incorrect since the measures $\|S|_t\|$ are already defined on the product set $\mathbb{R}^{1+d}$. It would be more correct to write "$\mathrm{d}t \otimes (\mathbf{p}_* \|S|_t\|)$" or "$\int \|S|_t\| \, \mathrm{d}t$", as in [10], for this measure. However, this is significantly more cumbersome, so will stick to the shorthand "$\mathrm{d}t \otimes \|S|_t\|$" with the above interpretation.

Given a space-time integral current $S \in \mathrm{I}_{1+k}(\mathbb{R}^{1+d})$ we recall from Section 2.4 that for $\|S\|$-almost every (t, x) the approximate tangent space $\mathrm{T}_{(t,x)}S$ as well as the approximate differential $\mathrm{D}^S\mathbf{t}(t,x)$ and the approximate gradient $\nabla^S\mathbf{t}(t,x)$ of the time projection $\mathbf{t}$ exist. Here and in the following we write $\mathrm{T}_{(t,x)}S$, $\mathrm{D}^S\mathbf{t}$, $\nabla^S\mathbf{t}$ instead of $\mathrm{T}_{(t,x)}R$, $\mathrm{D}^R\mathbf{t}$, $\nabla^R\mathbf{t}$ with R the $\mathcal{H}^{1+k}$-rectifiable set carrying the integral current S. We recall that $\nabla^S\mathbf{t}(t,x)$ is the projection of $\nabla\mathbf{t}(t,x)$ onto $\mathrm{T}_{(t,x)}S$ and $\mathrm{D}^S\mathbf{t}(t,x)$ is the restriction of the differential to $\mathrm{T}_{(t,x)}S$, which are well-defined $\|S\|$-almost everywhere by Proposition 2.23. Moreover, the **regular set** and the **critical set** of S are defined as

$$\begin{aligned}\mathrm{Reg}(S) &:= \big\{\,(t,x)\in\mathbb{R}^{1+d}\ :\ \nabla^S\mathbf{t}(t,x)\neq 0\,\big\},\\ \mathrm{Crit}(S) &:= \big\{\,(t,x)\in\mathbb{R}^{1+d}\ :\ \nabla^S\mathbf{t}(t,x)= 0\,\big\},\end{aligned}$$

respectively.

Lemma 6.8. *Let $S\in\mathrm{I}_{1+k}(\mathbb{R}^{1+d})$.*

(i) $\|S|_t\|(\mathrm{Crit}(S))=0$ *for $\mathcal{L}^1$-a.e. $t\in\mathbb{R}$;*

(ii) $\nabla^S\mathbf{t}\neq 0$ *almost everywhere with respect to* $\mathrm{d}t\otimes\|S|_t\|$*;*

(iii) for $\mathcal{L}^1$-a.e. $t\in\mathbb{R}$ and for any Borel set $A\subset\mathbb{R}^{1+d}$ with $\|S\|(A)<\infty$ it holds that

$$\int_A\frac{1}{|\nabla^S\mathbf{t}|}\,\mathrm{d}\|S|_t\|<\infty.$$

Proof. Ad (i). By the coarea formula for slices from Proposition 6.2, applied to $g:=\mathbb{1}_{\mathrm{Crit}(S)}$, we calculate

$$\int\|S|_t\|(\mathrm{Crit}(S))\,\mathrm{d}t=\int\int_{\{t\}\times\mathbb{R}^d}\mathbb{1}_{\mathrm{Crit}(S)}\,\mathrm{d}\|S|_t\|\,\mathrm{d}t=\int_{\mathrm{Crit}(S)}|\nabla^S\mathbf{t}|\,\mathrm{d}\|S\|=0,$$

whence $\|S|_t\|(\mathrm{Crit}(S))=0$ for $\mathcal{L}^1$-a.e. $t\in\mathbb{R}$.

Ad (ii). As a consequence of (i), it holds that

$$\mathbb{1}_{\mathrm{Crit}(S)}=0\qquad(\mathrm{d}t\otimes\|S|_t\|)\text{-a.e.},$$

from which the claim follows immediately.

Ad (iii). Using (ii) and the coarea formula for slices again, this time with $g:=|\nabla^S\mathbf{t}|^{-1}\mathbb{1}_{A\setminus\mathrm{Crit}(S)}$,

$$\begin{aligned}\int\int_A\frac{1}{|\nabla^S\mathbf{t}|}\,\mathrm{d}\|S|_t\|\,\mathrm{d}t&=\int\int_{\{t\}\times\mathbb{R}^d}\frac{1}{|\nabla^S\mathbf{t}|}\mathbb{1}_{A\cap\mathrm{Reg}(S)}\,\mathrm{d}\|S|_t\|\,\mathrm{d}t\\ &=\int_{A\cap\mathrm{Reg}(S)}\mathrm{d}\|S\|.\\ &<\infty\end{aligned}$$

This shows the claim. $\square$

The **displacement vector**

$$\xi(t,x) := \frac{\nabla^S \mathbf{t}(t,x)}{|\nabla^S \mathbf{t}(t,x)|} \in \mathbb{R}^d, \qquad (t,x) \in \mathrm{Reg}(S), \tag{6.18}$$

indicates the direction of motion for the slices $S|_t$. As a consequence of (ii) in the preceding lemma, ξ is well-defined almost everywhere with respect to the measure $\mathrm{d}t \otimes \|S|_t\|$. Moreover, since $\nabla\mathbf{t} = [1\,0\cdots 0]$, its projection onto the carrier set R must have non-negative first component, so that

$$\mathbf{t}\xi \geq 0. \tag{6.19}$$

From the slicing theory of Section 6.1 we obtain that for $\mathcal{L}^1$-almost every $t \in [0,T]$ the slice $S|_t$ exists and for $(\mathrm{d}t \otimes \|S|_t\|)$-almost every (t,x) the approximate tangent space $\mathrm{T}_{(t,x)}S$ (more precisely, the approximate tangent space of the carrier set of S) satisfies

$$\mathrm{T}_{(t,x)}S = \mathrm{T}_{(t,x)}S|_t \oplus \operatorname{span}\xi(t,x), \qquad \mathrm{T}_{(t,x)}S|_t \perp \xi(t,x),$$

with "$\oplus$" indicating the direct sum of vector spaces. Defining also the dual vector

$$\xi^*(t,x) := \frac{\mathrm{D}^S \mathbf{t}(t,x)}{|\mathrm{D}^S \mathbf{t}(t,x)|} \in \textstyle\bigwedge^1 \mathbb{R}^{1+d},$$

we may then represent

$$\vec{S}|_t = \vec{S} \llcorner \xi^* = |\mathrm{D}^S \mathbf{t}|^{-1}\, \vec{S} \llcorner \mathrm{d}t. \tag{6.20}$$

By Lemma 3.7 it furthermore holds that $\xi \wedge (\vec{S} \llcorner \xi^*) = \vec{S}$ since $\xi \wedge \vec{S} = 0$, whereby

$$\vec{S} = \xi \wedge \vec{S}|_t. \tag{6.21}$$

Remark 6.9. The displacement vector ξ, as defined in (6.18), is of fundamental importance in the space-time theory. If we considered instead of the space-time current S the path of slices $t \mapsto S|_t$, then ξ is not directly available (as the missing vector to write (6.21)) and hence the geometric derivative cannot be defined in a straightforward way. Attempting to define ξ by some form of limit procedure is also problematic since it is not clear which points on the respective curves to compare, and such an approach would also require additional smoothness to be well-defined.

In the following we fix $(t,x) \in \mathrm{Reg}(S)$ as above and suppress the arguments (t,x). Let ξ be defined as in (6.18). Since $|\xi| = 1$, we observe

$$|\nabla^S \mathbf{t}| = |(\mathrm{e}_0 \cdot \xi)\xi| = |\xi \cdot \mathrm{e}_0| = |(\xi \cdot \mathrm{e}_0)\mathrm{e}_0| = |\mathbf{t}\xi| = \mathbf{t}\xi.$$

Here, for the last equality we used (6.19). Moreover, since $|\vec{S}|_t| = 1$, we then infer from (6.20) that

$$\mathbf{t}\xi = |\mathbf{t}\xi| = |\nabla^S \mathbf{t}| = |\mathrm{D}^S \mathbf{t}| = |\vec{S} \llcorner \mathrm{d}t|. \tag{6.22}$$

As a particular consequence of (6.22),

$$\mathbf{t}\xi(t,x) \neq 0 \qquad \text{for all } (t,x) \in \mathrm{Reg}(S).$$

Based on this fact we may define a central quantity in the theory of space-time currents, namely the **geometric derivative** of S:

$$\frac{\mathrm{D}}{\mathrm{D}t} S(t,x) := \frac{\mathbf{p}\xi(t,x)}{\mathbf{t}\xi(t,x)}, \qquad (t,x) \in \mathrm{Reg}(S), \tag{6.23}$$

with ξ defined in (6.18). By (6.22), we may equivalently use any of the quantities appearing there as the denominator in this definition. One should think of the geometric derivative as the (normal) velocity of transport (or "advection") of the slices. This statement will be made much more precise later, most notably in the Advection Theorem 8.18.

Proposition 6.10. *For $S \in \mathrm{I}_{1+k}(\mathbb{R}^{1+d})$,*

$$\frac{\mathrm{D}}{\mathrm{D}t} S \in \mathrm{L}^1(\mathrm{d}t \otimes \|S|_t\|; \mathbb{R}^d)$$

and

$$\left| \frac{\mathrm{D}}{\mathrm{D}t} S \right| \leq \frac{1}{|\nabla^S \mathbf{t}|} \qquad (\mathrm{d}t \otimes \|S|_t\|)\text{-}a.e. \tag{6.24}$$

Proof. The fact that the geometric derivative $\frac{\mathrm{D}}{\mathrm{D}t} S$ is well-defined $(\mathrm{d}t \otimes \|S|_t\|)$-almost everywhere follows directly from Lemma 6.8 (i), which entails that $\xi(t,x)$ defined in (6.18) exists and is non-zero at $(\mathrm{d}t \otimes \|S|_t\|)$-almost every (t,x).

The estimate (6.24) is clear from (6.23). Combining this fact with the coarea formula for slices from Proposition 6.2 (with $g := |\nabla^S \mathbf{t}|^{-1}$), we estimate

$$\int_0^1 \int_{\{t\} \times \mathbb{R}^d} \left| \frac{\mathrm{D}}{\mathrm{D}t} S \right| \, \mathrm{d}\|S|_t\| \, \mathrm{d}t \leq \int_0^1 \int_{\{t\} \times \mathbb{R}^d} \frac{1}{|\nabla^S \mathbf{t}|} \, \mathrm{d}\|S|_t\| \, \mathrm{d}t = \mathbf{M}(S).$$

This directly implies the claimed integrability. □

Example 6.11. In the so-called Space-Time Framework of dislocation-driven plasticity, the fundamental objects are the **slip trajectories**, that is, 2-dimensional space-time integral currents

$$S^b \in \mathrm{I}_2([0,T] \times \overline{\Omega}),$$

one for every Burgers vector $b \in \mathcal{B} = \{\pm b_1, \dots \pm b_m\}$ ($m \in \mathbb{N}$), which are defined in the (closed) space-time cylinder $[0,T] \times \overline{\Omega}$, with $\Omega \subset \mathbb{R}^3$ an open bounded Lipschitz domain that is simply connected (e.g., contractible). We also require the boundary condition

$$\partial S^b \llcorner ((0,T) \times \Omega) = 0. \tag{6.25}$$

and the symmetry relation

$$S^{-b} = -S^b$$

for the family $(S^b)_{b\in\mathcal{B}}$. The corresponding **dislocation system** at almost every time $t \in [0,T]$ is given by

$$S^b(t) := \mathbf{p}_* S^b|_t, \qquad b \in \mathcal{B},$$

i.e., the pushforward under the space projection of the slice $S^b|_t$ of S^b at time t. The pushforward is applied to move the slice from $\{t\} \times \overline{\Omega}$ to $\overline{\Omega}$. The slicing theory of integral currents developed in recalled above entails that $S^b(t)$ is well-defined for ($\mathcal{L}^1$-)almost every $t \in [0,T]$ and is a 1-dimensional integral current in $\overline{\Omega}$. Further, (6.25) implies for such t that $S^b(t)$ is boundaryless in Ω, i.e.,

$$\partial S^b(t) \llcorner \Omega = 0.$$

This reflects the modeling assumption that dislocation lines are always closed loops inside the specimen Ω. Moreover, the symmetry relation

$$S^{-b}(t) = -S^b(t)$$

holds for the family $(S^b(t))_{b\in\mathcal{B}}$ and almost every $t \in [0,T]$. Finally, the geometric derivative $\frac{\mathrm{D}}{\mathrm{D}t} S^b$ corresponds to the **dislocation velocity**.

6.3 Variation

Let $S \in \mathrm{I}_{1+k}([\sigma,\tau] \times \overline{\Omega})$, where $\sigma < \tau$. We define the **space-time variation** (or just **variation**) and **(space-time) boundary variation** of S in the interval $I \subset [\sigma,\tau]$ via, respectively,

$$\mathrm{Var}(S; I) := \int_{I\times\mathbb{R}^d} \|\mathbf{p}\vec{S}\| \,\mathrm{d}\|S\|,$$
$$\mathrm{Var}(\partial S; I) := \int_{I\times\mathbb{R}^d} \|\mathbf{p}\overrightarrow{\partial S}\| \,\mathrm{d}\|\partial S\|.$$

Here, $\|\cdot\|$ denotes the mass norm of multi-vectors. The variation equals the total traversed (spatial) area with *absolute* multiplicity, that is, not letting opposite movements cancel each other.

If $[\sigma, \tau] = [0, 1]$, then we also write $\mathrm{Var}(S)$, $\mathrm{Var}(\partial S)$ for $\mathrm{Var}(S; [0,1])$, $\mathrm{Var}(\partial S; [0,1])$. Clearly, the variation is additive in the interval I, that is, for $\sigma \le r < s < t \le \tau$ it holds that

$$\begin{aligned}
&\mathrm{Var}(S; [r,t)) = \mathrm{Var}(S; [r,s)) + \mathrm{Var}(S; [s,t)),\\
&\mathrm{Var}(S; (r,t]) = \mathrm{Var}(S; (r,s]) + \mathrm{Var}(S; (s,t]).
\end{aligned}$$

Remark 6.12. Since we assume that S is integral, $\vec{S}$ and $\overrightarrow{\partial S}$ are simple multi-vectors. Since the mass norm agrees with the Euclidean norm $|\cdot|$ for simple vectors (see Lemma 3.9), we may replace the mass norm with the Euclidean norm above. However, when we define the variation also for normal currents, then we must use the mass norm.

We also define the **(space-time)** L^∞**-norm** of S and ∂S on I as follows:

$$\begin{aligned}
&\|S\|_{\mathrm{L}^\infty(I)} := \operatorname*{ess\,sup}_{t \in I} \mathbf{M}(S(t))\\
&\|S\|_{\mathrm{L}^\infty(I)} := \operatorname*{ess\,sup}_{t \in I} \mathbf{M}(\partial S(t)).
\end{aligned}$$

Let $\sigma, \tau \in \mathbb{R}$ with $\sigma < \tau$. We define the space of **Lipschitz integral** $(1+k)$**-currents** in $[\sigma, \tau] \times \mathbb{R}^d$ as follows:

$$\mathrm{I}^{\mathrm{Lip}}_{1+k}([\sigma,\tau] \times \mathbb{R}^d) := \left\{ S \in \mathrm{I}_{1+k}([\sigma,\tau] \times \mathbb{R}^d) : \begin{array}{l} \|S|_{\mathrm{L}^\infty([\sigma,\tau])} + \|S\|_{\mathrm{L}^\infty([\sigma,\tau])} < \infty, \\ \|S\|(\{\sigma,\tau\} \times \mathbb{R}^d) = 0, \\ t \mapsto \mathrm{Var}(S; (\sigma, t)) \in \mathrm{Lip}((\sigma,\tau)), \\ t \mapsto \mathrm{Var}(\partial S; (\sigma, t)) \in \mathrm{Lip}((\sigma,\tau)) \end{array} \right\},$$

where $\mathrm{Lip}([\sigma,\tau])$ denotes the space of scalar Lipschitz functions on the interval $[\sigma,\tau]$. Furthermore, if $K \subset \mathbb{R}^d$ is compact, we define

$$\mathrm{I}^{\mathrm{Lip}}_{1+k}([\sigma,\tau] \times K) := \mathrm{I}^{\mathrm{Lip}}_{1+k}([\sigma,\tau] \times \mathbb{R}^d) \cap \mathrm{I}_{1+k}([\sigma,\tau] \times K).$$

Fix $S \in \mathrm{I}^{\mathrm{Lip}}_{1+k}([\sigma,\tau] \times \mathbb{R}^d)$. We shall use $\mathrm{Lip}(S)$ to denote the Lipschitz constant of

$$t \mapsto \mathrm{Var}(S; (\sigma, t)) + \mathrm{Var}(\partial S; (\sigma, t))$$

in $\mathrm{Lip}((\sigma,\tau))$.

Via the slicing theory of currents, for $\mathcal{L}^1$-almost every $t \in [\sigma,\tau]$ we can define

$$S(t) := \mathbf{p}_* S|_t \in \mathrm{I}_k(\overline{\Omega}),$$

where $S|_t \in \mathrm{I}_k([\sigma,\tau] \times \overline{\Omega})$ denotes the slice of S with respect to time (i.e., with respect to $\mathbf{t}$). Note that if $S \in \mathrm{I}_{1+k}([\sigma,\tau] \times \overline{\Omega})$ has a jump at $t \in [\sigma,\tau]$, that is, $\|S\|(\{t\} \times \mathbb{R}^d) > 0$, then $S|_t$ does not exist and the vertical piece $S \llcorner (\{t\} \times \mathbb{R}^d)$ takes the role of a "jump transient", i.e., the specific surface connecting the endpoints of the jump.

Let us consider some examples to illustrate the above notions.

Example 6.13. Let $u \in \mathrm{BV}([0,1])$ (see [4]) and define the associated graph currents $S_u \in \mathrm{I}_1([0,1] \times \mathbb{R})$ via

$$S_u := \tau\, \mathcal{H}^1 \llcorner \operatorname{graph}(u)$$

with

$$\operatorname{graph}(u) := \bigl\{\, (t, u^\theta(t)) \;:\; t \in (0,1),\ \theta \in [0,1] \,\bigr\},$$

where $u^\theta(t) := (1-\theta)u^-(t) + \theta u^+(t)$ is the affine jump between the left and right limits $u^\pm(t) = u(t\pm)$ (which are equal to $u(t)$ if t is a continuity point), and τ is the forward-pointing unit tangent to $\operatorname{graph}(u)$ (with $\tau \cdot \mathrm{e}_0 \geq 0$). In this case,

$$\operatorname{Var}(S_u; I) = \operatorname{Var}(u; I) = \|Du\|(I)$$

for every interval $I \subset [0,1]$. This can be seen as follows: By a smoothing argument and Reshetnyak's Continuity Theorem 1.21 we may without loss of generality assume that $u \in \mathrm{C}^1([0,1])$. Then,

$$\tau = \frac{1}{\sqrt{1 + |\dot{u}|^2}} \begin{bmatrix} 1 \\ \dot{u} \end{bmatrix}$$

and hence

$$\begin{aligned} \operatorname{Var}(u; I) &= \int_I |\dot{u}| \;\mathrm{d}t \\ &= \int_{\operatorname{graph}(u) \cap (I \times \mathbb{R})} \frac{|\dot{u}|}{\sqrt{1 + |\dot{u}|^2}} \;\mathrm{d}\mathcal{H}^1 \\ &= \int_{\operatorname{graph}(u) \cap (I \times \mathbb{R})} |\mathbf{p}(\tau)| \;\mathrm{d}\mathcal{H}^1 \\ &= \operatorname{Var}(S_u; I), \end{aligned}$$

where we used the area formula, Theorem 2.24. Clearly, $S_u \in \mathrm{I}^{\mathrm{Lip}}_{1+0}([0,1] \times \mathbb{R})$ if and only if u is Lipschitz. Moreover, here the vector ξ defined in (6.18) satisfies $\xi = \tau$ and hence

$$\frac{\mathrm{D}}{\mathrm{D}t} S_u = \frac{\mathbf{p}\tau}{\mathbf{t}\tau} = \dot{u}.$$

In this sense, the classical notions of BV- and Lipschitz-functions (with scalar values) constitute the 0-dimensional case of our theory.

Example 6.14. Let Ω be star-shaped with vertex $p \in \Omega$, and let $T \in \mathrm{I}_k(\overline{\Omega})$. Define $H(t,x) := (1-t)p + tx$ and $\overline{H}(t,x) := (t, H(t,x))$. The **cone**

$$p \lhd T := \overline{H}_*(\llbracket (0,1) \rrbracket \times T) \in \mathrm{I}^{\mathrm{Lip}}_{1+k}([0,1] \times \overline{\Omega})$$

satisfies $\partial(p \lhd T) = \delta_1 \times T - p \lhd \partial T$.

Example 6.15. Let $f_C\colon [0,1] \to [0,1]$ be the Cantor function, see Example 4.10, and let S_C be the "Cantor cone", that is, the set in $\mathbb{R}^{1+2}$ obtained by rotating the graph of f_C around the time axis. As the graph of f_C is 1-rectifiable (with length 2), we get that S_C is 2-rectifiable. Hence, with a choice of orientation, $S_C \in \mathrm{I}_2(\mathbb{R}^3)$. Then, $S_C(t) \in \mathrm{I}_1(\mathbb{R}^2)$ for $t \in [0,1]$ is the circle lying around the origin with radius $f_C(t)$ and

$$\mathrm{Var}(S_C; [0,t]) = \pi f_C(t)^2.$$

Hence, $S_C \notin \mathrm{I}^{\mathrm{Lip}}_{1+1}([0,T] \times \mathbb{R}^2)$.

Like the classical variation, also our space-time variation is invariant with respect to time rescalings:

Lemma 6.16. *Let $S \in \mathrm{I}_{1+k}([\sigma,\tau] \times \overline{\Omega})$ and let $a \in \mathrm{Lip}([\sigma,\tau])$ be injective. Then,*

$$a_*S := [(t,x) \mapsto (a(t),x)]_*S \in \mathrm{I}_{1+k}(a([\sigma,\tau]) \times \overline{\Omega})$$

with

$$(a_*S)(a(t)) = \operatorname{sgn} a'(t) \cdot S(t), \qquad t \in [\sigma,\tau],$$

and

$$\begin{aligned}
\mathrm{Var}(a_*S; a([\sigma,\tau])) &= \mathrm{Var}(S; [\sigma,\tau]),\\
\mathrm{Var}(\partial(a_*S); a([\sigma,\tau])) &= \mathrm{Var}(\partial S; [\sigma,\tau]),\\
\|a_*S\|_{\mathrm{L}^\infty(a([\sigma,\tau]))} &= \|S\|_{\mathrm{L}^\infty([\sigma,\tau])},\\
\|\partial(a_*S)\|_{\mathrm{L}^\infty(a([\sigma,\tau]))} &= \|\partial S\|_{\mathrm{L}^\infty([\sigma,\tau])}.
\end{aligned}$$

If $S \in \mathrm{I}^{\mathrm{Lip}}_{1+k}([\sigma,\tau] \times \overline{\Omega})$, then also $a_*S \in \mathrm{I}^{\mathrm{Lip}}_{1+k}(a([\sigma,\tau]) \times \overline{\Omega})$.

Proof. If $S = m\,\vec{S}\,\mathcal{H}^{1+k} \llcorner R$ with a countably $(1+k)$-rectifiable set $R \subset \mathbb{R}^{1+d}$, we get (see, e.g., [23, (3) on p. 197])

$$a_*S = m \circ a^{-1}\, \frac{(\mathrm{D}^R a \circ a^{-1})[\vec{S} \circ a^{-1}]}{|(\mathrm{D}^R a \circ a^{-1})[\vec{S} \circ a^{-1}]|}\, \mathcal{H}^k \llcorner a(R),$$

where here and in the following we identify a with the space-time map $(t,x) \mapsto (a(t),x)$. Since a only transforms the time coordinate,

$$\mathbf{p}((\mathrm{D}^R a \circ a^{-1})[\vec{S} \circ a^{-1}]) = \mathbf{p}((Da \circ a^{-1})[\vec{S} \circ a^{-1}]) = \operatorname{sgn} a'(t) \cdot \mathbf{p}(\vec{S} \circ a^{-1})$$

and

$$\mathbf{J}^R_k a = |(\mathrm{D}^R a)[\vec{S}]|.$$

Hence,

$$\begin{aligned}
\mathrm{Var}(a_*S; a([\sigma,\tau])) &= \int_{a(R)} \left\| \mathbf{p}\Big(\frac{(\mathrm{D}^R a \circ a^{-1})[\vec{S} \circ a^{-1}]}{|(\mathrm{D}^R a \circ a^{-1})[\vec{S} \circ a^{-1}]|} \Big) \right\| \, m \circ a^{-1} \, \mathrm{d}\mathcal{H}^k \\
&= \int_{a(R)} \frac{\|\mathbf{p}(\vec{S} \circ a^{-1})\|}{|\mathbf{J}^R_k a \circ a^{-1}|} \, m \circ a^{-1} \, \mathrm{d}\mathcal{H}^k \\
&= \int_R \|\mathbf{p}\vec{S}\| \, m \, \mathrm{d}\mathcal{H}^k \\
&= \mathrm{Var}(S; [\sigma,\tau]),
\end{aligned}$$

where we used the area formula of Theorem 2.24. The equality for the boundary variation follows in the same way. The additional claim about Lipschitz integral currents is then also clear. □

Finally, we want to consider some estimates between mass and variation. First, we observe that since $\|\mathbf{p}\vec{S}\| \le 1$ the variation is dominated by the mass, that is,

$$\mathrm{Var}(S; I) \le \mathbf{M}(S \llcorner (I \times \mathbb{R}^d)) \le \mathbf{M}(S) \tag{6.26}$$

and likewise for the boundary variation.

The next **"Pythagoras" lemma** gives a reverse estimate to (6.26). For this, let $\Lambda(k,d)$ be the set of ordered k-tuples $(i_1,\dots,i_k) \in \{1,\dots,d\}^k$ with $i_1 < \cdots < i_k$; given $J = (i_1,\dots,i_k) \in \Lambda(k,d)$, we set $\mathrm{e}_J := \mathrm{e}_{i_1} \wedge \cdots \wedge \mathrm{e}_{i_k}$.

Lemma 6.17. *Let $S \in \mathrm{I}_{1+k}(\mathbb{R}^{1+d})$. Then,*

$$|\mathbf{t}\xi|^2 + |\mathbf{p}\vec{S}|^2 = 1 \qquad \|S\|\text{-a.e.} \tag{6.27}$$

and, for $I \subset \mathbb{R}$,

$$\|S\|(I \times \mathbb{R}^d) \le \int_I \mathbf{M}(S(t)) \, \mathrm{d}t + \mathrm{Var}(S; I) \le \mathcal{L}^1(I) \cdot \|S\|_{\mathrm{L}^\infty(I)} + \mathrm{Var}(S; I).$$

Proof. Any $\eta \in \bigwedge_{1+k} \mathbb{R}^{1+d}$ can be written in coordinates as

$$\eta = \sum_{J \in \Lambda(k,d)} \eta_{0,J} \, \mathrm{e}_0 \wedge \mathrm{e}_J + \sum_{J \in \Lambda(1+k,d)} \eta_J \, \mathrm{e}_J = \eta \llcorner \mathrm{d}t + \mathbf{p}\eta,$$

where $\eta_{0,J}, \eta_J \in \mathbb{R}$. Here we have used that

$$\begin{aligned}
\eta \llcorner \mathrm{d}t &= \sum_{J \in \Lambda(k,d)} \eta_{0,J} \, \mathrm{e}_J, \\
\mathbf{p}\eta &= \sum_{J \in \Lambda(1+k,d)} \eta_J \, \mathrm{e}_J.
\end{aligned}$$

This decomposition also shows that $\eta \llcorner \mathrm{d}$ and $\mathbf{p}\eta$ are orthogonal with respect to the scalar product on k-vectors, whereby

$$|\eta \llcorner \mathrm{d}t|^2 + |\mathbf{p}\eta|^2 = |\eta|^2.$$

Applying this pointwise to $\eta := \vec{S}$ and also relying on (6.22), whereby

$$|\vec{S} \llcorner \mathrm{d}t| = |\nabla^S \mathbf{t}| = |\mathbf{t}\xi|,$$

we obtain (6.27). Moreover,

$$\begin{aligned}
\mathbf{M}(S \llcorner I \times \mathbb{R}^d) &= \int_{I\times\mathbb{R}^d} \sqrt{|\vec{S} \llcorner \mathrm{d}t|^2 + |\mathbf{p}\vec{S}|^2} \ \mathrm{d}\|S\| \\
&\le \int_{I\times\mathbb{R}^d} |\nabla^S \mathbf{t}| + |\mathbf{p}\vec{S}| \ \mathrm{d}\|S\| \\
&= \int_I \mathbf{M}(S|_t) \ \mathrm{d}t + \mathrm{Var}(S; I),
\end{aligned}$$

where in the last line we have used the coarea formula for slices in the form of Proposition 6.2 and the definition of the variation together with Remark 6.12.

Alternatively, one could estimate the Euclidean norm by the mass norm, see Lemma 3.9, to obtain

$$1 = |\vec{S}| \le |\vec{S} \llcorner \mathrm{d}t| + |\mathbf{p}\vec{S}| \le \|\vec{S} \llcorner \mathrm{d}t\| + \|\mathbf{p}\vec{S}\| \qquad \|S\|\text{-a.e.}$$

Integrating this estimate over $I \times \mathbb{R}^d$ with respect to $\|S\|$ and applying the coarea formula we conclude again. $\square$

6.4 Operations on space-time currents

it is convenient to define the concatenation and reversal of space-time currents with boundaryless traces at the start end end points:

Lemma 6.18. *Let* $S_1, S_2 \in \mathrm{I}_{1+k}([0,1] \times \overline{\Omega})$ *with*

$$\partial S_1 = \delta_1 \times T_1 - \delta_0 \times T_0, \qquad \partial S_2 = \delta_1 \times T_2 - \delta_0 \times T_1,$$

where $T_0, T_1, T_2 \in \mathrm{I}_k(\overline{\Omega})$ *with* $\partial T_0 = \partial T_1 = \partial T_2 = 0$*. Then, there is* $S_2 \circ S_1 \in \mathrm{I}_{1+k}([0,1] \times \overline{\Omega})$*, called the **concatenation** of* S_1, S_2*, with*

$$\partial(S_2 \circ S_1) = \delta_1 \times T_2 - \delta_0 \times T_0$$

and

$$\begin{aligned}
\mathrm{Var}(S_2 \circ S_1) &= \mathrm{Var}(S_1) + \mathrm{Var}(S_2), \\
\|S_2 \circ S_1\|_{\mathrm{L}^\infty([0,1])} &= \max\Big\{ \|S_1\|_{\mathrm{L}^\infty([0,1])}, \ \|S_2\|_{\mathrm{L}^\infty([0,1])} \Big\}.
\end{aligned}$$

Furthermore, if $S_1, S_2 \in \mathrm{I}^{\mathrm{Lip}}_{1+k}([0,1] \times \overline{\Omega})$*, then also* $S_2 \circ S_1 \in \mathrm{I}^{\mathrm{Lip}}_{1+k}([0,1] \times \overline{\Omega})$*.*

Proof. We set

$$S := a^1_* S_1 + a^2_* S_2, \qquad a^i(t,x) := \left(\frac{i-1}{2} + \frac{t}{2}, x\right).$$

Then, all claimed properties follow directly from Lemma 6.16. $\square$

Lemma 6.19. *Let $S \in \mathrm{I}_{1+k}([0,1] \times \overline{\Omega})$ with*

$$\partial S = \delta_1 \times T_1 - \delta_0 \times T_0,$$

*where $T_0, T_1 \in \mathrm{I}_k(\overline{\Omega})$ with $\partial T_0 = \partial T_1 = 0$. Then, there is $S^{-1} \in \mathrm{I}_{1+k}([0,1] \times \overline{\Omega})$, called the **reversal** of S, with*

$$\partial S^{-1} = \delta_1 \times T_0 - \delta_0 \times T_1$$

and

$$\mathrm{Var}(S^{-1}) = \mathrm{Var}(S), \qquad \|S^{-1}\|_{\mathrm{L}^\infty([0,1])} = \|S\|_{\mathrm{L}^\infty([0,1])}.$$

Furthermore, if $S \in \mathrm{I}^{\mathrm{Lip}}_{1+k}([0,1] \times \overline{\Omega})$, then also $S^{-1} \in \mathrm{I}^{\mathrm{Lip}}_{1+k}([0,1] \times \overline{\Omega})$.

Proof. We set

$$S^{-1} := a_* S, \qquad a(t,x) := (1-t, x)$$

and again conclude by Lemma 6.16. $\square$

Next we consider progressive-in-time deformations of (boundaryless) integral currents. To see how one could generalize currents deformed via C^1-homotopies (or Lipschitz-homotopies), we first examine the classical situation: Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $H \in \mathrm{C}^1([0,1] \times \overline{\Omega}; \overline{\Omega})$ be a C^1-homotopy between the identity and $g \in \mathrm{C}^1(\overline{\Omega}; \overline{\Omega})$, i.e., $H(0,x) = x$ and $H(1,x) = g(x)$; we also set $\overline{H}(t,x) := (t, H(t,x))$. For $T \in \mathrm{I}_k(\overline{\Omega})$ with $\partial T = 0$ define the **deformation trajectory**

$$S := \overline{H}_*(\llbracket (0,1) \rrbracket \times T) \in \mathrm{I}_{1+k}([0,1] \times \overline{\Omega}),$$

where we have denoted by $\llbracket (0,1) \rrbracket$ the canonical current associated with the interval $(0,1)$. Then, by the formula for the boundary of products, see Lemma 4.19, we have

$$\partial S = \delta_1 \times g_* T - \delta_0 \times T.$$

Moreover, since H was assumed to possess C^1-regularity,

$$S|_t = \delta_t \times H(t, \cdot)_* T, \qquad t \in [0,1],$$

and $t \mapsto S|_t$ can be understood as a continuous deformation of T into g_*T.

Unfortunately, the class of C^1-homotopies is not closed in a topology suitable for our needs. Furthermore, C^1-homotopies do not allow to move overlapping or intersecting parts of currents into different directions since they represent deformations of the underlying space and not of the currents themselves. Our generalization of a deformation is thus based on the deformation trajectory S itself. In this spirit, the lemma below show that in the case of "essentially injective" homotopies (which do not reverse direction and have no overlaps) the variation measures precisely the mass of the pushforward of the space-time current under the space projection (the "slip surface" in the situation of dislocations).

Lemma 6.20. *Let $\Omega \subset \mathbb{R}^d$ and $\Omega' \subset \mathbb{R}^m$ be bounded Lipschitz domains, let $T \in \mathrm{I}_k(\overline{\Omega})$, and let $H \in \mathrm{Lip}([0,1] \times \overline{\Omega}; \overline{\Omega'})$ be a homotopy that is essentially injective in the sense that*

$$\begin{cases} \text{there is a Borel set } N \subset [0,1] \times \operatorname{supp} T \text{ with } \mathcal{H}^{1+k}(N) = 0 \text{ such that} \\ H \text{ is injective on } D := ([0,1] \times \operatorname{supp} T) \setminus (N \cup \{\mathbf{p}(D\overline{H}[\mathrm{e}_0 \wedge \vec{T}]) = 0\}), \end{cases} \tag{6.28}$$

where $\overline{H}(t,x) := (t, H(t,x))$. Set $S_H := \overline{H}_*(\llbracket(0,1)\rrbracket \times T)$. Then, for all intervals $[\sigma,\tau] \subset [0,1]$,

$$\mathrm{Var}(S_H; [\sigma,\tau]) = \mathbf{M}(H_*(\llbracket(\sigma,\tau)\rrbracket \times T)), \tag{6.29}$$

$$\mathrm{Var}(\partial S_H; (\sigma,\tau)) = \mathbf{M}(H_*(\llbracket(\sigma,\tau)\rrbracket \times \partial T)). \tag{6.30}$$

Proof. Let $T = m\,\vec{T}\,\mathcal{H}^k \llcorner R$ with a countably k-rectifiable carrier set R such that $\mathcal{H}^k(R) < \infty$, and a $\mathcal{H}^k$-measurable and integrable multiplicity function $m\colon R \to \mathbb{N}$. Fix an interval $[\sigma,\tau] \subset [0,1]$ and set $Z := (\sigma,\tau) \times R$, which is a countably $(1+k)$-rectifiable set. Assume furthermore that

$$S_H = m_H\,\vec{S}_H\,\mathcal{H}^{1+k} \llcorner R_H \in \mathrm{I}_{1+k}([0,1] \times \overline{\Omega'}).$$

We have for $\omega \in \mathcal{D}^{1+k}(\mathbb{R}^{1+m})$ that

$$\begin{aligned} &\bigl\langle \overline{H}_*(\llbracket(\sigma,\tau)\rrbracket \times T), \omega \bigr\rangle \\ &\quad = \int_\sigma^\tau \int_R \bigl\langle \mathrm{D}^Z\overline{H}(t,x)[\mathrm{e}_0 \wedge \vec{T}(x)], \omega(\overline{H}(t,x)) \bigr\rangle\, m(x)\,\mathrm{d}\mathcal{H}^k(x)\,\mathrm{d}t \\ &\quad = \int_{R_H \cap ([\sigma,\tau]\times\mathbb{R}^m)} \sum_{\{x \,:\, y = H(t,x)\}} m(x) \Bigl\langle \frac{\mathrm{D}^Z\overline{H}(t,x)[\mathrm{e}_0 \wedge \vec{T}(x)]}{|\mathrm{D}^Z\overline{H}(t,x)[\mathrm{e}_0 \wedge \vec{T}(x)]|}, \omega(t,y) \Bigr\rangle\,\mathrm{d}\mathcal{H}^{1+k}(t,y) \end{aligned}$$

by the representation formula for pushforwards, Proposition 5.2, and the area formula, Theorem 2.24. Note that $|\mathrm{D}^Z\overline{H}(t,x)[\mathrm{e}_0 \wedge \vec{T}(x)]|$ is precisely the modulus of the k-dimensional Jacobian of $\overline{H}$ at (t,x) with respect to Z. It follows, see, e.g., [23, eq. (7.29)]), that

$$m_H(t,y)\,\vec{S}_H(t,y) = \sum_{\{x \,:\, y = H(t,x)\}} m(x) \frac{\mathrm{D}^Z\overline{H}(t,x)[\mathrm{e}_0 \wedge \vec{T}(x)]}{|\mathrm{D}^Z\overline{H}(t,x)[\mathrm{e}_0 \wedge \vec{T}(x)]|}. \tag{6.31}$$

Define

$$\eta(t,x) := \mathbf{p}(\mathrm{D}^Z\overline{H}(t,x)[\mathrm{e}_0 \wedge \vec{T}(x)]).$$

Our assumption (6.28) now implies that whenever $\eta(t,x) \neq 0$, then in (6.31) we have

$$\|\mathbf{p}(\vec{S}_H(t,y))\|\, m_H(t,y) = \sum_{\{x\,:\, y=H(t,x)\}} m(x) \frac{\|\mathbf{p}(\mathrm{D}^Z\overline{H}(t,x)[\mathrm{e}_0 \wedge \vec{T}(x)])\|}{|\mathrm{D}^Z\overline{H}(t,x)[\mathrm{e}_0 \wedge \vec{T}(x)]|}$$

on a set of full measure (there is only one term in the sum). Thus,

$$\begin{aligned}
\mathrm{Var}(S_H;[\sigma,\tau]) &= \int_{R_H\cap([\sigma,\tau]\times\mathbb{R}^m)} \|\mathbf{p}(\vec{S}_H)\|\;\mathrm{d}\|S_H\| \\
&= \int_{R_H\cap([\sigma,\tau]\times\mathbb{R}^m)} \sum_{\{x\,:\, y=H(t,x)\}} m(x) \frac{\|\mathbf{p}(\mathrm{D}^Z\overline{H}(t,x)[\mathrm{e}_0 \wedge \vec{T}(x)])\|}{|\mathrm{D}^Z\overline{H}(t,x)[\mathrm{e}_0 \wedge \vec{T}(x)]|}\;\mathrm{d}\mathcal{H}^{1+k}(t,y) \\
&= \int_\sigma^\tau \int_R |\eta(t,x)|\; m(x)\;\mathrm{d}\mathcal{H}^k(x)\;\mathrm{d}t. \qquad (6.32)
\end{aligned}$$

By similar arguments as before, for $\omega \in \mathcal{D}^{1+k}(\mathbb{R}^m)$ it also holds that

$$\begin{aligned}
&\big\langle H_*([\![(\sigma,\tau)]\!] \times T), \omega \big\rangle \\
&\quad = \int_\sigma^\tau \int_R \big\langle \eta(t,x), \omega(H(t,x)) \big\rangle\; m(x)\;\mathrm{d}\mathcal{H}^k(x)\;\mathrm{d}t \\
&\quad = \iint_D \big\langle \eta(t,x), \omega(H(t,x)) \big\rangle\; m(x)\;\mathrm{d}\mathcal{H}^k(x)\;\mathrm{d}t.
\end{aligned}$$

We now find a measurable k-covector field $\widetilde{\omega}\colon D \to \bigwedge^k \mathbb{R}^m$ with $|\widetilde{\omega}| \leq 1$ and $\langle \eta(t,x), \widetilde{\omega}(t,x)\rangle = |\eta(t,x)|$. Then, by (6.28), there exists a measurable k-covector field $\widehat{\omega}\colon \overline{\Omega} \to \bigwedge^k \mathbb{R}^m$ satisfying $|\widehat{\omega}| \leq 1$ and

$$\big\langle \eta(t,x), \widehat{\omega}(H(t,x)) \big\rangle = \big\langle \eta(t,x), \widetilde{\omega}(t,x) \big\rangle = |\eta(t,x)|$$

for $(t,x) \in D$. By a standard smoothing argument we thus obtain

$$\mathbf{M}(H_*([\![(\sigma,\tau)]\!] \times T)) \geq \int_\sigma^\tau \int_R |\eta(t,x)|\; m(x)\;\mathrm{d}\mathcal{H}^k(x)\;\mathrm{d}t$$

and the other inequality "$\leq$" is easily seen to be true as well. Consequently, using this equality in (6.32),

$$\mathrm{Var}(S_H;[\sigma,\tau]) = \mathbf{M}(H_*([\![(\sigma,\tau)]\!] \times T)).$$

This shows (6.29); the boundary estimate (6.30) follows in the same way. $\square$

The prototypical class of deformation trajectories is defined via affine homotopies:

Lemma 6.21. *Let $\Omega \subset \mathbb{R}^d$ and $\Omega' \subset \mathbb{R}^m$ be bounded Lipschitz domains and let H be an affine homotopy between $f, g \in \mathrm{Lip}(\overline{\Omega}; \overline{\Omega'})$, i.e.,*

$$H(t,x) := (1-t)f(x) + tg(x), \qquad (t,x) \in [0,1] \times \overline{\Omega}.$$

Let $T \in \mathrm{I}_k(\overline{\Omega})$ and set $S_H := \overline{H}_*(⟦(0,1)⟧ \times T)$, where $\overline{H}(t,x) := (t, H(t,x))$. Then,

$$S_H \in \mathrm{I}^{\mathrm{Lip}}_{1+k}([0,1] \times \overline{\Omega'}) \tag{6.33}$$

and, for all intervals $[\sigma,\tau] \subset [0,1]$ and almost every $t \in [0,1]$,

$$\begin{aligned}
\mathrm{Var}(S_H; [\sigma,\tau]) &\le \|g-f\|_\infty \cdot V^k(f,g,T) \cdot |\sigma-\tau|, && (6.34)\\
\mathrm{Var}(\partial S_H; (\sigma,\tau)) &\le \|g-f\|_\infty \cdot V^{k-1}(f,g,\partial T) \cdot |\sigma-\tau|, && (6.35)\\
\mathbf{M}(S_H(t)) &\le V^k(f,g,T), && (6.36)\\
\mathbf{M}(\partial S_H(t)) &\le V^{k-1}(f,g,\partial T), && (6.37)
\end{aligned}$$

where, for $l = k-1, k$,

$$V^l(f,g,T) := \int |Df|^l + |Dg|^l \,\mathrm{d}\|T\| \le \bigl(\|Df\|^l_{\mathrm{L}^\infty} + \|Dg\|^l_{\mathrm{L}^\infty}\bigr)\mathbf{M}(T).$$

and the L^∞-norms may be taken over the support of T.

Proof. We use the same notation as in the proof of Lemma 6.20. Recalling (6.31) (which holds independently of the injectivity hypothesis (6.28)), we can estimate

$$\|\mathbf{p}(\vec{S}_H(t,y))\|\, m_H(t,y) \le \sum_{\{x \,:\, y=(1-t)f(x)+tg(x)\}} m(x) \frac{\|\mathbf{p}(\mathrm{D}^Z\overline{H}(t,x)[\mathrm{e}_0 \wedge \vec{T}(x)])\|}{|\mathrm{D}^Z\overline{H}(t,x)[\mathrm{e}_0 \wedge \vec{T}(x)]|}.$$

A computation shows

$$D^Z\overline{H}(t,x)[\mathrm{e}_0 \wedge \vec{T}(x)] = \begin{bmatrix} 1 \\ g(x)-f(x) \end{bmatrix} \wedge \left((1-t) \begin{bmatrix} 0 \\ Df(x) \end{bmatrix} + t \begin{bmatrix} 0 \\ Dg(x) \end{bmatrix} \right) [\vec{T}(x)]$$

and then

$$\begin{aligned}
\|\mathbf{p}(\mathrm{D}^Z\overline{H}(t,x)[\mathrm{e}_0 \wedge \vec{T}(x)])\| &= \bigl\|(g(x)-f(x)) \wedge ((1-t)Df(x) + tDg(x))[\vec{T}(x)]\bigr\| \\
&\le \|g-f\|_\infty \cdot (|Df(x)|^k + |Dg(x)|^k).
\end{aligned}$$

So,

$$\begin{aligned}
&\operatorname{Var}(S_H;[\sigma,\tau]) \\
&\quad = \int_{R_H\cap([\sigma,\tau]\times\mathbb{R}^m)} \|\mathbf{p}(\vec{S}_H(t,y))\|\, m_H(t,y)\ \mathrm{d}\mathcal{H}^{1+k}(t,y) \\
&\quad \le \int_{R_H\cap([\sigma,\tau]\times\mathbb{R}^m)} \sum_{\{x\,:\,y=(1-t)f(x)+tg(x)\}} \\
&\qquad\qquad m(x)\frac{\|\mathbf{p}(\mathrm{D}^Z\overline{H}(t,x)[\mathrm{e}_0\wedge\vec{T}(x)])\|}{|\mathrm{D}^Z\overline{H}(t,x)[\mathrm{e}_0\wedge\vec{T}(x)]|}\ \mathrm{d}\mathcal{H}^{1+k}(t,y) \\
&\quad \le \int_{R_H\cap([\sigma,\tau]\times\mathbb{R}^m)} \sum_{\{x\,:\,y=(1-t)f(x)+tg(x)\}} \\
&\qquad\qquad m(x)\frac{\|g-f\|_\infty\cdot(|Df(x)|^k+|Dg(x)|^k)}{|\mathrm{D}^Z\overline{H}(t,x)[\mathrm{e}_0\wedge\vec{T}(x)]|}\ \mathrm{d}\mathcal{H}^{1+k}(t,y) \\
&\quad = \int_\sigma^\tau \int_R \|g-f\|_\infty\cdot(|Df(x)|^k+|Dg(x)|^k)\ m(x)\ \mathrm{d}\mathcal{H}^k(x)\ \mathrm{d}t \\
&\quad = \|g-f\|_\infty\cdot\left(\int |Df|^k+|Dg|^k\ \mathrm{d}\|T\|\right)\cdot|\sigma-\tau|,
\end{aligned}$$

where we have used the area formula again in the second-to-last equality. This shows (6.34).

For the boundary variation $\operatorname{Var}(\partial S_H;(\sigma,\tau))$ we observe via the formula for the boundary of products, see Lemma 4.19, together with the fact that boundaries and pushforwards commute, see Lemma 4.26, the formula

$$\partial S_H = \delta_1\times g_*T - \delta_0\times f_*T - \overline{H}_*([\![(0,1)]\!]\times\partial T).$$

We can argue in a similar fashion to above to obtain (6.35) (note that the interval is open, so that the endpoint terms are not counted). For (6.36) we use that for almost every $t\in(0,1)$ it holds that

$$S_H(t) = H(t,\bullet)_*T.$$

This follows from the cylinder formula (6.13). Then, for $\omega\in\mathcal{D}^k(\mathbb{R}^m)$,

$$\langle S_H(t),\omega\rangle = \int_R \langle \mathrm{D}^R H(t,\bullet)[\vec{T}(x)],\omega(H(t,x))\rangle\ m(x)\ \mathrm{d}\mathcal{H}^k(x).$$

Taking the supremum over all $\omega\in\mathcal{D}^k(\mathbb{R}^m)$ with $|\omega|\le 1$ and employing a similar estimate as above yields (6.36); likewise for (6.37). Then, also (6.33) follows. □

Remark 6.22. Note that for an affine homotopy from f to g as in the preceding lemma,

$$\begin{aligned}
\eta(t,x) &= \mathbf{p}(\mathrm{D}^Z\overline{H}(t,x)[\mathrm{e}_0\wedge\vec{T}(x)]) \\
&= (g(x)-f(x))\wedge((1-t)Df(x)+tDg(x))[\vec{T}(x)],
\end{aligned}$$

which is zero in particular where $f = g$ (that is, where the affine homotopy "stands still"). So, in this case, the assumption (6.28) in Lemma 6.20 is implied by the more restrictive, but easier to check, condition

$$\begin{cases} \text{there is a Borel set } N \subset [0,1] \times \operatorname{supp} T \text{ with } \mathcal{H}^{1+k}(N) = 0 \text{ such that} \\ H \text{ is injective on } D := ([0,1] \times \operatorname{supp} T) \setminus (N \cup \{f = g\}). \end{cases}$$

6.5 Weak* convergence and compactness

We say that $(S_j) \subset \mathrm{I}_{1+k}([\sigma,\tau] \times \overline{\Omega})$ **converges BV-weakly*** to $S \in \mathrm{I}_{1+k}([\sigma,\tau] \times \overline{\Omega})$ as $j \to \infty$, in symbols "$S_j \overset{*}{\rightharpoonup} S$ in BV", if

$$\begin{cases} \quad S_j \overset{*}{\rightharpoonup} S & \text{in } \mathrm{I}_{1+k}([\sigma,\tau] \times \overline{\Omega}), \\ S_j(t) \overset{*}{\rightharpoonup} S(t) & \text{in } \mathrm{I}_k(\overline{\Omega}) \text{ for } \mathcal{L}^1\text{-almost every } t \in [\sigma,\tau]. \end{cases}$$

For this convergence we have the following compactness principle in the spirit of the classical Helly selection principle for BV-maps:

Theorem 6.23. *Assume that the sequence $(S_j) \subset \mathrm{I}_{1+k}([\sigma,\tau] \times \overline{\Omega})$ satisfies*

$$\|S_j\|_{\mathrm{L}^\infty([\sigma,\tau])} + \|\partial S_j\|_{\mathrm{L}^\infty([\sigma,\tau])} + \mathrm{Var}(S_j;[\sigma,\tau]) + \mathrm{Var}(\partial S_j;[\sigma,\tau]) \le C < \infty$$

for all $j \in \mathbb{N}$. Then, there exists $S \in \mathrm{I}_{1+k}([\sigma,\tau] \times \overline{\Omega})$ and a (not relabelled) subsequence such that

$$S_j \overset{*}{\rightharpoonup} S \quad \textit{in BV.}$$

Moreover,

$$\begin{aligned} \|S\|_{\mathrm{L}^\infty([\sigma,\tau])} &\le \liminf_{j\to\infty} \|S_j\|_{\mathrm{L}^\infty([\sigma,\tau])}, \\ \|\partial S\|_{\mathrm{L}^\infty([\sigma,\tau])} &\le \liminf_{j\to\infty} \|\partial S_j\|_{\mathrm{L}^\infty([\sigma,\tau])}, \\ \mathrm{Var}(S;[\sigma,\tau]) &\le \liminf_{j\to\infty} \mathrm{Var}(S_j;[\sigma,\tau]), \\ \mathrm{Var}(\partial S;(\sigma,\tau)) &\le \liminf_{j\to\infty} \mathrm{Var}(\partial S_j;(\sigma,\tau)). \end{aligned}$$

If additionally $(S_j) \subset \mathrm{I}^{\mathrm{Lip}}_{1+k}([\sigma,\tau] \times \overline{\Omega})$ such that the Lipschitz constants L_j of the scalar maps $t \mapsto \mathrm{Var}(S_j;[\sigma,t]) + \mathrm{Var}(\partial S_j;(\sigma,t))$ are uniformly bounded, then also $S \in \mathrm{I}^{\mathrm{Lip}}_{1+k}([\sigma,\tau] \times \overline{\Omega})$ with Lipschitz constant bounded by $\liminf_{j\to\infty} L_j$. Moreover, in this case, $S_j(t) \overset{*}{\rightharpoonup} S(t)$ in $\mathrm{I}_k(\overline{\Omega})$ for every $t \in [\sigma,\tau)$.

Proof. From the assumptions we infer a uniform bound on the masses $\mathbf{M}(S_j)$ and $\mathbf{M}(\partial S_j)$ via Lemma 6.17. Then, the first convergence $S_j \overset{*}{\rightharpoonup} S$ in $\mathrm{I}_{1+k}([\sigma,\tau]\times\overline{\Omega})$, up to selecting a subsequence, follows directly from the Federer–Fleming compactness theorem in $\mathrm{I}_{1+k}([\sigma,\tau]\times\overline{\Omega})$, see Theorem 5.6.

By the cylinder formula (6.13),

$$S|_t = \partial(S \llcorner \{\mathbf{t} < t\}) - (\partial S) \llcorner \{\mathbf{t} < t\}$$

and likewise for $S_j|_t$. If $\|S_j\| + \|\partial S_j\| \overset{*}{\rightharpoonup} \nu$ in $\mathcal{M}^+([\sigma,\tau]\times\mathbb{R}^d)$ (for a subsequence), then standard results in measure theory (see, e.g., [4, Theorem 1.62 (b)]) imply that for all $t \in [\sigma,\tau]$ with $\nu(\{t\}\times\mathbb{R}^d) = 0$ it holds that

$$\big\langle S_j, \mathbb{1}_{\{\mathbf{t}<t\}} \wedge \mathrm{d}\omega \big\rangle \to \big\langle S, \mathbb{1}_{\{\mathbf{t}<t\}} \wedge \mathrm{d}\omega \big\rangle, \qquad \big\langle \partial S_j, \mathbb{1}_{\{\mathbf{t}<t\}} \wedge \omega \big\rangle \to \big\langle \partial S, \mathbb{1}_{\{\mathbf{t}<t\}} \wedge \omega \big\rangle.$$

for all $\omega \in \mathcal{D}^k(\mathbb{R}^{1+d})$. Thus, for these t,

$$\partial\big(S_j \llcorner \{\mathbf{t} < t\}\big) \overset{*}{\rightharpoonup} \partial\big(S \llcorner \{\mathbf{t} < t\}\big), \qquad \partial S_j \llcorner \{\mathbf{t} < t\} \overset{*}{\rightharpoonup} \partial S \llcorner \{\mathbf{t} < t\}.$$

Since there are only at most countably many t's with $\nu(\{t\}\times\mathbb{R}^d) > 0$, we obtain that

$$S_j|_t \overset{*}{\rightharpoonup} S|_t \qquad \text{for } \mathcal{L}^1\text{-almost every } t.$$

This shows the second convergence $S_j(t) \overset{*}{\rightharpoonup} S(t)$ in $\mathrm{I}_k(\overline{\Omega})$ for $\mathcal{L}^1$-almost every $t \in [\sigma,\tau]$.

The lower semicontinuity of the mass and variation follow in the usual way from the weak* convergences. Indeed, the variation $\mathrm{Var}(S_j;[\sigma,\tau])$ is lower semicontinuous by Reshetnyak's Lower Semicontinuity Theorem 1.22, i.e.,

$$\begin{aligned}
\mathrm{Var}(S;[\sigma,\tau]) &= \int_{[\sigma,\tau]\times\Omega} \|\mathbf{p}\vec{S}\| \,\mathrm{d}\|S\| \\
&\le \liminf_{j\to\infty} \int_{[\sigma,\tau]\times\Omega} \|\mathbf{p}(\vec{S}_j)\| \,\mathrm{d}\|S_j\| \\
&= \liminf_{j\to\infty} \mathrm{Var}(S_j;[\sigma,\tau])
\end{aligned}$$

since the integrand $\|\mathbf{p}(\cdot)\|$ is positively 1-homogeneous, convex (as the composition of a convex and a linear map on $\bigwedge_{1+k}\mathbb{R}^{1+d}$), and continuous.

Finally, assume that $(S_j) \subset \mathrm{I}^{\mathrm{Lip}}_{1+k}([\sigma,\tau]\times\overline{\Omega})$ and the Lipschitz constants L_j of the scalar maps $t \mapsto \mathrm{Var}(S_j;[\sigma,t]) + \mathrm{Var}(\partial S_j;(\sigma,t))$ are uniformly bounded by $L^* > 0$. Then, by Lemma 6.17, for almost every $s,t \in (\sigma,\tau)$ and every $j \in \mathbb{N}$,

$$\begin{aligned}
\|S_j\|([s,t]\times\mathbb{R}^d) &= \mathbf{M}(S_j \llcorner ([s,t]\times\mathbb{R}^d)) \\
&\le |s-t| \cdot \operatorname*{ess\,sup}_{r\in[s,t]} \mathbf{M}(S_j(r)) + \mathrm{Var}(S_j;[s,t]) \\
&\le (C + L^*)|s-t|.
\end{aligned}$$

Likewise, we obtain $\|\partial S_j\|([s,t] \times \mathbb{R}^d) \le (C + L^*)|s-t|$. Then, for the measure ν defined above it holds that $\nu(\{t\} \times \mathbb{R}^d) = 0$ for *all* $t \in [\sigma, \tau)$. Consequently, the same argument as before yields that $S_j(t) \overset{*}{\rightharpoonup} S(t)$ in $\mathrm{I}_k(\overline{\Omega})$ at every $t \in [\sigma, \tau)$.

Finally, for $s, t \in [\sigma, \tau]$ it holds that

$$\mathrm{Var}(S; [s,t]) \le \liminf_{j\to\infty} \mathrm{Var}(S_j; [s,t]) \le L|s-t|$$

by the same argument based on Reshetnyak's theorem as above, where $L := \liminf_{j\to\infty} L_j$; similarly for the boundary variation. In particular, $S \in \mathrm{I}^{\mathrm{Lip}}_{1+k}([\sigma,\tau] \times \overline{\Omega})$ with Lipschitz constant bounded by L. □

Corollary 6.24. *Assume that* $S_j \overset{*}{\rightharpoonup} S$ *in* $\mathrm{I}_{1+k}([\sigma,\tau] \times \overline{\Omega})$ *and*

$$\|S_j\|_{\mathrm{L}^\infty([\sigma,\tau])} + \|\partial S_j\|_{\mathrm{L}^\infty([\sigma,\tau])} \le C < \infty$$

for all $j \in \mathbb{N}$*. Then,* $S_j \overset{*}{\rightharpoonup} S$ *in BV.*

Proof. Since $S_j \overset{*}{\rightharpoonup} S$ and $\partial S_j \overset{*}{\rightharpoonup} \partial S$ in the sense of measures (which follows from the weak* convergence as currents), we have that $\mathbf{M}(S_j) + \mathbf{M}(\partial S_j) \le C$ for some constant $C > 0$. Then also $\mathrm{Var}(S_j; [\sigma,\tau]) + \mathrm{Var}(\partial S_j; [\sigma,\tau]) \le C$ by (6.26) and so the assumptions of the preceding theorem are satisfied. Since the limit is already determined, we get $S_j \overset{*}{\rightharpoonup} S$ in BV. □

6.6 Deformation and connectivity theorems

In this section we establish a version of the deformation theorem (see [19, 4.2.9] or [23, Section 7.7]) that is adapted to our BV-theory of integral currents. Let us emphasize that this theorem requires the current being approximated to be integral and boundaryless. Also recall our standing assumption that $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain.

Theorem 6.25 (Space-time deformation). *Let* $T \in \mathrm{I}_k(\overline{\Omega})$ *with* $\partial T = 0$*. Then, for all* $\varrho > 0$ *there exists* $S \in \mathrm{I}^{\mathrm{Lip}}_{1+k}([0,1] \times \overline{\Omega'})$*, where* $\Omega' := \Omega + B_{(\sqrt{d}+1)\varrho}$*, such that*

$$\partial S = \delta_1 \times P - \delta_0 \times T, \qquad P = \sum_{F \in \mathcal{F}_k(\varrho)} p_F [\![F]\!], \qquad \partial P = 0.$$

Here, $[\![F]\!]$ *is the integral current associated to an oriented* k*-face* $F \in \mathcal{F}_k(\varrho)$ *of one of the cubes* $\varrho z + (0,\varrho)^d$ *(with unit multiplicity and a fixed choice of orientation),* $z \in \mathbb{Z}^d$*, and* $p_F \in \mathbb{Z}$*. Moreover,*

$$\begin{aligned}\mathbf{M}(P) &\le C\mathbf{M}(T),\\ \mathrm{Var}(S) &\le C\varrho\mathbf{M}(T),\\ \|S\|_{\mathrm{L}^\infty([0,1])} &\le C\mathbf{M}(T).\end{aligned}$$

Here, the constant $C > 0$ depends only on the dimensions.

Proof. It suffices to prove the theorem for $\varrho = 1$; the general case is reduced to $\varrho = 1$ by scaling. Indeed, setting $r^\alpha(x) := \alpha x$ for $\alpha > 0$ and $x \in \mathbb{R}^d$, we may apply the result in the version for $\varrho = 1$ to $\widetilde{T} := r^{1/\varrho}_* T$ (in a suitable domain) to obtain $\widetilde{P}, \widetilde{S}$ as in the statement of the theorem for $\varrho = 1$. Then, set $P := r^\varrho_* \widetilde{P}$, $S := r^\varrho_* \widetilde{S}$ (or, more verbosely, $S := [(t,x) \mapsto (t, \varrho x)]_* \widetilde{S}$). These P, S satisfy the conclusion of the theorem for our ϱ since P, T and $S(t)$ (for a.e. $t \in [0,1]$) have the same dimension k and S has dimension $1+k$, whereby

$$\mathrm{Var}(S) = \varrho^{1+k}\,\mathrm{Var}(\widetilde{S}) \le C\varrho^{1+k}\mathbf{M}(\widetilde{T}) = C\varrho\mathbf{M}(T).$$

One quick way to see the first equality is to observe that

$$\mathrm{Var}(S) = \varrho^{1+k}\,\mathrm{Var}\big([(t,x) \mapsto (t/\varrho, x/\varrho)]_* S; [0, 1/\varrho]\big) = \varrho^{1+k}\,\mathrm{Var}(\widetilde{S})$$

by the area formula and Lemma 6.16.

So, in the following let $\varrho = 1$. Inspecting the proof of the standard deformation theorem, in the version of [23, Sections 7.7, 7.8], say, we observe that in the present situation of *boundaryless* integral currents the proof proceeds by constructing a homotopy from T to a P of the form

$$P = \sum_{F \in \mathcal{F}_k(1)} p_F [\![F]\!]$$

with

$$\partial P = 0, \qquad \mathbf{M}(P) \le C\mathbf{M}(T).$$

We remark in particular that in the last step of the proof of the deformation theorem (as in [23, Section 7.8]) we do not need to modify the retraction onto any k-face since $\partial T = 0$ (by the constancy theorem, see [23, Proposition 7.3.5]) and P is indeed a homotopical image of T. The homotopy constructed is seen to be the concatenation of two affine homotopies: The first affine homotopy, call it H_1, goes from the identity to a translation $t^a(x) := x + a$ ($|a| < 1$). The second affine homotopy, H_2, goes from the identity to the "radial" retraction ψ onto the k-skeleton (defined in [23, Section 7.7]).

We have $H_1(1, \cdot)_* T = t^a_* T$ and

$$S_{H_1} := (\overline{H}_1)_* T \in \mathrm{I}^{\mathrm{Lip}}_{1+k}([0,1] \times \overline{\Omega + B_1}),$$

where $\overline{H}_1(t,x) := (t, H_1(t,x))$. From Lemma 6.21 we obtain

$$\mathrm{Var}(S_{H_1}) \le C\mathbf{M}(T).$$

Moreover, it can be shown (see [23, top of p. 218]) that a may be chosen such that

$$\int |D\psi|^k \,\mathrm{d}\|t^a_* T\| \le C\mathbf{M}(T).$$

Thus, from Lemma 6.21 we get for

$$S_{H_2} := (\overline{H}_2)_*[t^a_* T] \in \mathrm{I}^{\mathrm{Lip}}_{1+k}([0,1] \times \overline{\Omega + B_{1+\sqrt{d}}}),$$

where $\overline{H}_2(t,x) := (t, H_2(t,x))$, that also

$$\mathrm{Var}(S_{H_2}) \le C\mathbf{M}(T).$$

Once we concatenate S_{H_1} and S_{H_2} via Lemma 6.18, we obtain that for

$$S := S_{H_2} \circ S_{H_1} \in \mathrm{I}^{\mathrm{Lip}}_{1+k}([0,1] \times \overline{\Omega'})$$

it holds that $\partial S = \delta_1 \times P - \delta_0 \times T$ and

$$\mathrm{Var}(S) \le C\mathbf{M}(T).$$

The statement about the essential mass bound on $S(t)$ also follows from the estimates of Lemmas 6.21, 6.18. This finishes the proof. □

As a corollary, we obtain the following version of the isoperimetric inequality:

Theorem 6.26 (Isoperimetric inequality). *Let $T \in \mathrm{I}_k(\overline{\Omega})$, $k \ge 1$, with $\partial T = 0$. Then, there exists $S \in \mathrm{I}^{\mathrm{Lip}}_{1+k}([0,1] \times \overline{\Omega'})$, where $\Omega' := \Omega + B_{C\mathbf{M}(T)^{1/k}}$, such that*

$$\partial S = -\delta_0 \times T$$

and

$$\mathrm{Var}(S;[0,1]) \le C\mathbf{M}(T)^{(k+1)/k}, \qquad \|S\|_{\mathrm{L}^\infty([0,1])} \le C\mathbf{M}(T).$$

Here, the constant $C > 0$ depends only on the dimensions.

Proof. The proof is similar to the one for the classical isoperimetric inequality and follows immediately from the deformation theorem: Assuming that $T \neq 0$, we let P, S as in the deformation theorem with

$$\varrho := [2C\mathbf{M}(T)]^{1/k},$$

where $C > 0$ is the constant from said theorem. By a scaling argument, $\mathbf{M}(P) = N(\varrho)\varrho^k$ for some nonnegative integer $N(\varrho)$. From the estimates in the deformation theorem we have $\mathbf{M}(P) \le C\mathbf{M}(T)$ and thus

$$N(\varrho) \cdot 2C\mathbf{M}(T) = \mathbf{M}(P) \le C\mathbf{M}(T).$$

So, $2N(\varrho) \le 1$, whereby $N(\varrho) = 0$, and hence $P = 0$. This immediately yields all the claimed statements. □

The following is a metric measuring the distance between two boundaryless integral k-currents via progressive-in-time deformations, namely Lipschitz integral currents: For $T_0, T_1 \in \mathrm{I}_k(\overline{\Omega})$ with $\partial T_0 = \partial T_1 = 0$, the **(Lipschitz) deformation distance** between T_0 and T_1 is

$$\mathrm{dist}_{\mathrm{Lip},\overline{\Omega}}(T_0, T_1) := \inf\Big\{ \mathrm{Var}(S) \,:\, S \in \mathrm{I}^{\mathrm{Lip}}_{1+k}([0,1] \times \overline{\Omega}) \text{ with } \partial S = \delta_1 \times T_1 - \delta_0 \times T_0 \Big\}.$$

That $\mathrm{dist}_{\mathrm{Lip},\overline{\Omega}}(\cdot,\cdot)\colon \mathrm{I}_k(\overline{\Omega}) \times \mathrm{I}_k(\overline{\Omega}) \to [0,\infty]$ is positive definite, symmetric, and obeys the triangle inequality follows immediately from Lemmas 6.18, 6.19 and the fact that $\mathrm{Var}(S) = 0$ for $S \in \mathrm{I}^{\mathrm{Lip}}_{1+k}([0,1] \times \overline{\Omega})$ with $\partial S = \delta_1 \times T_1 - \delta_0 \times T_0$ implies that $T_0 = T_1$. We remark that $\mathrm{dist}_{\mathrm{Lip},\overline{\Omega}}(\cdot,\cdot)$ is not necessarily finite if $\overline{\Omega}$ has holes that can be detected by boundaryless integral k-currents.

With regard to the notion of convergence induced by the (Lipschitz) deformation distance, we have the following **space-time connectivity theorem**:

Theorem 6.27. *For every $M > 0$ and T_j, T ($j \in \mathbb{N}$) in the set*

$$\big\{ T \in \mathrm{I}_k(\overline{\Omega}) \,:\, \partial T = 0,\ \mathbf{M}(T) \le M \big\}$$

the following equivalence holds (as $j \to \infty$):

$$\mathrm{dist}_{\mathrm{Lip},\overline{\Omega}}(T_j, T) \to 0 \qquad \textit{if and only if} \qquad T_j \overset{*}{\rightharpoonup} T \quad \textit{in } \mathrm{I}_k(\overline{\Omega}).$$

Moreover, in this case, for all j from a subsequence of the j's, there are $S_j \in \mathrm{I}^{\mathrm{Lip}}_{1+k}([0,1] \times \overline{\Omega})$ with

$$\partial S_j = \delta_1 \times T - \delta_0 \times T_j, \qquad \mathrm{dist}_{\mathrm{Lip},\overline{\Omega}}(T_j, T) \le \mathrm{Var}(S_j) \to 0,$$

and

$$\limsup_{j\to\infty} \|S_j\|_{\mathrm{L}^\infty([0,1])} \le C \cdot \limsup_{l\to\infty} \mathbf{M}(T_l).$$

Here, the constant $C > 0$ depends only on the dimensions and on Ω.

Proof. For the first direction, assume $\mathrm{dist}_{\mathrm{Lip},\overline{\Omega}}(T_j, T) \to 0$. For any $S \in \mathrm{I}^{\mathrm{Lip}}_{1+k}([0,1] \times \overline{\Omega})$ with $\partial S = \delta_1 \times T_j - \delta_0 \times T_0$ it holds that $\partial \mathbf{p}_* S = T_j - T_0$ and so,

$$\mathbf{FI}^\circ(T_j - T) \le \mathbf{M}(\mathbf{p}_* S) \le \mathrm{Var}(S)$$

by the definition of the variation. Taking the supremum over all such S,

$$\mathbf{F}^\circ(T_j - T) \le \mathbf{FI}^\circ(T_j - T) \le \mathrm{dist}_{\mathrm{Lip},\overline{\Omega}}(T_j, T) \to 0.$$

Then, the claim $T_j \overset{*}{\rightharpoonup} T$ follows as the first (easier) part of Proposition 5.12, namely as follows: For $\omega \in \mathcal{D}^k(\mathbb{R}^d)$,

$$\bigl|\langle T_j - T, \omega\rangle\bigr| \le \mathbf{F}^\circ(T_j - T) \cdot \|\mathrm{d}\omega\|_\infty \to 0.$$

For the other direction, assume $T_j \overset{*}{\rightharpoonup} T$ in $\mathrm{I}_k(\overline{\Omega})$ with $\partial T_j = \partial T = 0$ and $M := \sup_j \mathbf{M}(T_j) < \infty$. We need to show that

$$\mathrm{dist}_{\mathrm{Lip},\overline{\Omega}}(T_j, T) \to 0. \tag{6.38}$$

The first step is to observe that for all $N \in \mathbb{N}$ sufficiently large there exists a finite collection $\mathbf{P}_N \subset \mathrm{I}_k(\overline{\Omega})$ such that for all $\widehat{T} \in \mathrm{I}_k(\overline{\Omega})$ with $\partial\widehat{T} = 0$ and $\mathbf{M}(\widehat{T}) \le M$ it holds that

$$\mathrm{dist}_{\mathrm{Lip},\overline{\Omega}}(\widehat{T}, P) < C2^{-N} \text{ for some } P \in \mathbf{P}_N, \tag{6.39}$$

where the constant $C > 0$ and the lower bound for N depend only on the dimensions and the domain Ω. We first claim that (6.39) holds (with $C = 1$ and for all $N \in \mathbb{N}$) for $\mathrm{dist}_{\mathrm{Lip},\overline{\Omega'}}$, where we have set $\Omega' := \Omega + B_{(\sqrt{d}+1)\varrho} \Supset \Omega$ is as in our deformation theorem, Theorem 6.25, with $\varrho := 2^{-N}/(C_{d,k}M)$ (with $C_{d,k}$ the constant from the deformation theorem). Indeed, for $\mathbf{P}_N$ we take the collection of all polyhedral chains P that can possibly satisfy the conclusion of the deformation theorem for a $\widehat{T}$ as above, which is clearly a finite set. Thus, (6.39) is established in $\overline{\Omega'}$.

Next, for N sufficiently large (how large only depending on Ω), we may retract $\overline{\Omega'}$ to $\overline{\Omega}$. In this context recall that Ω is always assumed to be a bounded Lipschitz domain and hence a Lipschitz neighborhood retract, i.e., there exists a Lipschitz map that retracts some neighborhood of $\overline{\Omega}$ onto $\overline{\Omega}$. In fact, since Ω was assumed to be bounded and to have a (strong) Lipschitz boundary, one may proceed by observing that Ω is a Lipschitz manifold and thus a Lipschitz neighborhood retract, see, e.g., [25, Theorem 5.13 and Remark 3.2 (3)]). Thus, (6.39) also holds for $\mathrm{dist}_{\mathrm{Lip},\overline{\Omega}}$ and with $\mathbf{P}_N$ containing the retracts of the polyhedral chains. Note that the retraction itself only contributes a bounded factor to the estimate of the variation.

Returning to our sequence (T_j), for every $N \in \mathbb{N}$ sufficiently large we find a $P \in \mathbf{P}_N$ such that $\mathrm{dist}_{\mathrm{Lip},\overline{\Omega}}(T_j, P) < C2^{-N}$ for infinitely many j's. Applying this argument repeatedly and selecting a subsequence at every step (such that the constraint holds for all elements of that subsequence), we may find a diagonal subsequence, still denoted by (T_j), such that

$$\mathrm{dist}_{\mathrm{Lip},\overline{\Omega}}(T_l, P_j) < 2^{-(j+1)} \qquad \text{for all } l \ge j \text{ and a } P_j \in \bigcup_N \mathbf{P}_N \subset \mathrm{I}_k(\overline{\Omega}).$$

By the construction above P_j is the Lipschitz retract of a polyhedral chain. Then, via the triangle inequality,

$$\mathrm{dist}_{\mathrm{Lip},\overline{\Omega}}(T_j, T_{j+1}) < 2^{-j}.$$

Hence, there exists an $R_j \in \mathrm{I}^{\mathrm{Lip}}_{1+k}([0,1] \times \overline{\Omega})$ with

$$\partial R_j = \delta_1 \times T_{j+1} - \delta_0 \times T_j, \qquad \mathrm{Var}(R_j) < 2^{-j}.$$

Using the space-time currents constructed in the proof of the deformation theorem as witnesses for $\mathrm{dist}_{\mathrm{Lip},\overline{\Omega}}(T_j, P_j) < 2^{-(j+1)}$ and $\mathrm{dist}_{\mathrm{Lip},\overline{\Omega}}(P_j, T_{j+1}) < 2^{-(j+1)}$ and concatenating them via Lemma 6.18 to obtain R_j, we may further require

$$\|R_j\|_{\mathrm{L}^\infty([0,1])} \leq C \cdot \max\bigl\{\mathbf{M}(T_j), \mathbf{M}(T_{j+1})\bigr\}.$$

For the concatenation of the R_l for $l = j, \dots, j+m-1$, that is,

$$S_j^m := R_{j+m-1} \circ R_{j+m-2} \circ \cdots \circ R_j,$$

see again Lemma 6.18, it holds that

$$\partial S_j^m = \delta_1 \times T_{j+m} - \delta_0 \times T_j.$$

and

$$\begin{aligned} \mathrm{Var}(S_j^m) &= \sum_{l=0}^{m-1} \mathrm{Var}(R_{j+l}) \leq 2^{-j+1}, \\ \mathrm{Var}(\partial S_j^m) &= \mathbf{M}(T_j) + \mathbf{M}(T_{j+m}) \leq 2M, \\ \|S_j^m\|_{\mathrm{L}^\infty([0,1])} &\leq C \cdot \max_{l=j,\dots,j+m} \mathbf{M}(T_l) \leq C \cdot \sup_{l \geq j} \mathbf{M}(T_l). \end{aligned}$$

Moreover, via Lemma 6.16 we may rescale S_j^m in time (which we do not make explicit in our notation) to assume

$$\mathrm{Var}(S_j^m; [0,t]) = t \, \mathrm{Var}(S_j^m), \qquad t \in [0,1].$$

In this way, also the Lipschitz constants of S_j^m are uniformly in m bounded by 2^{-j+1}.

We now pass to the limit $m \to \infty$. Via Theorem 6.23 this yields $S_j \in \mathrm{I}^{\mathrm{Lip}}_{1+k}([0,1] \times \overline{\Omega})$ with

$$\partial S_j = \delta_1 \times \Bigl(\operatorname*{w^*\text{-}lim}_{m\to\infty} T_{j+m}\Bigr) - \delta_0 \times T_j = \delta_1 \times T - \delta_0 \times T_j$$

and

$$\begin{aligned} \mathrm{Var}(S_j) &\leq 2^{-j+1}, \\ \|S_j\|_{\mathrm{L}^\infty([0,1])} &\leq C \cdot \sup_{l \geq j} \mathbf{M}(T_l). \end{aligned}$$

Our S_j is admissible in the definition of the metric $\mathrm{dist}_{\mathrm{Lip},\overline{\Omega}}(\cdot,\cdot)$ and so,

$$\mathrm{dist}_{\mathrm{Lip},\overline{\Omega}}(T_j, T) \leq \mathrm{Var}(S_j) \to 0 \qquad \text{as } j \to \infty.$$

In this way we can find for every subsequence of the original sequence $(T_j)_j$ (before taking the repeated subsequences above) a further subsequence that converges in the $\mathrm{dist}_{\mathrm{Lip},\overline{\Omega}}$-metric to T. Hence, also $\mathrm{dist}_{\mathrm{Lip},\overline{\Omega}}(T_j, T) \to 0$ for the original sequence, proving our claim (6.38).

Finally, taking the upper limit of the mass estimate,

$$\limsup_{j\to\infty} \|S_j\|_{\mathrm{L}^\infty([0,1])} \leq C \cdot \limsup_{l\to\infty} \mathbf{M}(T_l).$$

This finishes the proof. $\square$

Chapter 7

Fine Properties of Space-Time Currents

In this chapter we explore in detail the relationship between a space-time current and its slices. In particular, there are two different decompositions of a space-time current, namely the one given by current slicing and the one given by the measure-theoretic disintegration (see Theorem 1.5 and Remark 1.7). Comparing them, we can understand whether or not a given space-time current is essentially just a collection of slices, or if there are other singular phenomena present (which may indeed happen).

7.1 Disintegration

Given a space-time integral current $S \in \mathrm{I}_{1+k}(\mathbb{R}^{1+d})$, define the positive measure

$$\lambda := \mathcal{L}^1 + \lambda^s, \qquad \lambda^s := (\mathbf{t}_\# \|S\|)^s.$$

Trivially, $\mathbf{t}_\# \|S\| \ll \lambda$, so that we may consider the (unique) disintegration of the mass measure $\|S\|$ with respect to $\mathbf{t}$ and λ according to Theorem 1.5 and Remark 1.7, namely

$$\|S\| = \lambda(\mathrm{d}t) \otimes \mu_t \tag{7.1}$$

with

$$\mu_t \in \mathcal{M}\big(\{t\} \times \mathbb{R}^d; \textstyle\bigwedge_{1+k} \mathbb{R}^{1+d}\big) \qquad \text{for } \lambda\text{-a.e. } t \in \mathbb{R}.$$

Also set

$$\mu_t^* := \mu_t \llcorner \mathrm{Crit}(S).$$

Theorem 7.1 (Disintegration structure). *Let $S \in \mathrm{I}_{1+k}(\mathbb{R}^{1+d})$ and let $\|S\| = \lambda(\mathrm{d}t) \otimes \mu_t$ be the disintegration of the mass measure as in* (7.1). *Then:*

(i) For $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ the measure μ_t^* is singular with respect to $|\nabla^S \mathbf{t}|^{-1} \|S|_t\|$ and μ_t can be decomposed as

$$\mu_t = |\nabla^S \mathbf{t}|^{-1} \|S|_t\| + \mu_t^*,$$

(ii) For λ^s-a.e. $t \in \mathbb{R}$ the measure μ_t is concentrated on $\mathrm{Crit}(S)$, so $\mu_t^* = \mu_t$.

Therefore, the disintegration structure formula

$$\|S\| = \mathrm{d}t \otimes \left(|\nabla^S \mathbf{t}|^{-1} \|S|_t\| + \mu_t^*\right) + \lambda^s(\mathrm{d}t) \otimes \mu_t^* \tag{7.2}$$

holds.

To establish this theorem, we first prove the following two lemmas.

Lemma 7.2. *Let $S \in \mathrm{I}_{1+k}(\mathbb{R}^{1+d})$. For the measures*

$$\gamma := |\nabla^S \mathbf{t}| \, \|S\|, \qquad \Gamma := \nabla^S \mathbf{t} \, \|S\|$$

the following statements hold:

(i) $\mathbf{t}_\# \gamma \ll \mathcal{L}^1$ and $\gamma = \mathrm{d}t \otimes \|S|_t\|$.

(ii) $\mathbf{t}_\# \|\Gamma\| \ll \mathcal{L}^1$ and $\Gamma = \mathrm{d}t \otimes (\xi \, \|S|_t\|)$.

Proof. Ad (i). Let $N \subset \mathbb{R}$ be an $\mathcal{L}^1$-null set. Then, applying the coarea formula for slices from Proposition 6.2 with $g := \mathbb{1}_{N \times \mathbb{R}^d}$,

$$\begin{aligned}
\mathbf{t}_\# \gamma(N) &= \gamma(N \times \mathbb{R}^d) \\
&= \int_{\mathbb{R}^{1+d}} \mathbb{1}_{N \times \mathbb{R}^d} \, |\nabla^S \mathbf{t}| \, \mathrm{d}\|S\| \\
&= \int_{\mathbb{R}} \int_{\{t\} \times \mathbb{R}^d} \mathbb{1}_{N \times \mathbb{R}^d} \, \mathrm{d}\|S|_t\| \, \mathrm{d}t \\
&= \int_{\mathbb{R}} \mathbb{1}_N(t) \left(\int_{\{t\} \times \mathbb{R}^d} \mathrm{d}\|S|_t\| \right) \mathrm{d}t \\
&= 0, ,
\end{aligned}$$

Hence, $\mathbf{t}_\# \gamma \ll \mathcal{L}^1$.

By the disintegration procedure of Theorem 1.5 and the remark following it (for $\lambda = \mathcal{L}^1$), we may write

$$\gamma = \mathrm{d}t \otimes \gamma_t.$$

Combining this with the coarea formula for slices from Proposition 6.2, we obtain for all Borel sets $A \subset \mathbb{R}$, $B \subset \mathbb{R}^d$ that

$$\int_A \|S|_t\|(B)\,\mathrm{d}t = \int_{A\times B} |\nabla^S \mathbf{t}|\,\mathrm{d}\|S\| = \gamma(A \times B) = \int_A \gamma_t(B)\,\mathrm{d}t.$$

Since A, B are arbitrary, (i) follows from the uniqueness of the disintegration.

Ad (ii). Clearly, $\|\Gamma\| = \gamma$, whereby $\mathbf{t}_\#\|\Gamma\| = \mathbf{t}_\#\gamma \ll \mathcal{L}^1$ follows directly from (i). For the second claim we observe for any $\varphi \in \mathrm{C}_c(\mathbb{R}^{1+d}; \mathbb{R} \times \mathbb{R}^d)$ tand with ξ defined in (6.18) that

$$\begin{aligned}\int_{\mathbb{R}^{1+d}} \varphi \cdot \mathrm{d}\Gamma &= \int_{\mathbb{R}^{1+d}} \varphi \cdot \nabla^S \mathbf{t}\,\mathrm{d}\|S\| \\ &= \int_{\mathbb{R}^{1+d}} \varphi \cdot \xi\,|\nabla^S \mathbf{t}|\,\mathrm{d}\|S\| \\ &= \int \int_{\{t\}\times\mathbb{R}^d} \varphi \cdot \xi\,\mathrm{d}\|S|_t\|\,\mathrm{d}t,\end{aligned}$$

where the last line follows from (i). We conclude again via the uniqueness of the disintegration. □

Lemma 7.3. *Let $S \in \mathrm{I}_{1+k}(\mathbb{R}^{1+d})$. For the measures*

$$\sigma := \|S\| \llcorner \mathrm{Reg}(S), \qquad \Sigma := S \llcorner \mathrm{Reg}(S)$$

the following statements hold:

(i) $\mathbf{t}_\#\sigma \ll \mathcal{L}^1$ *and* $\sigma = \mathrm{d}t \otimes (|\nabla^S \mathbf{t}|^{-1}\,\|S|_t\|)$.

(ii) $\mathbf{t}_\#\|\Sigma\| \ll \mathcal{L}^1$ *and* $\Sigma = \mathrm{d}t \otimes (|\nabla^S \mathbf{t}|^{-1}\,\xi \wedge S|_t)$.

Proof. Ad (i). Observe first that

$$\mathbb{1}_{\mathrm{Reg}(S)} = 1 \qquad (\mathrm{d}t \otimes \|S|_t\|)\text{-a.e.}$$

by virtue of Lemma 6.8 (i). Similarly to the proof of the previous lemma, we then apply the coarea formula for slices from Proposition 6.2 with $g := |\nabla^S \mathbf{t}|^{-1}\mathbb{1}_{(N\times\mathbb{R}^d)\cap\mathrm{Reg}(S)}$ to compute for any Borel set $A \subset \mathbb{R}^{1+d}$,

$$\begin{aligned}\sigma(A) &= \|S\|(A \cap \mathrm{Reg}(S)) \\ &= \int \int_{\{t\}\times\mathbb{R}^d} \mathbb{1}_{A\cap\mathrm{Reg}(S)}\,|\nabla^S \mathbf{t}|^{-1}\,\mathrm{d}\|S|_t\|\,\mathrm{d}t \\ &= \int \int_{\{t\}\times\mathbb{R}^d} \mathbb{1}_A\,|\nabla^S \mathbf{t}|^{-1}\,\mathrm{d}\|S|_t\|\,\mathrm{d}t.\end{aligned}$$

In particular,

$$\mathbf{t}_{\#}\sigma = \big(|\nabla^S \mathbf{t}|^{-1}\, \|S|_t\|\big)\,(\mathbb{R}\times\mathbb{R}^d)\,\mathcal{L}^1(\mathrm{d}t)$$

and hence $\mathbf{t}_{\#}\sigma \ll \mathcal{L}^1$. Then,

$$\sigma = \mathrm{d}t \otimes \big(|\nabla^S \mathbf{t}|^{-1}\, \|S|_t\|\big)$$

must be the (unique) disintegration of σ with respect to $\mathcal{L}^1$, which concludes the proof of (i).

Ad (ii). Since $\|\Sigma\| = \sigma$, from (i) we immediately see that $\mathbf{t}_{\#}\|\Sigma\| = \mathbf{t}_{\#}\sigma \ll \mathcal{L}^1$. Then take $\varphi \in \mathrm{C}_c(\mathbb{R}^{1+d};\mathbb{R}\times\mathbb{R}^d)$ and compute

$$\begin{aligned}\int_{\mathbb{R}^{1+d}} \varphi\, \mathrm{d}\Sigma &= \int_{(\mathbb{R}^{1+d})\cap \mathrm{Reg}(S)} \varphi\, \mathrm{d}S \\ &= \int_{\mathbb{R}^{1+d}} \varphi\cdot\vec{S}\, \mathrm{d}\sigma.\end{aligned}$$

By (i) we thus obtain for the disintegration $\Sigma = \mathrm{d}t\otimes\Sigma_t$ with

$$\Sigma_t = \vec{S}\,|\nabla^S \mathbf{t}|^{-1}\,\|S|_t\| = (\xi\wedge\vec{S}|_t)\,|\nabla^S \mathbf{t}|^{-1}\,\|S|_t\| = \frac{1}{|\nabla^S \mathbf{t}|}\,\xi\wedge S|_t$$

for $\mathcal{L}^1$-a.e. $t\in\mathbb{R}$, where we have also used (6.21). This shows the second claim of (ii). $\square$

Proof of Theorem 7.1. Summing up the the insights from the preceding two lemmas, we have the following relations:

$$\begin{aligned}|\nabla^S \mathbf{t}|\,\|S\| &= \mathrm{d}t\otimes\|S|_t\|, & \|S\|\llcorner\mathrm{Reg}(S) &= \mathrm{d}t\otimes\big(|\nabla^S \mathbf{t}|^{-1}\,\|S|_t\|\big),\\ \nabla^S \mathbf{t}\,\|S\| &= \mathrm{d}t\otimes(\xi\,\|S|_t\|), & S\llcorner\mathrm{Reg}(S) &= \mathrm{d}t\otimes\big(|\nabla^S \mathbf{t}|^{-1}\,\xi\wedge S|_t\big).\end{aligned}$$

Write $\|S\| = \sigma+\nu$ with $\sigma := \|S\|\llcorner\mathrm{Reg}(S)$ and $\nu := \|S\|\llcorner\mathrm{Crit}(S)$. From Lemma 7.3 we obtain the relations

$$\begin{aligned}\sigma_t &= |\nabla^S \mathbf{t}|^{-1}\,\|S|_t\| && \text{for } \mathcal{L}^1\text{-a.e. } t\in\mathbb{R},\\ \sigma_t &= 0 && \text{for } \lambda^s\text{-a.e. } t\in\mathbb{R}.\end{aligned}$$

Denoting by $\nu = \mathrm{d}t\otimes\nu_t$ the disintegration of ν with respect to λ,

$$\begin{aligned}\|S\| &= \sigma+\nu\\ &= \mathrm{d}t\otimes\big(|\nabla^S \mathbf{t}|^{-1}\,\|S|_t\|\big) + \lambda(\mathrm{d}t)\otimes\nu_t\\ &= \mathrm{d}t\otimes\big(|\nabla^S \mathbf{t}|^{-1}\,\|S|_t\| + \nu_t\big) + \lambda^s(\mathrm{d}t)\otimes\nu_t.\end{aligned}$$

Comparing this to (7.1), we obtain

$$\begin{aligned}\mu_t &= |\nabla^S \mathbf{t}|^{-1}\,\|S|_t\| + \nu_t && \text{for } \mathcal{L}^1\text{-a.e. } t\in\mathbb{R},\\ \mu_t &= \nu_t && \text{for } \lambda^s\text{-a.e. } t\in\mathbb{R}.\end{aligned}$$

In particular, since the ν_t are concentrated in $\mathrm{Crit}(S)$ for λ^s-a.e. $t \in \mathbb{R}$, the same holds for μ_t, which immediately yields (ii).

For the absolutely continuous parts, observe that $\mu_t^* = \nu_t$ and that this measure is singular with respect to $|\nabla^S \mathbf{t}|^{-1} \|S|_t\|$ by Lemma 6.8 (i). So,

$$\mu_t = |\nabla^S \mathbf{t}|^{-1} \|S|_t\| + \mu_t^*.$$

This shows (i). Finally, (7.2) is a direct consequence of (i) and (ii). □

Remark 7.4. One can also show that μ_t^* is singular with respect to $\mathcal{H}^k$ for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$. For this, notice that the set $\mathrm{Crit}(S)$ is $(k+1)$-rectifiable, being a subset of (the carrier of) S. By the coarea formula of Theorem 2.7, the set

$$C_t := \mathrm{Crit}(S) \cap (\{t\} \times \mathbb{R}^d)$$

is k-rectifiable for $\mathcal{L}^1$-a.e. t and

$$\int \mathcal{H}^k(C_t)\, \mathrm{d}t = \int_C |\nabla^S \mathbf{t}|\, \mathrm{d}\mathcal{H}^{k+1} = 0.$$

So, $\mathcal{H}^k(C_t) = 0$ for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$. Since, by definition, μ_t^* is concentrated on $\mathrm{Crit}(S) \cap (\{t\} \times \mathbb{R}^d) = C_t$, we conclude that μ_t^* is indeed singular to $\mathcal{H}^k$ for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$. Moreover μ_t^* is concentrated on $\mathrm{Crit}(S) \cap (\{t\} \times \mathbb{R}^d)$, which is $\mathcal{H}^k$-null for $\mathcal{L}^1$-a.e. t.

7.2 Slices and AC space-time currents

An important consideration in much of the theory of space-time integral currents is the relationship between a space-time currents and its slices. Since it is the most important case we will only consider $S \in \mathrm{N}_{k+1}([0,1] \times \mathbb{R}^d)$ with the additional property $\partial W \llcorner (0,1) \times \mathbb{R}^d = 0$, which entails that $\mathcal{L}^1$-almost all time slices are boundaryless.

Intuitively, we have purported that space-time currents are a model for the evolution of its slices. This is clearly not the whole picture since we need to take into account singularities. The simplest ones are jumps, as in the following example.

Example 7.5. Consider $S \in \mathrm{I}_2(\mathbb{R} \times \mathbb{R}^2)$ defined by $S := S_1 + S_2$ for

$$S_1 := [\![0,1]\!] \times C_1, \qquad S_2 := [\![1,2]\!] \times C_2 + \delta_1 \times W,$$

where C_1 and C_2 are circles of radius 1 and 2, respectively, and W is the annulus between them. Hence, $\partial W = C_2 - C_1$ and $\partial S \llcorner (0,1) \times \mathbb{R}^2 = 0$. Then,

$$\mathbf{t}_\# \|S\| = \mathcal{L}^1 \llcorner [0,2] + \delta_1.$$

Consequently, in the disintegration structure formula (7.2), λ^s is non-zero.

If all slices vanish, so we are considering "piecewise constant" evolutions, then jumps are indeed the only type of singularity that can occur:

Lemma 7.6 (Zero-slices lemma). *Let $S \in \mathrm{I}_{1+k}([0,1] \times \mathbb{R}^d)$ with $\partial S \llcorner (0,1) \times \mathbb{R}^d = 0$. Then, the following are equivalent:*

(i) $S|_t = 0$ *for $\mathcal{L}^1$-a.e. $t \in [0,1]$.*

(ii) With $\kappa := \mathbf{t}_\# \|S\|$, for κ-a.e. t there exists $S_t \in \mathrm{I}_{1+k}(\{t\} \times \mathbb{R}^d)$ such that

$$S = \kappa(\mathrm{d}t) \otimes S_t \qquad \text{and} \qquad \partial S_t = 0 \quad \text{for } \kappa\text{-a.e. } t.$$

Moreover, if one of these conditions holds, then κ is purely atomic, so that for an at most countable collection of points $t_j \in [0,1]$,

$$S = \sum_j S_{t_j}, \qquad \vec{S}_{t_j} \in \textstyle\bigwedge_k(\{0\} \times \mathbb{R}^d) \quad \|S_{t_j}\|\text{-a.e.},$$

Proof. Assume first that (i) holds. Disintegrating S with respect to the temporal projection $\mathbf{t}$ via Theorem 1.5 and Remark 1.7, we write

$$S = \kappa(\mathrm{d}t) \otimes S_t$$

with $\kappa := \mathbf{t}_\# \|S\|$ and $S_t \in \mathcal{M}(\{t\} \times \mathbb{R}^d; \bigwedge_{k+1} \mathbb{R}^{1+d})$ for κ-almost every $t \in [0,1]$. By Remark 1.6, for κ-a.e. $t \in (0,1)$,

$$S_t = \mathrm{w}^*\text{-}\lim_{h \to 0} \frac{S \llcorner (t-h, t+h) \times \mathbb{R}^d}{\kappa((t-h, t+h))}.$$

By the cylinder formula (6.13) and the assumption (i) there is a sequence $h_j \to 0$ with

$$\partial \big[S \llcorner (t-h_j, t+h_j) \times \mathbb{R}^d \big] = S|_{t+h_j} - S|_{t-h_j} = 0.$$

Passing to the limit (and using that the boundary operator is weakly* continuous), $\partial S_t = 0$. Since S is integral, it is supported on a $(k+1)$-rectifiable set R, and thus there can be only countably many t's for which $\|S_t\|(R) > 0$ (there can only be finitely many t's such that $\|S_t\|(R) > 1/m$ for any $m \in \mathbb{N}$). Therefore, κ must be atomic, say with carrier set $\{t_1, t_2, \ldots\}$. Since S_t is supported in $\{t\} \times \mathbb{R}^d$, its $\vec{S}_t$ must belong to $\bigwedge_k(\{0\} \times \mathbb{R}^d)$. We can also rescale, so that $\kappa(\{t_j\}) = 1$ for all j. This shows the claimed decompositions of S. As S is integral, so must be the S_{t_j}. More precisely, we can write S_{t_j} as the weak*-limit of a sequence of integral currents as above and apply the Federer–Fleming Theorem 5.6.

For the other implication, assume the decomposition (ii). We use the cylinder formula (6.13) to infer that for $\mathcal{L}^1$-a.e. $t \in [0,1]$ and any $\omega \in \mathcal{D}^k(\mathbb{R}^d)$,

$$\begin{aligned}\langle S|_t, \omega\rangle &= \langle \partial(S \llcorner \{\mathbf{t} < t\}) - (\partial S) \llcorner \{\mathbf{t} < t\}, \omega\rangle \\ &= \langle S \llcorner \{\mathbf{t} < t\}, \mathrm{d}\omega\rangle \\ &= \sum_j \langle S_{t_j}, \mathrm{d}\omega\rangle \\ &= 0\end{aligned}$$

because $\partial S_t = 0$ for κ-a.e. t and hence also $\partial S = 0$. □

Example 7.7. One non-zero $S \in \mathrm{I}_2([0,1] \times \mathbb{R}^3)$ that satisfies the assumptions of the previous lemma is

$$S = \delta_{1/2} \times [\![\mathbb{S}^2]\!],$$

where $[\![\mathbb{S}^2]\!]$ is the (oriented) 2-sphere in $\mathbb{R}^3$.

In the zero-slices lemma we have that $\mathbf{t}_\#(\|S\| \llcorner \mathrm{Crit}(S))$ is singular with respect to $\mathcal{L}^1$ and $\mathrm{Var}(S; \{t_j\}) = \mathbf{M}(S_{t_j}) > 0$.

As the preceding example shows, this does not exclude transient effects, where a current flashes in and out of existence in an instant. To avoid this, we say that an integral current $S \in \mathrm{I}_{1+k}([0,1] \times \mathbb{R}^d)$ satisfies the **AC (absolute-continuity) condition** if

$$\mathbf{t}_\#(\|S\| \llcorner \mathrm{Crit}(S)) \ll \mathcal{L}^1,$$

where $\mathrm{Crit}(S)$ is the critical set of S. In the disintegration structure formula (7.2) of Theorem 7.1 this corresponds to $\lambda^s = 0$.

The AC condition indeed prevents jumps in the variation:

Lemma 7.8. *Let $S \in \mathrm{I}_{1+k}([0,1] \times \mathbb{R}^d)$ with $\partial S \llcorner (0,1) \times \mathbb{R}^d = 0$. Then,*

$$S \text{ is AC} \qquad \text{if and only if} \qquad \mathrm{Var}(S; \cdot) \ll \mathcal{L}^1.$$

Proof. Assume first that $\mathrm{Var}(S; \cdot) \ll \mathcal{L}^1$. For every $(t,x) \in \mathrm{Crit}(S)$ we have $|\mathbf{p}(\vec{S}(t,x))| = 1$. So, for every open interval $(a,b) \subset [0,1]$,

$$\begin{aligned}\mathbf{t}_\#(\|S\| \llcorner \mathrm{Crit}(S))((a,b)) &= \|S\|(\mathrm{Crit}(S) \cap ((a,b) \times \mathbb{R}^d)) \\ &= \int_{\mathrm{Crit}(S) \cap ((a,b) \times \mathbb{R}^d)} \|\mathbf{p}(\vec{S})\| \,\mathrm{d}\|S\| \\ &\le \int_{(a,b) \times \mathbb{R}^d} \|\mathbf{p}(\vec{S})\| \,\mathrm{d}\|S\| \\ &= \mathrm{Var}(S, (a,b)).\end{aligned}$$

This immediately implies the claim

Suppose now that S is AC, i.e., $\mathbf{t}_\#(\|S\| \llcorner \mathrm{Crit}(S)) \ll \mathcal{L}^1$. Then, for every Borel set $A \subset [0,1]$ we have

$$\begin{aligned}
\mathrm{Var}(S;A) &= \int_{A\times\mathbb{R}^d} \|\mathbf{p}(\vec{S})\| \,\mathrm{d}\|S\| \\
&\leq \|S\|(A\times\mathbb{R}^d) \\
&= (\|S\| \llcorner \mathrm{Reg}(S))(\mathbf{t}^{-1}(A)) + (\|S\| \llcorner \mathrm{Crit}(S))(\mathbf{t}^{-1}(A)) \\
&= \mathbf{t}_\#(\|S\| \llcorner \mathrm{Reg}(S))(A) + \mathbf{t}_\#(\|S\| \llcorner \mathrm{Crit}(S))(A),
\end{aligned}$$

whence

$$\mathrm{Var}(S;\,\bullet) \leq \mathbf{t}_\#(\|S\| \llcorner \mathrm{Reg}(S)) + \mathbf{t}_\#(\|S\| \llcorner \mathrm{Crit}(S)).$$

The first term is always absolutely continuous with respect to $\mathcal{L}^1$ by Theorem 7.1 (or just Lemma 7.3) and the second term is so by assumption. □

In fact, AC space-time currents are in a sense orthogonal to the ones that occurred in the zero-slices lem:

Lemma 7.9. *Let $S \in \mathrm{I}_{1+k}([0,1]\times\mathbb{R}^d)$ be AC with $\partial S \llcorner (0,1)\times\mathbb{R}^d = 0$. Then, $\|S\|$ is singular to $\|W\|$ for any $W \in \mathrm{I}_{1+k}([0,1]\times\mathbb{R}^d)$ of the form*

$$W = \sum_j W_{t_j}$$

where $W_{t_j} \in \mathrm{I}_{1+k}(\{t_j\}\times\mathbb{R}^d)$ with

$$\partial W_{t_j} = 0, \qquad \vec{W}_{t_j} \in {\textstyle\bigwedge_k}(\{0\}\times\mathbb{R}^d) \quad \|W_{t_j}\|\text{-a.e.}$$

for an at most countable collection of points $t_j \in [0,1]$.

Proof. Let

$$R := \left\{ x \in \mathbb{R}^{1+d} \;:\; \limsup_{\varrho\to 0} \frac{\|S\|(B(x,\varrho))}{\varrho^{k+1}} > 0 \right\},$$

which is a $(k+1)$-rectifiable carrier set of $\|S\|$. As S is an integral $(k+1)$-current,

$$\|W_{t_j}\| \llcorner R \ll \mathcal{H}^{k+1} \llcorner R \leq \|S\| \llcorner R.$$

Now, $\mathrm{Var}(S;\,\bullet)$ is non-atomic by Lemma 7.8, from which we know that $\|S\|(\{t_j\}\times\mathbb{R}^d) = 0$ for every t_j. It follows that $\|W_{t_j}\|(R) = 0$. Hence $\|W\|(R) = 0$ and so $\|W\|$ is singular to $\|S\|$. □

Corollary 7.10. *Let $S, S' \in \mathrm{I}_{1+k}([0,1] \times \mathbb{R}^d)$ be AC with $\partial S \llcorner (0,1) \times \mathbb{R}^d = 0 = \partial S' \llcorner (0,1) \times \mathbb{R}^d$ and suppose that*

$$S|_t = S'|_t \qquad \text{for } \mathcal{L}^1\text{-a.e. } t.$$

Then, $S = S'$.

Proof. By Lemma 7.6,

$$S - S' = \sum_j W_{t_j} =: W$$

satisfies the assumptions of Lemma 7.9. Thus, $\|W\|$ must be singular with respect to $\|S\| + \|S'\|$, from which we conclude that $W = 0$. □

Example 7.11. Without AC, the slices alone do not determine the space-time current. Consider, for instance the non-AC space-time current given in Example 7.7, which has zero slices for $\mathcal{L}^1$-almost every point in time, but is not zero.

While space-time integral currents $S \in \mathrm{I}_{1+k}(\mathbb{R}^{1+d})$ satisfying the AC condition are determined by their slices, it seems natural to conjecture that also $\|S\| \ll \mathrm{d}t \otimes \|S|_t\|$, so that we could call S "slice-dominated". However, inspecting the distrintegration structure formula (7.2), we can only obtain that

$$\|S\| = \mathrm{d}t \otimes \left(|\nabla^S \mathbf{t}|^{-1} \, \|S|_t\| + \mu_t^* \right)$$

with μ_t^* concentrated on $\mathrm{Crit}(S)$. Interestingly, the μ_t^* may be non-zero and *singular* with respect to $\|S|_t\|$. An example demonstrating this fact will be shown in the next section. Thus, the above conjecture is *false* and in general $\|S\| \not\ll \mathrm{d}t \otimes \|S|_t\|$, even if S is AC.

The problem with S being "slice-dominated" is that μ_t^* may contain singularities that are smeared out in *both* space and time. To avoid this kind of pathology, we define the following stronger condition: A space-time integral current $S \in \mathrm{I}_{1+k}(\mathbb{R}^{1+d})$ is said to satisfy the **NC (negligible-criticality) condition** if

$$\|S\| \llcorner \mathrm{Crit}(S) = 0.$$

Equivalently, we could require that for for $\|S\|$-a.e. (t, x) it holds that

$$\operatorname{span} \vec{S}(t,x) \not\subset \{0\} \times \mathbb{R}^d$$

or

$$\vec{S}(t,x) \llcorner \mathrm{d}\mathbf{t}(t,x) \neq 0,$$

meaning that at such (t, x) the approximate tangent space to S has a non-trivial time component. So, under NC the geometric derivative $\frac{\mathrm{D}}{\mathrm{D}t} S$ is well-defined $\|S\|$-almost everywhere.

Lemma 7.12. *Let $S \in \mathrm{I}_{1+k}(\mathbb{R}^{1+d})$.*

(i) If S is NC, then S is AC.

(ii) If S is AC and $\mathbf{t}_\#(\|S\| \llcorner \mathrm{Crit}(S))$ is singular with respect to $\mathcal{L}^1$, then S is NC.

Proof. These claims are trivial. □

Remark 7.13. In [10], S is said to have the "weak Sard property" if $\mathbf{t}_\#(\|S\| \llcorner \mathrm{Crit}(S))$, which is equivalent to $\mu_t^* = 0$ for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, in the disintegration structure formula (7.2).

Lemma 7.14. *Let $S \in \mathrm{I}_{1+k}([0,1] \times \mathbb{R}^d)$ with $\partial S \llcorner (0,1) \times \mathbb{R}^d = 0$. Then, S is NC if and only if*

$$\|S\| \ll \mathrm{d}t \otimes \|S|_t\|$$

Proof. Assume first that S is NC. Then, in the disintegration structure formula (7.2), λ must be singular with respect to $\mathcal{L}^1$ and therefore the disintegration of $\|S\| \llcorner \mathrm{Crit}(S)$ with respect to $\mathbf{t}$ and λ has the form

$$\|S\| \llcorner \mathrm{Crit}(S) = \lambda^s(\mathrm{d}t) \otimes \mu_t^*.$$

Thus, $\mu_t^* = 0$ for $\mathcal{L}^1$-a.e. t, whereby $\mathbf{t}_\#(\|S\| \llcorner \mathrm{Crit}(S)) \ll \lambda^s$ is singular with respect to $\mathcal{L}^1$. We also know that $\lambda^s = 0$. Combining this with (7.2), we obtain

$$\|S\| = \int_0^1 |\nabla^S \mathbf{t}|^{-1} \, \|S|_t\| \, \mathrm{d}t,$$

whence the claim follows.

For the other direction it suffices to observe that by Lemma 6.8 (i) we have $\|S|_t\|(\mathrm{Crit}(S)) = 0$ for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$. Thus,

$$\int_0^1 \|S|_t\|(\mathrm{Crit}(S)) \, \mathrm{d}t = 0$$

and this implies $\|S\|(\mathrm{Crit}(S)) = 0$ by the hypothesis. □

Remark 7.15. If $k = 0$, for every $S \in \mathrm{I}_1(\mathbb{R} \times \mathbb{R}^d)$ by the area formula it holds that $\mathcal{H}^1(\mathbf{t}(\mathrm{Crit}(S))) = 0$. So, $\mathbf{t}_\#(\|S\| \llcorner \mathrm{Crit}(S))$ is singular to $\mathcal{L}^1$. Thus, by Lemma 7.12 (ii), S is automatically NC. This means that phenomena involving "critical singularities", such as the ones to be investigated in the next section, are a genuinely novel higher-dimensional feature of the space-time theory, and do not occur in the classical theory of functions of bounded variation (BV) or absolutely continuous functions on $\mathbb{R}$, which can be recovered as the $k = 0$ endpoint of the space-time theory (cf. Example 6.13).

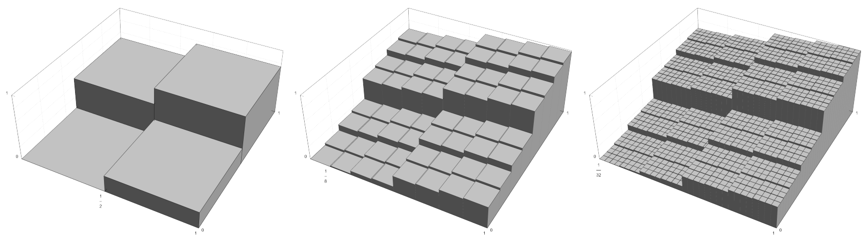

Figure 7.1: The construction of the Flat Mountain (steps 1, 3, 5).

7.3 The Flat Mountain

In this section we construct the so-called "Flat Mountain", which is an AC integral space-time current that does not possess the NC property. This space-time current $S_{\mathrm{FM}} \in \mathrm{I}_2^{\mathrm{Lip}}(\mathbb{R} \times \mathbb{R}^2)$ will be given as the graph of a BV-map $u\colon [0,1]^2 \to \mathbb{R}$ (the "mountain"). Here we use the convention that time goes *upwards*, i.e., the slices of S_{FM} are given by intersection with horizontal planes. For the theory of functions of bounded variation (BV) see [4] and also Example 6.13.

One particular property of the construction is that the absolutely continuous part of the derivative is zero, that is, $\nabla u \equiv 0$ ("the mountain is flat"), yet the function is far from constant. This property is somewhat reminiscent of the Cantor function f_C, defined in Example 4.10, for which $f_C' \equiv 0$ (the absolutely continuous part of the BV-derivative, or even classically via differential quotients almost everywhere), but f_C increases monotonically from 0 to 1 in the interval $[0,1]$. The precise form of the construction was furthermore inspired by the ideas from [2]. The result will be:

Proposition 7.16. *There is a current $S_{\mathrm{FM}} \in \mathrm{I}_2([0,1]^2 \times \mathbb{R})$ with $\partial S \llcorner \mathbb{R}^2 \times (0,1) = 0$ (the time-axis being the* last *component here) and such that for the time-slices*

$$S(t) := \mathbf{p}_* S|_t,$$

where we have employed the modified definitions $\mathbf{t}(x,t) := t$ and $\mathbf{p}(x,t) := x$, it holds that

$$\mathbf{FI}^\circ(S_{\mathrm{FM}}(t) - S_{\mathrm{FM}}(s)) = \mathrm{Var}(S_{\mathrm{FM}}; [s,t]) = |t-s|, \qquad s < t,$$

and

$$\|S_{\mathrm{FM}}\|(\mathrm{Crit}(S_{\mathrm{FM}})) = 1.$$

In particular, S_{FM} is AC, but not NC.

The proof will be accomplished in the remainder of this section via several lemmas.

The construction of the Flat Mountain is iterative and proceeds by successively subdividing the square $[0,1]^2$ and raising the subsquares by a carefully chosen amount; see Fig. 7.1 for an illustration. Then one passes to

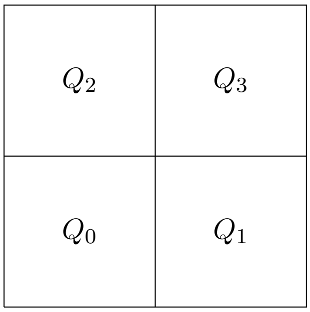

Figure 7.2: The squares Q_0, Q_1, Q_2, Q_3.

We first define the cubes

$$Q_0 := [0, \tfrac{1}{2}) \times [0, \tfrac{1}{2}), \qquad Q_1 := [\tfrac{1}{2}, 1) \times [0, \tfrac{1}{2}),$$
$$Q_2 := [0, \tfrac{1}{2}) \times [\tfrac{1}{2}, 1), \qquad Q_3 := [\tfrac{1}{2}, 1) \times [\tfrac{1}{2}, 1),$$

Figure 7.2 for an illustration, and corresponding maps $A_i \colon Q_i \to [0,1)^2$,

$$A_0(x) := 2x, \qquad A_1(x) := 2x - (1,0),$$
$$A_2(x) := 2x - (0,1), \qquad A_3(x) := 2x - (1,1),$$

which have as their inverse the maps $B_i \colon [0,1)^2 \to Q_i$, given as

$$B_0(x) := \tfrac{1}{2}x, \qquad B_1(x) := \tfrac{1}{2}x + (\tfrac{1}{2}, 0),$$
$$B_2(x) := \tfrac{1}{2}x + (0, \tfrac{1}{2}), \qquad B_3(x) := \tfrac{1}{2}x + (\tfrac{1}{2}, \tfrac{1}{2}).$$

Define the function $\overline{u} \colon [0,1)^2 \to \mathbb{R}$ via

$$\overline{u} := \sum_{i=0}^{3} \frac{i}{4} \mathbb{1}_{Q_i}.$$

and the map $H \colon \mathrm{L}^1([0,1]^2) \to \mathrm{L}^1([0,1]^2)$ via

$$H[v] := \overline{u} + \frac{1}{4} \sum_{i=0}^{3} (v \circ A_i)\, \mathbb{1}_{Q_i}. \tag{7.3}$$

Lemma 7.17. *The sequence* $(u_n)_n \subset \mathrm{BV}((0,1)^2)$ *defined by* $u_0 := 0$ *and*

$$u_{n+1} := H[u_n], \qquad n \in \mathbb{N},$$

converges in L^∞ *and in* BV *to a limit* $u \in \mathrm{BV}((0,1)^2)$ *with* $\nabla u \equiv 0$.

Proof. For any two $v, w \in \mathrm{BV}((0,1)^2)$ it holds that

$$\|H[v] - H[w]\|_{\mathrm{L}^\infty} = \frac{1}{4} \max_{i=0,1,2,3} \|(v \circ A_i - w \circ A_i)\|_{\mathrm{L}^\infty(\overline{Q_i})} \leq \frac{1}{4} \|v - w\|_{\mathrm{L}^\infty}$$

and

$$\begin{aligned} \|DH[v] - DH[w]\|([0,1]^2) &\leq \frac{1}{4} \sum_{i=0}^{3} \|D(v \circ A_i - w \circ A_i)\|(\overline{Q_i}) \\ &= \frac{1}{4} \sum_{i=0}^{3} \frac{1}{2} \|D(v - w)\|([0,1]^2) \\ &= \frac{1}{2} \|Dv - Dw)\|([0,1]^2). \end{aligned}$$

Thus, H is a contraction in L^∞ and BV, so the iteratively-defined sequence $(u_n)_n$ is Cauchy in these spaces and hence converges to $u \in \mathrm{BV}((0,1)^2)$, as claimed.

Since Du_n converges strongly to Du in the sense of measures and Du_n is supported on the set

$$D_n \times D_n, \qquad \text{where} \qquad D_n := \left\{ \frac{j}{2^{n+1}} \, : \, j = 0, \ldots, 2^{n+1} \right\},$$

the measure Du must be carried by the set

$$J := \bigcup_{n=1}^{\infty} D_n \times D_n.$$

This set is σ-finite with respect to $\mathcal{H}^1$ and the general properties of BV-maps, see [4, Proposition 3.92], thus imply that u must be a pure jump map, that is, $Du = D^j u$, and thus $\nabla u \equiv 0$. □

Having defined u on $(0,1)^2$ above, we now consider it to be extended by zero to all of $\mathbb{R}^2$. In this way, new jumps are added on the boundary $\partial(0,1)^2$. We will define the Flat Mountain as the 2-current S_{FM} associated with the graph of u, after completing the jump parts. For this, define the 2-rectifiable sets

$$\Gamma := \bigcup_{x \in [0,1]^2 \setminus J_u} \{(x, u(x))^\mathsf{T}\}, \qquad \Delta := \bigcup_{x \in J_u} \{x\} \times [u^-(x), u^+(x)],$$

where $[u^-(x), u^+(x)]$ denotes the segment between the two traces of u at the jump at $x \in J_u$. Then let

$$S_{\mathrm{FM}} := (\mathrm{e}_1 \wedge \mathrm{e}_2)\, \mathcal{H}^2 \llcorner (\Gamma \cup \Delta).$$

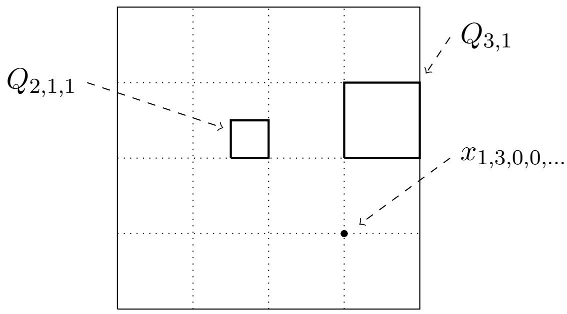

Figure 7.3: A few subsquares and points identified by a sequence of numbers in $\{0, 1, 2, 3\}$.

Denote by $(Du)^\perp$ the (divergence-free) measure that is obtained from Du by rotating the gradient clockwise by $\pi/2$, that is,

$$(Du)^\perp = \begin{bmatrix} -\partial_2 u \\ \partial_1 u \end{bmatrix}.$$

The layer cake decomposition and coarea formula for BV functions then imply that

$$u(x)\,\mathrm{e}_1 \wedge \mathrm{e}_2 = \int_0^1 \mathbb{1}_{\{u \geq t\}}(x)\,\mathrm{d}t \qquad \text{and} \qquad (Du)^\perp = \int_0^1 \partial(\mathbb{1}_{\{u \geq t\}}\,\mathrm{e}_1 \wedge \mathrm{e}_2)\,\mathrm{d}t.$$

In order to derive the properties claimed about S_{FM} in Proposition 7.16, we need perform a more precise analysis of the approximating maps u_n. For

For any finite sequence $I = (i_1, \ldots, i_n) \in \{0, 1, 2, 3\}^n$, $n \in \mathbb{N}$, let Q_I be the dyadic cube

$$Q_I = Q_{i_1,\ldots,i_n} := B_{i_1} \circ \ldots \circ B_{i_n}([0, 1)^2), \tag{7.4}$$

which has side length 2^{-n}. Intuitively, this corresponds to descending to finer and finer subcubes, with the index determining which subcube to choose at each step (with $0, 1, 2, 3$ giving the SW, SE, NW and NE subcubes, respectively); see Fig. 7.3 for an illustration.

For any dyadic cube Q_I, its dyadic subcubes are exactly those of the form $Q_{I'}$, where I' extends I (that is, the first n indices i'_j in I' agree with those in I).

Moreover, for an infinite sequence $J \in \{0, 1, 2, 3\}^{\mathbb{N}}$ we let $x_J \in [0, 1)^2$ be the unique point satisfying

$$\{x_J\} = \bigcap_{n \in \mathbb{N}} \overline{Q}_{i_1,\ldots,i_n}.$$

Conversely, for any $x \in [0, 1)^2$ denote by $J(x)$ the (unique) index sequence such that $x_{J(x)} = x$.

Lemma 7.18. *On each dyadic cube Q_I for $I = (i_1, \dots, i_n)$, it holds that*

$$u_n \equiv \sum_{j=1}^{n} \frac{i_j}{4^j}. \tag{7.5}$$

Moreover, if we write $J(x) = (i_1(x), i_2(x), \dots)$, then

$$u(x) = \sum_{j=1}^{\infty} \frac{i_j(x)}{4^j}. \tag{7.6}$$

Proof. The first claim is clear if $n = 1$. Suppose now that (7.5) holds for n. We need to show the claim for $n+1$. For this, let $x \in Q_{i_1,\dots,i_{n+1}}$. In particular, $x \in Q_{i_1}$ and by (7.4) together with the fact that A_i and B_i are inverses of each other, we have

$$A_i(x) \in Q_{i_2,\dots,i_{n+1}}.$$

Therefore, by (7.3),

$$u_{n+1}(x) = \overline{u}(x) + \frac{1}{4} \sum_{i=0}^{3} u_n(A_i(x))\, \mathbb{1}_{Q_i}(x) = \frac{i_1}{4} + \frac{1}{4} u_n(A_{i_1}(x)).$$

From the induction hypothesis it then follows that

$$u_{n+1}(x) = \frac{i_1}{4} + \frac{1}{4} \sum_{j=2}^{n+1} \frac{i_j}{4^j} = \sum_{j=1}^{n+1} \frac{i_j}{4^j},$$

which is the first claim for $n+1$.

The second claim follows since $u_n \to u$ in L^∞ and hence we may pass to the limit $n \to \infty$ in (7.5) to obtain (7.6). $\square$

The projected time-slices of S_{FM} are given as

$$S_{\mathrm{FM}}(t) := \mathbf{p}_* S_{\mathrm{FM}}|_t = \partial(\mathbb{1}_{\{u \geq t\}} \mathrm{e}_1 \wedge \mathrm{e}_2).$$

The above equality should be intuitively clear, but a more precise argument may proceed via the cylinder formula (6.13) in Proposition 6.4 (adapted to our present convention of time direction) together with the definition of S_{FM}.

Lemma 7.19. *For every $t, s \in [0, 1]$ it holds that*

$$\mathbf{FI}^\circ(S_{\mathrm{FM}}(t) - S_{\mathrm{FM}}(s)) = \mathrm{Var}(S_{\mathrm{FM}}; [s, t]) = |t - s|.$$

Proof. It is clear from (7.5) that the superlevel sets $\{u_n \geq t\}$ only change when t is a multiple of 4^{-n}. In fact, for all $m, n \in \mathbb{N}$ with $m \leq n$ and all $j = 0, \ldots, 4^n$ it holds that

$$\left\{u_m \geq \frac{j}{4^n}\right\} = \left\{u_n \geq \frac{j}{4^n}\right\} = \left\{u \geq \frac{j}{4^n}\right\}. \tag{7.7}$$

The first equality is verified via (7.5) and the fact that

$$u_m \leq u_n < u_m + \frac{1}{4^m}.$$

The second equality follows from the L^∞-convergence of u_n to u, which was proved in Lemma 7.17.

If $x \in Q_{i_1,\ldots,i_n}$, then by (7.6) in Lemma 7.18,

$$\sum_{j=1}^{n} \frac{i_j}{4^j} \leq u(x) = \sum_{j=1}^{n} \frac{i_j}{4^j} + \sum_{j=n+1}^{\infty} \frac{i_j}{4^j} < \sum_{j=1}^{n} \frac{i_j}{4^j} + \frac{1}{4^n}.$$

The left-most inequality is an equality only at the point x_0 for which it holds that $J(x_0) = (i_1, \ldots, i_n, 0, 0, \ldots)$. Thus,

$$Q_{i_1,\ldots,i_n} \setminus \{x_0\} \subset \left\{\sum_{j=1}^{n} \frac{i_j}{4^j} < u < \sum_{j=1}^{n} \frac{i_j}{4^j} + \frac{1}{4^n}\right\}. \tag{7.8}$$

On the other hand, let $x \in Q_{\ell_1,\ldots,\ell_n}$, with $(\ell_1, \ldots, \ell_n) \neq (i_1, \ldots, i_n)$. Then either

$$\sum_{j=1}^{n} \frac{\ell_j}{4^j} \leq \sum_{j=1}^{n} \frac{i_j}{4^j} - \frac{1}{4^n} \qquad \text{or} \qquad \sum_{j=1}^{n} \frac{\ell_j}{4^j} \geq \sum_{j=1}^{n} \frac{i_j}{4^j} + \frac{1}{4^n}.$$

By Lemma 7.18, these cases imply either

$$u(x) < \sum_{j=1}^{n} \frac{i_j}{4^j} \qquad \text{or} \qquad u(x) \geq \sum_{j=1}^{n} \frac{i_j}{4^j} + \frac{1}{4^n}.$$

Combining this with (7.8), we obtain

$$Q_{i_1,\ldots,i_n} = \left\{\sum_{j=1}^{n} \frac{i_j}{4^j} < u < \sum_{j=1}^{n} \frac{i_j}{4^j} + \frac{1}{4^n}\right\} \cup \{x_0\}. \tag{7.9}$$

Given $h, k \in \{0, \ldots, 4^n\}$ with $h \leq k$,

$$\left\{\frac{h}{4^n} < u < \frac{k}{4^n}\right\} = N \cup \bigcup_{m=h}^{k-1} \left\{\frac{m}{4^n} < u < \frac{m}{4^n} + \frac{1}{4^n}\right\},$$

where $\mathcal{L}^2(N) = 0$. By (7.9), each set on the right-hand side corresponds to precisely one dyadic cube of order n, up to all null set. So, after redefining N, it holds that

$$\left\{ \frac{h}{4^n} < u < \frac{k}{4^n} \right\} = N \cup \bigcup_{I \in \mathcal{I}(h,k)} Q_I,$$

where $\mathcal{I}(h,k)$ is a set of n-indices with exactly $k - h$ elements. Since the squares corresponding to these indices in $\mathcal{I}(h,k)$ are essentially disjoint,

$$\mathcal{L}^2 \left(\left\{ \frac{h}{4^n} < u < \frac{k}{4^n} \right\} \right) = \sum_{I \in \mathcal{I}(h,k)} \mathcal{L}^2(Q_I) = \frac{k-h}{4^n}.$$

We now observe

$$\mathbf{FI}^\circ(S_{\mathrm{FM}}(t) - S_{\mathrm{FM}}(s)) = |t - s|$$

for all $s, t \in [0,1]$ that are multiples of 4^{-n} for any n. Indeed, from the definition of slices for S_{FM}, $\mathbf{FI}^\circ(S_{\mathrm{FM}}(t) - S_{\mathrm{FM}}(s))$ must equal the $\mathcal{L}^2$-measure of the set between $\{u = 4^{-n}h\}$ and $\{u = 4^{-n}k\}$ (the corresponding current is admissible for $\mathbf{FI}^\circ$ and it is also the only such current by the Constancy Theorem 4.23). By the above relation, this $\mathcal{L}^2$-measure equals $|t - s|$.

A similar argument yields that

$$\mathrm{Var}(S_{\mathrm{FM}}; [s,t]) = |t - s|,$$

where we also use that $\|\mathbf{p}(\vec{S}_{\mathrm{FM}})\| = \|\vec{S}_{\mathrm{FM}}\| = 1$ since $\nabla u \equiv 1$.

Since these relations hold for all s, t on a dense set, a continuity argument shows the conclusion for every $t, s \in [0,1]$. □

Proof of Proposition 7.16. The formula for the homogeneous integral flat norm and the space-time variation was established in the preceding lemma. By Lemma 7.8 we see that S_{FM} is AC. On the other hand,

$$\|S_{\mathrm{FM}}\|(\mathrm{Crit}(S_{\mathrm{FM}})) = 1$$

because $\nabla u(x) = 0$ for a.e. $x \in [0,1]^2$. Thus, S_{FM} is not NC and the proof of Proposition 7.16 is complete. □

Chapter 8

The Geometric Transport Equation

So far we have seen how one can represent the motion of geometric objects represented by currents via space-time trajectories. Now we want to investigate a complimentary description of current transport via the Geometric Transport Equation. One may think of this as writing down an evolution law for the slices of a space-time current. Specializing this equation to different dimension, we will recover many known transport-type PDEs, like the continuity and transport equations. This chapter will also discuss existence and uniqueness theorems as well as various results on the relationship with the "variational" space-time approach that was introduced previously.

8.1 Lie derivative and Geometric Transport Equation

Recall the definition of the Lie derivative of a k-form $\omega \in \mathcal{D}^k(\mathbb{R}^d)$ by a velocity vector field $v \in \mathrm{C}^\infty(\mathbb{R}^d; \mathbb{R}^d)$ in (3.4). Moreover, by Cartan's Magic Formula from Proposition 3.19, we have that

$$\mathcal{L}_v \omega = \mathrm{d}(\iota_v \omega) + \iota_v(\mathrm{d}\omega). \tag{8.1}$$

Dualizing this, we will in the following be able to define the Lie derivative also for currents.

So, given $T \in \mathcal{D}_k(\mathbb{R}^d)$, let

$$v \in \mathrm{L}^1(\|T\| + \|\partial T\|; \mathbb{R}^d),$$

that is, $(\|T\| + \|\partial T\|)$-integrable (which we understand to contain the requirement that $v \colon \mathbb{R}^d \to \mathbb{R}^d$ is measurable with respect to $\|T\| + \|\partial T\|$). We now define the **Lie derivative of a current** $\mathcal{L}_v T \in \mathcal{D}_k(\mathbb{R}^d)$ with respect to such a v via

$$\mathcal{L}_v T := v \wedge \partial T + \partial(v \wedge T). \tag{8.2}$$

This definition is motivated by the following result:

Proposition 8.1. *For $T \in \mathcal{D}_k(\mathbb{R}^d)$ and $v \in \mathrm{C}^\infty(\mathbb{R}^d;\mathbb{R}^d)$ it holds that*

$$\big\langle \mathcal{L}_v T, \omega \big\rangle = \big\langle T, \mathcal{L}_v \omega \big\rangle, \qquad \omega \in \mathcal{D}^k(\mathbb{R}^d). \tag{8.3}$$

Proof. We compute, using (3.8) and Cartan's Magic Formula (8.1) (originally from Proposition 3.19), as follows:

$$\big\langle \mathcal{L}_v T, \omega \big\rangle = \big\langle v \wedge \partial T + \partial(v \wedge T), \omega \big\rangle = \big\langle T, \mathrm{d}(\iota_v \omega) + \iota_v(\mathrm{d}\omega) \big\rangle = \big\langle T, \mathcal{L}_v \omega \big\rangle.$$

This yields the claim. □

While it is tempting to define $\mathcal{L}_v T$ via (8.3) and think of (8.2) as **(dual) Cartan formula** for currents, our way of using (8.2) as the definition has the advantage that it works for a larger class of velocities v without any approximation arguments. This will be useful in many instances.

Finally, for *non-autonomous* vector fields, i.e., $v\colon [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$, we write $v_t(x) := v(t,x)$ and we define the Lie derivative of T with respect to v_t as

$$\mathcal{L}_{v_t} T := v_t \wedge \partial T + \partial(v_t \wedge T),$$

whenever this is defined.

Remark 8.2. In contrast to [8–10] in (8.2) we use the "positive" sign convention in the definition of the Lie derivative of a current (which also then necessitates a sign change in the Geometric Transport equation relative to *loc. cit.*), which seems somewhat more natural. Indeed, we will see in Theorem 8.14, whereby for any $\overline{T} \in \mathrm{N}_k(\mathbb{R}^d)$ it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t} \big\langle \Phi^t_* \overline{T}, \omega \big\rangle \bigg|_{t=0} = \big\langle \mathcal{L}_v \overline{T}, \omega \big\rangle, \qquad \omega \in \mathcal{D}^k(\mathbb{R}^d),$$

with Φ^t_t the flow for a smooth vector field $v \in \mathrm{C}^\infty(\mathbb{R}^d;\mathbb{R}^d)$, which seems like a natural requirement. Moreover, the "positive" sign convention reduces the number of overall minus signs.

The Geometric Transport Equation is a differential equation for a **path of normal currents**

$$[0,1] \ni t \mapsto T_t \in \mathrm{N}_k(\mathbb{R}^d),$$

where we recall the definition of the space $\mathrm{N}_k(\mathbb{R}^d)$ of normal currents on $\mathbb{R}^d$ from (4.3), namely

$$\mathrm{N}_k(\mathbb{R}^d) := \big\{\, T \in \mathcal{D}_k(\mathbb{R}^d) \;:\; \mathbf{M}(T) + \mathbf{M}(\partial T) < \infty \,\big\}.$$

We will always tacitly assume the **weak* measurability** of the path, that is, for every $\omega \in \mathcal{D}^k(\mathbb{R}^d)$ the scalar mapping $t \mapsto \big\langle T_t, \omega \big\rangle$ is assumed to be Lebesgue-measurable. Let us remark that the time interval is only $[0,1]$ to simplify notation, but all results hold, with due modifications, for a general interval.

Lemma 8.3. *Let $[0,1] \ni t \mapsto T_t \in \mathrm{N}_k(\mathbb{R}^d)$ be a path of normal currents with*

$$\int_I \mathbf{M}(T_t)\,\mathrm{d}t < \infty \qquad \text{for all } I \Subset (0,1). \tag{8.4}$$

Then, the following statements are equivalent:

(i) $T_t = 0$ *for* $\mathcal{L}^1$*-a.e.* $t \in \mathbb{R}$.

(ii) For every $\eta \in \mathcal{D}^k((0,1)\times\mathbb{R}^d)$,

$$\int_0^1 \big\langle \delta_t \times T_t, \eta \big\rangle \,\mathrm{d}t = 0.$$

(iii) For every $\omega \in \mathcal{D}^k(\mathbb{R}^d)$ *and for every function* $\varphi \in \mathrm{C}^\infty_c(\mathbb{R})$,

$$\int_0^1 \varphi(t)\,\big\langle T_t, \omega \big\rangle \,\mathrm{d}t = 0.$$

Moreover, in (ii) and (iii), we may equivalently replace $\mathcal{D}^k((0,1)\times\mathbb{R}^d)$, $\mathcal{D}^k(\mathbb{R}^d)$ *and* $\mathrm{C}^\infty_c(\mathbb{R})$ *with countable dense (with respect to the supremum norm) subsets thereof.*

Proof. It is trivial to see that (i) implies (ii) and (ii) implies (iii), so we only need to show that (iii) implies (i). We consider right from the start the additional claim and assume that (iii) holds for countable dense sets $\{\omega_m\}_m$ and $\{\varphi^n\}_n$. For each n, the map $t \mapsto \langle T_t, \omega_m \rangle$ is integrable over any open time interval $I \Subset [0,1]$ by (8.4). By (iii), approximating $\mathbb{1}_I$ with a suitable sequence of φ^n's, we see that $\langle T_t, \omega_m \rangle = 0$ for $\mathcal{L}^1$-a.e. $t \in I$. Since I was arbitrary, we can thus find a full measure set $J_m \subset \mathbb{R}$ such that $\big\langle T_t, \omega_m \big\rangle = 0$ for $t \in J_m$. Hence, for $t \in J := \bigcap_m J_m$, $\big\langle T_t, \omega_m \big\rangle = 0$ for every $m \in \mathbb{N}$. By the density of the φ^m's we conclude (i). $\square$

Remark 8.4. In (ii) it suffices to consider forms $\eta \in \mathcal{D}^k((0,1)\times\mathbb{R}^d)$ of the form $\eta(t,x) = \alpha(t)\beta(x)$ with $\alpha \in \mathrm{C}^\infty_c(\mathbb{R})$ and $\beta \in \mathcal{D}^k(\mathbb{R}^d)$ such that $\beta = \sum_I \beta_I \mathrm{d}x^I$, where $I \in \mathbb{N}^k_0$ ranges over all multi-indices of order k in just the spatial variables since any term involving $\mathrm{d}t$ is not seen by $\langle \mathbf{p}_* T_t, \eta \rangle$.

Given a velocity vector field $v\colon [0,1]\times\mathbb{R}^d \to \mathbb{R}^d$, we say that the path of currents $t \mapsto T_t \in \mathrm{N}_k(\mathbb{R}^d)$ ($t \in [0,1]$) is a **weak solution** of the **Geometric Transport Equation (GTE)**, written as

$$\frac{\mathrm{d}}{\mathrm{d}t} T_t - \mathcal{L}_{v_t} T_t = 0, \tag{8.5}$$

if $v \in \mathrm{L}^1(\mathrm{d}t \otimes (\|T_t\| + \|\partial T_t\|); \mathbb{R}^d)$, that is,

$$\int_0^1 \int_{\mathbb{R}^d} |v_t| \,\mathrm{d}(\|T_t\| + \|\partial T_t\|)\,\mathrm{d}t < \infty \tag{8.6}$$

and the Geometric Transport Equation (8.5) holds distributionally, that is,

$$\int_0^1 \langle T_t, \omega \rangle \, \dot{\psi}(t) + \langle \mathcal{L}_{v_t} T_t, \omega \rangle \, \psi(t) \, \mathrm{d}t = 0 \tag{8.7}$$

for every $\psi \in \mathrm{C}_c^\infty((0,1))$ and every $\omega \in \mathcal{D}^k(\mathbb{R}^d)$.

Note that assuming that v is globally bounded, then we a stronger condition than (8.6) is that the currents T_t are L^1**-normal**, that is

$$\int_0^1 \mathbf{M}(T_t) + \mathbf{M}(\partial T_t) \, \mathrm{d}t < \infty,$$

or even that they are **uniformly normal**, that is,

$$\sup_{t \in [0,1]} \big[\mathbf{M}(T_t) + \mathbf{M}(\partial T_t) \big] < \infty. \tag{8.8}$$

The following lemmas give equivalent ways to express the validity of (8.5).

Lemma 8.5. *Assuming* (8.6)*, condition* (8.7) *is equivalent to*

$$\langle T_t - T_s, \omega \rangle = \int_s^t \langle \mathcal{L}_{v_\tau} T_\tau, \omega \rangle \, \mathrm{d}\tau, \qquad \omega \in \mathcal{D}^k(\mathbb{R}^d). \tag{8.9}$$

Moreover, $t \mapsto \langle T_t, \omega \rangle$ is absolutely continuous and differentiable $\mathcal{L}^1$-almost everywhere with

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle T_t, \omega \rangle = \langle \mathcal{L}_{v_t} T_t, \omega \rangle. \tag{8.10}$$

Proof. Fix $\omega \in \mathcal{D}^k(\mathbb{R}^d)$. Observe first that (8.6) implies

$$\begin{aligned} \int_0^1 \big| \langle \mathcal{L}_{v_t} T_t, \omega \rangle \big| \, \mathrm{d}t &\le \|\mathrm{d}\omega\|_\infty \int_0^1 \int_{\mathbb{R}^d} |v_t| \, \mathrm{d}\|T_t\| \, \mathrm{d}t + \|\omega\|_\infty \int_0^1 \int_{\mathbb{R}^d} |v_t| \, \mathrm{d}\|\partial T_t\| \, \mathrm{d}t \\ &< \infty. \end{aligned}$$

So we may define

$$I(t) := \int_0^t \langle \mathcal{L}_{v_\tau} T_\tau, \omega \rangle \, \mathrm{d}\tau, \qquad t \in [0,1].$$

By Lebesgue's Differentiation Theorem 1.16, this function is absolutely continuous and by Rademacher's Theorem 1.18 it is $\mathcal{L}^1$-almost everywhere differentiable with (8.10). Via an integration by parts, for every $\psi \in \mathrm{C}_c^\infty((0,1))$ it holds that

$$\int_0^1 I(t) \, \dot{\psi}(t) + \langle \mathcal{L}_{v_t} T_t, \omega \rangle \, \psi(t) \, \mathrm{d}t = 0.$$

Now, if (8.7) holds, that is,

$$\int_0^1 \langle T_t, \omega \rangle \,\dot{\psi}(t) + \langle \mathcal{L}_{v_t} T_t, \omega \rangle \,\psi(t) \,\mathrm{d}t = 0,$$

we thus obtain that

$$\int_0^1 I(t)\,\dot{\psi}(t)\,\mathrm{d}t = \int_0^1 \langle T_t, \omega \rangle \,\dot{\psi}(t)\,\mathrm{d}t.$$

This implies that $I(t) = \langle T_t, \omega \rangle + \alpha$, with a constant $\alpha \in \mathbb{R}$, whereby (8.9) follows from the definition of I.

Conversely, if (8.9) holds, then $t \mapsto \langle T_t, \omega \rangle$ is absolutely continuous and an integration by parts yields (8.7). □

Lemma 8.6. *Suppose that v is autonomous and smooth. Assuming* (8.6)*, condition* (8.7) *is equivalent to*

$$\int_0^1 \langle \delta_t \times T_t, \dot{\eta} + \mathcal{L}_v \eta \rangle \,\mathrm{d}t = 0, \qquad \eta \in \mathcal{D}^k((0,1) \times \mathbb{R}^d). \tag{8.11}$$

Proof. To see that (8.11) implies (8.7), just choose $\eta(t,x) := \psi(t)\omega(x)$. For the other direction, one first observes that the subclass of k-forms given by $\psi\omega := (\mathbf{t}^*\psi) \wedge (\mathbf{p}^*\omega)$, with $\psi \in \mathrm{C}_c^\infty((0,1))$ and $\omega \in \mathcal{D}^k(\mathbb{R}^d)$ are dense in the space of k-forms in $\mathbb{R}^{1+d}$ and that they, when written in coordinates $\mathrm{d}t, \mathrm{d}x^1, \ldots, \mathrm{d}x^d$, do not contain terms involving $\mathrm{d}t$ (see, e.g., Section 4.1.8 in [19] for a similar argument). Then, (8.7) implies (8.11) via Lemma 8.3, taking into account Remark 8.4. □

We would like to complement the Geometric Transport Equation with an initial condition. For this we need to verify that a solution path has a suitable continuity property. A path $[0,1] \ni t \mapsto T_t \in \mathrm{N}_k(\mathbb{R}^d)$ of normal currents is called

(a) **weakly* continuous** if for every equence $t_j \to t$ in $[0,1]$ it holds that $T_{t_j} \overset{*}{\rightharpoonup} T_t$;

(b) **F-absolutely continuous** if for all $s < t$ it holds that

$$\mathbf{F}(T_s - T_t) \leq \int_s^t g(r)\,\mathrm{d}r \tag{8.12}$$

for some function $g \in \mathrm{L}^1([0,1])$;

(c) **F-Lipschitz** if for all $s < t$ it holds that

$$\mathbf{F}(T_s - T_t) \leq C|s-t|.$$

It is straightforward to see that $\mathbf{F}$-Lipschitz continuity implies $\mathbf{F}$-absolute continuity, which in turn implies finite $\mathbf{F}$-variation. We can of course also define absolute continuity and Lipschitz continuity with respect to the other flat norms $\mathbf{F}^\circ$, $\mathbf{FI}$, $\mathbf{FI}^\circ$.

Lemma 8.7. *Let $[0,1] \ni t \mapsto T_t \in \mathrm{N}_k(\mathbb{R}^d)$ be a weak solution to the Geometric Transport Equation for a velocity field $v\colon [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$. Then, there exists another path $[0,1] \ni t \mapsto \widetilde{T}_t \in \mathrm{N}_k(\mathbb{R}^d)$ such that the following conditions hold:*

(i) $T_t = \widetilde{T}_t$ *for a.e.* $t \in [0,1]$.

(ii) $t \mapsto \widetilde{T}_t$ *is weakly* continuous.*

(iii) $t \mapsto \widetilde{T}_t$ *is* $\mathbf{F}$*-absolutely continuous.*

(iv) If, in addition,

$$B(t) := \int_{\mathbb{R}^d} |v_t| \, \mathrm{d}(\|T_t\| + \|\partial T_t\|)$$

is uniformly bounded on $[0,1]$ (e.g., if v is uniformly bounded), then $t \mapsto \widetilde{T}_t$ is $\mathbf{F}$*-Lipschitz.*

Proof. Let $\{\omega_m\}_m$ be a countable dense set in $\mathcal{D}^k(\mathbb{R}^d)$. By Lemma 8.5, for each m the function $t \mapsto \langle T_t, \omega_m \rangle$ admits an absolutely continuous representative, that is, we can find a set $J_m \subset [0,1]$ with $\mathcal{L}^1([0,1] \setminus J_m) = 0$ such that $t \mapsto \langle T_t, \omega_m \rangle$ is uniformly continuous on J_m and

$$\frac{\mathrm{d}}{\mathrm{d}t} \big\langle T_t, \omega \big\rangle = \big\langle \mathcal{L}_{v_t} T_t, \omega \big\rangle$$

for $t \in J_m$.

Set $J := \bigcap_m J_m$. Then for each $\omega = \omega_m$ from our dense collection, and all $s, t \in [0,1]$ with $s < t$, we obtain the estimate

$$\big| \big\langle T_t - T_s, \omega \big\rangle \big| \leq \int_s^t \frac{\mathrm{d}}{\mathrm{d}\tau} \big\langle T_\tau, \omega \big\rangle \, \mathrm{d}\tau \leq (\|\omega\|_\infty + \|\mathrm{d}\omega\|_\infty) \int_s^t B(\tau) \, \mathrm{d}\tau, \tag{8.13}$$

with B defined in (iv). This then extends to every $\omega \in \mathcal{D}^k(\mathbb{R}^d)$ by density. Thus, the restriction $J \ni t \mapsto T_t$ is uniformly weakly* continuous path.

Moreover, by the flat norm duality formula of Theorem 5.11,

$$\mathbf{F}(T_t - T_s) = \sup\big\{ \big\langle T_t - T_s, \omega \big\rangle \;:\; \omega \in \mathcal{D}^k(\mathbb{R}^d), \ \|\omega\|_\infty \leq 1, \ \|\mathrm{d}\omega\|_\infty \leq 1 \big\},$$

the inequality (8.13) shows that also

$$\mathbf{F}(T_t - T_s) \leq 2 \int_s^t B(\tau) \, \mathrm{d}\tau \tag{8.14}$$

for $s, t \in J$. This proves that the restriction $J \ni t \mapsto T_t$ is $\mathbf{F}$-absolutely continuous, or even $\mathbf{F}$-Lipschitz if B is uniformly bounded.

It remains to show that we can extend $J \ni t \mapsto T_t$ to the whole interval $[0,1]$ without destroying these continuity properties. For this, fix $t \in [0,1]$ and pick a sequence $(t_i)_i \subset J$ with $t_i \to t$. Then, $(T_{t_i})_i$ is a Cauchy sequence with respect to $\mathbf{F}$ by (8.14). Hence, by Proposition 4.16 and Lemma 5.10, $T_{t_i} \overset{*}{\rightharpoonup} \widetilde{T}_t$ for some $\widetilde{T}_t \in \mathrm{N}_k(\mathbb{R}^d)$, and $\widetilde{T}_t$ does not depend on the choice of the sequence $(t_i)_i$. For this extended sequence, (i)–(iv) hold by passing to the limit in the relevant estimates. We only make this explicit for (iv): Observe that for $s, t \in [0,1]$ with approximating sequences $(s_i)_i, (t_i)_i \subset J$ such that $s_i \to s$, $t_i \to t$, it holds that

$$\begin{aligned}\mathbf{F}(\widetilde{T}_t - \widetilde{T}_s) &\le \mathbf{F}(\widetilde{T}_t - T_{t_i}) + \mathbf{F}(T_{t_i} - T_{s_i}) + \mathbf{F}(T_{s_i} - \widetilde{T}_s)\\ &\le \mathbf{F}(\widetilde{T}_t - T_{t_i}) + 2\int_{s_i}^{t_i} B(\tau)\,\mathrm{d}\tau + \mathbf{F}(T_{s_i} - \widetilde{T}_s)\end{aligned}$$

and sending $t_i \to t$ and $s_i \to s$ we obtain that (8.14) holds for $t \mapsto \widetilde{T}_t$ at all $s, t \in [0,1]$ with $s < t$. □

As a consequence of the preceding lemma, one may complement the Geometric Transport Equation (8.5) with an initial condition: Given $\overline{T} \in \mathrm{N}_k(\mathbb{R}^d)$ the initial-value problem for the Geometric Transport Equation reads as follows:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} T_t - \mathcal{L}_{v_t} T_t = 0, \\ \qquad\qquad T_0 = \overline{T}, \end{cases} \tag{8.15}$$

with the initial condition understood via a continuous representative of $t \mapsto T_t$. In this context note that T_0 is uniquely defined because for two different weakly* continuous representatives $\widetilde{T}, \widetilde{T}'$ we can always find a sequence $t_j \to 0$ such that $\widetilde{T}_{t_j} = \widetilde{T}'_{t_j} = T_{t_j}$ and hence

$$\widetilde{T}_0 = \operatorname*{w^*\text{-}lim}_{j\to\infty} \widetilde{T}_{t_j} = \operatorname*{w^*\text{-}lim}_{j\to\infty} \widetilde{T}'_{t_j} = \widetilde{T}'_0.$$

One may also understand (8.15) in the following equivalent way: For every $\omega \in \mathcal{D}^k(\mathbb{R}^d)$ and for every $\psi \in C_c^\infty([0,1])$ and $\omega \in \mathcal{D}^k(\mathbb{R}^d)$, it holds that

$$\int_0^1 \langle T_t, \omega\rangle\, \dot{\psi}(t) + \langle \mathcal{L}_{v_t} T_t, \omega\rangle\, \psi(t)\,\mathrm{d}t = \langle \widetilde{T}_1, \omega\rangle\, \psi(1) - \langle \widetilde{T}_0, \omega\rangle\, \psi(0),$$

where $\widetilde{T}_t$ denotes the continuous representative defined in Lemma 8.7. Thus, we may express (8.15) by requiring that for all $\psi \in C_c^\infty([0,1))$ (note the half-open interval $[0,1)$ here) it holds that

$$\int_0^1 \langle T_t, \omega\rangle\, \dot{\psi}(t) + \langle \mathcal{L}_{v_t} T_t, \omega\rangle\, \psi(t)\,\mathrm{d}t = -\langle \overline{T}, \omega\rangle\, \psi(0).$$

In the following we will usually denote the initial value by T_0 by a slight abuse of notation, which should not cause any confusion due to Lemma 8.7.

If the velocity v is smooth, then by similar arguments as in the proof of Lemma 8.6, we see furthermore that the initial-value problem is equivalent to

$$\int_0^1 \big\langle \delta_t \times T_t, \dot{\eta} + \mathcal{L}_v \eta \big\rangle \,\mathrm{d}t = \big\langle T_1, \eta(1, \cdot) \big\rangle - \big\langle T_0, \eta(0, \cdot) \big\rangle, \tag{8.16}$$

for every $\eta \in C^\infty([0,1] \times \mathbb{R}^d; \bigwedge^k(\mathbb{R}^{1+d}))$ (note that in there is no vanishing condition on η at $\{0,1\}$).

8.2 Particular dimensions

We now want to investigate special cases of the Geometric Transport Equation (8.5) for a time-indexed path $t \mapsto T_t \subset \mathrm{N}_k(\mathbb{R}^d)$, where $t \in \mathbb{R}$, that is,

$$\frac{\mathrm{d}}{\mathrm{d}t} T_t = \mathcal{L}_{v_t} T_t = \partial(v_t \wedge T_t) + v_t \wedge \partial T_t,$$

which is always to be understood in the weak sense (8.7). It is instructive to consider the concrete form of this equation for different values of the dimension k. In this way it turns out that the Geometric Transport Equation contains several classical transport-type PDEs as special cases.

Example 8.8 (0-currents). In the case $k = 0$, the normal currents $T_t \in \mathrm{N}_0(\mathbb{R}^d)$ can be identified with measures $\mu_t \in \mathcal{M}(\mathbb{R}^d; \mathbb{R}^d)$, see Proposition 4.12. Then, analogously to the arguments in Example 4.9, and also taking into account that $\partial T_t = 0$ (by convention, all 0-currents are boundaryless), we obtain that

$$\mathcal{L}_{v_t} T_t = \partial(v_t \wedge T_t) = -\operatorname{div} \mu_t.$$

Therefore, the Geometric Transport Equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} \mu_t + \operatorname{div}(v_t \mu_t) = 0,$$

which is the classical **continuity equation**.

Example 8.9 (1-currents). Let us assume that we have a time-indexed path $t \mapsto T_t \in \mathrm{N}_1(\mathbb{R}^d)$ satisfying the Geometric Transport Equation in the weak sense and such that, for almost every t, the current T_t is absolutely continuous, i.e. $T_t \ll \mathcal{L}^d$. Then $T_t = \tau_t \mathcal{L}^d$ for a time-dependent vector field $\tau_t = (\tau_t^1, \ldots, \tau_t^d)^\mathsf{T}$ on $\mathbb{R}^d$.

For every smooth compactly supported 1-form $\omega = \sum_{j=1}^d \omega_j \,\mathrm{d}x^j$, we have

$$\big\langle T_t, \omega \big\rangle = \int_{\mathbb{R}^d} \sum_{j=1}^d \tau_t^j \omega_j \,\mathrm{d}x.$$

Moreover, for the boundary and contraction (see (3.7)) of ω in coordinates we obtain

$$\mathrm{d}\omega = \sum_{i,j=1}^{d} \partial_i \omega_j \, \mathrm{d}x^i \wedge \mathrm{d}x^j, \qquad \iota_{v_t}\omega = \sum_{i=1}^{d} v_t^i \omega_i,$$

and also, using $\iota_{v_t}(\mathrm{d}x^i \wedge \mathrm{d}x^j) = v_t^i \mathrm{d}x^j - v_t^j \mathrm{d}x^i$,

$$\iota_{v_t}(\mathrm{d}\omega) = \sum_{i,j=1}^{d} \partial_i \omega_j \, (v_t^i \mathrm{d}x^j - v_t^j \mathrm{d}x^i) = \sum_{i,j=1}^{d} v_t^i \, (\partial_i \omega_j - \partial_j \omega_i) \, \mathrm{d}x^j.$$

Hence,

$$\begin{aligned}
\big\langle \partial(v_t \wedge T_t), \omega \big\rangle &= \big\langle T_t, \iota_{v_t}(\mathrm{d}\omega) \big\rangle \\
&= \int_{\mathbb{R}^d} \sum_{i,j=1}^{d} \tau_t^j v_t^i \, (\partial_i \omega_j - \partial_j \omega_i) \, \mathrm{d}x \\
&= \int_{\mathbb{R}^d} \sum_{i,j=1}^{d} (\tau_t^j v_t^i - v_t^j \tau_t^i) \, \partial_i \omega_j \, \mathrm{d}x \\
&= - \int_{\mathbb{R}^d} \sum_{j=1}^{d} \left[\sum_{i=1}^{d} \partial_i (\tau_t^j v_t^i - v_t^j \tau_t^i) \right] \omega_j \, \mathrm{d}x \\
&= - \int_{\mathbb{R}^d} \sum_{j=1}^{d} \operatorname{div}(\tau_t \otimes v_t - v_t \otimes \tau_t)_j \, \omega_j \, \mathrm{d}x.
\end{aligned}$$

Also, for every $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^d)$,

$$\big\langle \partial T_t, \varphi \big\rangle = \big\langle T_t, \mathrm{d}\varphi \big\rangle = \int_{\mathbb{R}^d} \tau_t \cdot \nabla \varphi \, \mathrm{d}x = - \int_{\mathbb{R}^d} \operatorname{div}(\tau_t) \varphi \, \mathrm{d}x,$$

and therefore

$$\begin{aligned}
\big\langle v_t \wedge \partial T_t, \omega \big\rangle &= \big\langle \partial T_t, \iota_{v_t}\omega \big\rangle \\
&= - \int_{\mathbb{R}^d} \operatorname{div}(\tau_t) \, \iota_{v_t}\omega \, \mathrm{d}x \\
&= - \int_{\mathbb{R}^d} \operatorname{div}(\tau_t) \sum_{i=1}^{d} v_t^i \omega_i \, \mathrm{d}x.
\end{aligned}$$

We conclude that the Geometric Transport Equation in this case reads as

$$\frac{\mathrm{d}}{\mathrm{d}t} \tau_t + \operatorname{div}(\tau_t \otimes v_t - v_t \otimes \tau_t) + v_t \operatorname{div} \tau_t = 0.$$

Example 8.10 (1**-currents, boundaryless).** If in the situation of the previous example we further assume that almost every T_t is boundaryless, or equivalently, $\operatorname{div} \tau_t = 0$, we obtain instead the PDE

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau_t + \operatorname{div}(\tau_t \otimes v_t - v_t \otimes \tau_t) = 0.$$

Moreover, in $\mathbb{R}^3$, we have the standard vector-calculus identity

$$\operatorname{div}(f \otimes g - g \otimes f) = -\operatorname{curl}(g \times f),$$

where $\times$ denotes the vector product; this can be verified by a straightforward calculation. Hence, we may rewrite the PDE as

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau_t - \operatorname{curl}(v_t \times \tau_t) = 0.$$

This equation is connected to some models of incompressible magnetohydrodynamics [13, 14] and is also the basis for line transport in dislocation motion [1, 21].

Example 8.11 (($d-1$)**-currents, boundaryless).** The following duality between $(d-1)$-currents and 1-forms is quite well-known: Given $\alpha \in \mathcal{D}^1(\mathbb{R}^d)$, define the current $T_\alpha \in \mathcal{D}_{d-1}(\mathbb{R}^d)$ by

$$\langle T_\alpha, \omega \rangle := \int_{\mathbb{R}^d} \alpha \wedge \omega = \int_{\mathbb{R}^d} \langle \mathrm{e}_1 \wedge \cdots \wedge \mathrm{e}_d, \alpha \wedge \omega \rangle \,\mathrm{d}x, \qquad \omega \in \mathcal{D}^{d-1}(\mathbb{R}^d),$$

where $\mathrm{e}_1, \dots, \mathrm{e}_d$ is the standard basis in $\mathbb{R}^d$. This T_α is clearly representable by integration with a smooth density. One can then establish the following facts from the Leibniz rule in Lemma 3.10 together with some computations:

$$\partial T_\alpha = T_{\mathrm{d}\alpha}, \qquad \partial(v \wedge T_\alpha) = \partial T_{\iota_v(\alpha)},$$

for any vector field $v \colon \mathbb{R}^d \to \mathbb{R}^d$ (with sufficient smoothness). In fact, the so-defined mapping $\alpha \mapsto T_\alpha$ is even invertible: For *any* absolutely continuous $(d-1)$-current $T = \tau\mathcal{L}^d$ with $\tau \in \mathrm{C}_c^\infty(\mathbb{R}^d; \bigwedge_{d-1}(\mathbb{R}^d))$ define

$$\alpha := \tau \,\lrcorner\, (\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d) \in \mathrm{C}_c^\infty(\mathbb{R}^d; \textstyle\bigwedge^1(\mathbb{R}^d)),$$

where "$\lrcorner$" denotes the interior product defined in (3.6). Then, a computation shows that $\langle \mathrm{e}_1 \wedge \cdots \wedge \mathrm{e}_d, \alpha \wedge \omega \rangle = \langle \tau, \omega \rangle$ and hence $T = T_\alpha$. Geometrically, the interpretation of this duality is that if one thinks of $\alpha(x)$ as a vector (not a covector), then $\vec{T}_\alpha(x)$ spans the orthogonal plane at almost every x.

For a path of boundaryless $(d-1)$-currents $t \mapsto T_t = T_{\alpha_t}$ and a smooth velocity field $v = (v^1, \dots, v^d)^\mathsf{T} \colon \mathbb{R}^d \to \mathbb{R}^d$, we obtain for the corresponding Lie derivative that

$$\langle \mathcal{L}_v T_t, \omega \rangle = \langle \partial(v \wedge T_t), \omega \rangle = \langle T_{\iota_v(\alpha_t)}, \mathrm{d}\omega \rangle.$$

Writing $\alpha = \sum_i \alpha_i \,\mathrm{d}x^i$ and $\omega = \sum_{j=1}^d \omega_j \,\widehat{\mathrm{d}x^j}$, where $\widehat{\mathrm{d}x^j} := \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{j-1} \wedge \mathrm{d}x^{j+1} \wedge \cdots \wedge \mathrm{d}x^d$, we thus get

$$\begin{aligned}
\big\langle \mathcal{L}_v T_t, \omega \big\rangle &= \big\langle T_{\iota_v(\alpha_t)}, \mathrm{d}\omega \big\rangle \\
&= \int_{\mathbb{R}^d} \iota_v(\alpha_t) \wedge \mathrm{d}\omega \\
&= \int_{\mathbb{R}^d} (\vec{\alpha}_t \cdot v) \sum_{j=1}^d (-1)^{j+1} \frac{\partial \omega_j}{\partial x_j} \,\mathrm{d}x \\
&= -\int_{\mathbb{R}^d} \sum_{j=1}^d (-1)^{j+1} \frac{\partial}{\partial x_j} (\vec{\alpha}_t \cdot v)\, \omega_j \,\mathrm{d}x,
\end{aligned}$$

where we have defined $\vec{\alpha} := (\alpha_1, \ldots, \alpha_d)^{\mathsf{T}}$. On the other hand,

$$\frac{\mathrm{d}}{\mathrm{d}t} \big\langle T_\alpha, \omega \big\rangle = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \alpha \wedge \omega = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \sum_{j=1}^d (-1)^{j+1} \alpha_j \omega_j \,\mathrm{d}x,$$

so that the Geometric Transport Equation (8.5) here translates to the system of conservation laws

$$\frac{\mathrm{d}}{\mathrm{d}t} \vec{\alpha} + \nabla(\vec{\alpha} \cdot v) = 0.$$

Example 8.12 (d-currents). For a path $t \mapsto T_t \in \mathcal{D}_d(\mathbb{R}^d)$ of top-dimensional currents, we may write $T_t = u_t \,\mathrm{e}_1 \wedge \cdots \wedge \mathrm{e}_d \,\mathcal{L}^d$ with a scalar function u_t. Then, $v_t \wedge T_t = 0$ and hence the Geometric Transport Equation (8.5) reads as

$$\frac{\mathrm{d}}{\mathrm{d}t} T_t - v_t \wedge \partial T_t = 0.$$

Every d-form $\omega \in \mathcal{D}^d(\mathbb{R}^d)$ can be written as $\omega = \overline{\omega} \,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d$ with $\overline{\omega} \in \mathrm{C}_c^\infty(\mathbb{R}^d)$, and hence

$$\iota_{v_t} \omega = \sum_{j=1}^d (-1)^{j-1} v_t^j \,\overline{\omega}\, \widehat{\mathrm{d}x^j},$$

whereby

$$\begin{aligned}
\big\langle v_t \wedge \partial T_t, \omega \big\rangle &= \big\langle T_t, \mathrm{d}(\iota_{v_t} \omega) \big\rangle \\
&= \int_{\mathbb{R}^d} u_t \sum_{j=1}^d (-1)^{j-1} \frac{\partial}{\partial x_j} \big((-1)^{j-1} v_t^j \overline{\omega} \big) \,\mathrm{d}x \\
&= -\int_{\mathbb{R}^d} \nabla u_t \cdot v_t \,\overline{\omega} \,\mathrm{d}x.
\end{aligned}$$

Therefore, in this case we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}u_t + v_t \cdot \nabla u_t = 0,$$

which is the **(classical) transport equation**.

Remark 8.13. The work [12] (see also [16, 17]) takes a dual point of view to ours and instead considers the general transport equation for differential k-forms. The transport equation is now recovered for the case of 0-forms (scalars) and the continuity equation for the case of top-dimensional forms (which represent volumes). However, currents are in a sense the more natural objects to transport since they have a clearer geometric interpretation. Moreover, the theory of currents allows to treat lower-regularity situations without any additional effort. A concrete situation is the following: It is quite common to model continuously-distributed dislocation fields via the torsion 2-form of an additional "material" connection on a manifold. This additional connection is assumed to be flat (at least, if no *disclinations* are present), but with non-zero torsion, turning a body manifold into a so-called Weitzenböck manifold; see, e.g., [18] for an overview of this. The torsion of the material connection is then represented via the *torsion* 2*-form*, which represents the dislocations. However, this approach only makes sense in the smooth setting since it is unclear (or at least not classically defined) what "singular" forms should be. Thus, individual dislocation lines cannot be represented. If one instead uses the duality in Example 8.11, one obtains 1-currents. This not only corresponds better to the intuition of dislocation *lines*, but also immediately opens the door to a low-regularity theory. In this way, The Geometric Transport Equation gives a rigorous interpretation of the "transport of singular torsion".

8.3 Existence of solutions

When our velocity vector field v is autonomous and Lipschitz, we have the following existence theorem

Theorem 8.14 (Existence of weak solutions [9, 10]). *Let $v\colon \mathbb{R}^d \to \mathbb{R}^d$ be a globally bounded and Lipschitz vector field and let Φ be its (unique) flow. Let $T_0 \in \mathrm{N}_k(\mathbb{R}^d)$ be a given initial value. Then, the path of normal currents defined by*

$$t \mapsto T_t := \Phi^t_* T_0$$

is a weak solution to (8.15) *and* (T_t) *is uniformly normal, that is,* (8.8) *holds.*

Remark 8.15. Note that the explicit formula for the solution in particular implies that if $\overline{T}$ is integral, then so is T_t for all $t \in (0, 1)$.

Proof. Step 1. Assume first that v is smooth. Then, the path $t \mapsto T_t$ satisfies the mass bound (8.8) by Proposition 4.30. The bound (8.6) is trivially satisfied due to the global boundedness of v. We will verify the condition in Lemma 8.6: Fix $\omega \in \mathcal{D}^k(\mathbb{R}^d)$. For $s < t$ it holds that

$$\begin{aligned}\bigl|\langle T_t, \omega\rangle - \langle T_s, \omega\rangle\bigr| &= \bigl|\langle (\Phi^t - \Phi^s)_* T_0, \omega\rangle\bigr| \\ &\le \|\omega\|_\infty \int \bigl\|\mathrm{D}(\Phi^t - \Phi^s)\vec{T}_0\bigr\| \,\mathrm{d}\|T_0\| \\ &\le C\mathbf{M}(T_0)\|\omega\|_\infty \|\mathrm{D}(\Phi^t - \Phi^s))\|_\infty^k \\ &\le C|t-s|\end{aligned}$$

since the map $t \mapsto \Phi^t$ is Lipschitz with values in $\mathrm{C}^1(\mathbb{R}^d; \mathbb{R}^d)$ by Proposition 3.16 (iv).

On the other hand, by Proposition 8.1,

$$\begin{aligned}\langle \mathcal{L}_v T_t, \omega\rangle &= \langle T_t, \mathcal{L}_v \omega\rangle \\ &= \left\langle T_t, \lim_{h\to 0} \frac{(\Phi^h)^*\omega - \omega}{h}\right\rangle \\ &= \left\langle \Phi^t_* T_0, \lim_{h\to 0} \frac{(\Phi^h)^*\omega - \omega}{h}\right\rangle \\ &= \lim_{h\to 0} \frac{1}{h}\langle T_0, (\Phi^{t+h})^*\omega - (\Phi^t)^*\omega\rangle \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\langle T_0, (\Phi^t)^*\omega\rangle \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\langle T_t, \omega\rangle,\end{aligned}$$

where we have used the group property (3.3). The validity of the initial condition is obvious.

Step 2. In the general case, we cannot perform the manipulations in Step 1. However, we can argue by mollification as follows: Let $(\varrho_\delta)_{\delta>0} \subset \mathrm{C}_c^\infty(\mathbb{R}^d)$ be a family of mollifiers with $\varrho_\delta \to \delta_0$ as $\delta \to 0$. Then set

$$v^\delta := \varrho_\delta * v,$$

which converges to v locally uniformly and with uniformly bounded Lipschitz constants. Moreover, if we denote by $(\Phi^t_\delta)_t$ the flow of v^δ, then

$$T_t := \Phi^t_* T_0 = \operatorname*{w^*\text{-}lim}_{\delta\to 0} T_t^\delta, \qquad T_t^\delta := (\Phi^t_\delta)_* T_0,$$

by Proposition 4.30.

From Step 1 we know that $(T_t^\delta)_t$ is a weak solution to the Geometric Transport Equation. It is clear from the properties of pushforwards of normal currents (see Proposition 4.30) that the condition (8.6) to the limit $\delta \to 0$. Moreover, (8.7) in condition (8.7) for v^δ reads as

$$\int_0^1 \langle T_t^\delta, \omega\rangle\, \dot{\psi}(t) + \langle v^\delta \wedge \partial T_t^\delta, \omega\rangle\, \psi(t) + \langle v^\delta \wedge T_t^\delta, \mathrm{d}\omega\rangle\, \psi(t)\, \mathrm{d}t = 0$$

for every $\psi \in \mathrm{C}_c^\infty((0,1))$ and every $\omega \in \mathcal{D}^k(\mathbb{R}^d)$. Here we have written out the definition (8.2) of the Lie derivative, namely,

$$\mathcal{L}_{v^\delta} T_t^\delta = v^\delta \wedge \partial T_t^\delta + \partial(v^\delta \wedge T_t^\delta)$$

Now, since $v^\delta \to v$ uniformly and $T_t^\delta \overset{*}{\rightharpoonup} T_t$ (which implies also $\partial T_t^\delta \overset{*}{\rightharpoonup} \partial T_t$), the above weak formulation of the Geometric Transport Equation passes to the limit, showing (8.7) for $(T_t)_t$ and velocity v, that is, condition (8.7). □

A result for non-autonomous (time-dependent) velocity fields can be found in [8].

8.4 Uniqueness of solutions

Theorem 8.16 (Uniqueness of weak solutions [9, 10]). *Let $v\colon \mathbb{R}^d \to \mathbb{R}^d$ be a globally bounded and Lipschitz vector field and let Φ be its (unique) flow. Let $T_0 \in \mathrm{N}_k(\mathbb{R}^d)$ be a given initial value. Then, the weak solution to* (8.15) *defined in Theorem 8.14 is the unique weak solution in the class of normal currents.*

The proof for a smooth velocity field v is relatively straightforward:

Proof for a smooth velocity. We will employ a duality strategy. By the linearity of the Geometric Transport Equation, it is enough to show that if $T_0 = 0$ then for the solution $t \mapsto T_t$ it holds that $T_t = 0$ for $\mathcal{L}^1$-a.e. $t \in (0,1)$. Fix $\zeta \in \mathcal{D}^k([0,1] \times \mathbb{R}^d)$ and consider the family of k-forms on $\mathbb{R}^d$ given for $t \in (0,1)$ by

$$\eta_t := -\int_t^1 (\Phi^{s-t})^* \zeta_s \, \mathrm{d}s,$$

where $\zeta_s := \iota_s^* \zeta \in \mathcal{D}^k(\mathbb{R}^d)$ is the pullback under the immersion $\iota_s\colon x \mapsto (s,x)$. Then,

$$\dot{\eta}_t = \zeta_t - \int_t^1 \frac{\mathrm{d}}{\mathrm{d}t} (\Phi^{s-t})^* \zeta_s \, \mathrm{d}s, \tag{8.17}$$

which is to be understood as a pointwise equality in $x \in \mathbb{R}^d$. On the other hand, by linearity of the pullback and the dominated convergence theorem,

$$\begin{aligned}
\mathcal{L}_v \eta_t &= \lim_{h \to 0} \frac{(\Phi^h)^* \eta_t - \eta_t}{h} \\
&= -\int_t^1 \lim_{h \to 0} \frac{(\Phi^h)^*(\Phi^{s-t})^* \zeta_s - (\Phi^{s-t})^* \zeta_s}{h} \, \mathrm{d}s \\
&= \int_t^1 \frac{\mathrm{d}}{\mathrm{d}t} (\Phi^{s-t})^* \zeta_s \, \mathrm{d}s.
\end{aligned}$$

Combining this with (8.17), we arrive at

$$\dot{\eta} + \mathcal{L}_v \eta = \zeta.$$

Since $(T_t)_t$ solves the Geometric Transport Equation with initial value $T_0 = 0$, we infer from (8.16) that

$$\int_0^1 \langle \delta_t \times T_t, \zeta \rangle \, \mathrm{d}t = \int_0^1 \langle \delta_t \times T_t, \dot{\eta} + \mathcal{L}_v \eta \rangle \, \mathrm{d}t = \langle T_1, \eta_1 \rangle - \langle T_0, \eta_0 \rangle = 0,$$

where for the last equality we have also used $\eta_1 = 0$. From Lemma 8.3 (ii) we thus infer that $T_t = 0$ for $\mathcal{L}^1$-a.e. $t \in (0,1)$, as required. $\square$

The general case is complicated by the fact that pushforwards of currents under Lipschitz maps are defined via an approximation procedure (see Proposition 4.30) and that the gradient of Φ^h may not be defined everywhere on the carrier set of T_0. Indeed, Rademacher's Theorem 1.18 only shows the existence of $\mathrm{D}\Phi^h$ almost everywhere with respect to $\mathcal{L}^{1+d}$, but this may not be enough to define it almost everywhere with respect to the mass measure of the current at hand. Thus, the above arguments are not applicable since (8.16) is only sufficient in the case of a smooth velocity field.

Proof of Theorem 8.16 for a general velocity. Without loss of generality we assume that $(T_t)_{t \in (0,1)} \subset \mathrm{N}_k(\mathbb{R}^d)$ is a weakly* continuous solution to the Geometric Transport Equation (see Lemma 8.7 for the weak* continuity).

Step 1. We will first prove the theorem under the additional assumption that $\partial T_t = 0$ for $t \in [0,1)$. In Step 5 we will extend the proof to the general case.

Define

$$\vec{T}(t,x) := (0, \vec{T}_t(x)) \in \textstyle\bigwedge_k \mathbb{R}^{1+d} \qquad \text{for } \mathrm{d}t \otimes \|T_t\|\text{-a.e. } (t,x),$$

which is a shorthand for the more precise definition $\vec{T}(t,x) := \mathrm{D}e_t[\vec{T}_t(x)]$ for $e_t(x) := (t,x)$. Note also that here we write $\vec{T}(t,x)$ as a row instead of as a column vector (which is our usual convention) to avoid a proliferation of transpositions.

We claim that

$$U = u\mu \qquad \text{with} \qquad u := (1,v) \wedge \vec{T}, \quad \mu := \mathrm{d}t \otimes \|T_t\| \in \mathcal{M}(\mathbb{R}^{1+d})$$

is a normal $(k+1)$-current in $(0,1) \times \mathbb{R}^d$, and

$$\partial U \llcorner (0,1) \times \mathbb{R}^d = 0. \tag{8.18}$$

It is easy to see that

$$\mathbf{M}(U) \le C \int_0^1 \int_{\mathbb{R}^d} |v_t| \, \mathrm{d}\|T_t\| \, \mathrm{d}t < \infty,$$

where we used the integrability condition (8.6) from the definition of weak solutions. Thus, the normality of U will hold as soon as we have verified (8.18).

To show (8.18), we employ the density of linear combinations of tensor forms (see, e.g., [19, 4.1.3 / 4.1.8]) to observe that it suffices to verify

$$\big\langle U, \mathrm{d}\eta \big\rangle = 0 \tag{8.19}$$

for $\eta = \mathbf{t}^*\alpha \wedge \mathbf{p}^*\beta$, where either (I) $\alpha \in \mathcal{D}^0((0,1))$ and $\beta \in \mathcal{D}^k(\mathbb{R}^d)$, or (II) $\alpha \in \mathcal{D}^1((0,1))$ and $\beta \in \mathcal{D}^{k-1}(\mathbb{R}^d)$. For case (I) let $\alpha \in \mathcal{D}^0((0,1))$ and $\beta \in \mathcal{D}^k(\mathbb{R}^d)$. By the Leibniz rule for the exterior differential,

$$\mathrm{d}\eta = \mathbf{t}^*\mathrm{d}\alpha \wedge \mathbf{p}^*\beta + \mathbf{t}^*\alpha \wedge \mathbf{p}^*\mathrm{d}\beta.$$

We compute

$$\begin{aligned}
\big\langle U, \mathbf{t}^*\mathrm{d}\alpha \wedge \mathbf{p}^*\beta \big\rangle &= \int_0^1 \int_{\mathbb{R}^d} \big\langle (1,v) \wedge \vec{T}, \mathbf{t}^*\mathrm{d}\alpha \wedge \mathbf{p}^*\beta \big\rangle \, \mathrm{d}\|T_t\| \, \mathrm{d}t \\
&= \int_0^1 \int_{\mathbb{R}^d} \dot{\alpha}(t) \, \big\langle \vec{T}_t, \beta \big\rangle \, \mathrm{d}\|T_t\| \, \mathrm{d}t \\
&= \int_0^1 \dot{\alpha}(t) \, \big\langle T_t, \beta \big\rangle \, \mathrm{d}t,
\end{aligned}$$

where we have invoked Lemma 3.8, using that $\mathrm{span}(\vec{T}) \subset \{0\} \times \mathbb{R}^d$ is space-like and $\mathbf{t}^*\mathrm{d}\alpha = \dot{\alpha}(t)\,\mathrm{d}t$ is time-like.

On the other hand,

$$\begin{aligned}
\big\langle U, \mathbf{t}^*\alpha \wedge \mathbf{p}^*\mathrm{d}\beta \big\rangle &= \int_0^1 \int_{\mathbb{R}^d} \big\langle (1,v) \wedge \vec{T}, \mathbf{t}^*\alpha \wedge \mathbf{p}^*\mathrm{d}\beta \big\rangle \, \mathrm{d}\|T_t\| \, \mathrm{d}t \\
&= \int_0^1 \int_{\mathbb{R}^d} \alpha(t) \, \big\langle (1,v) \wedge \vec{T}, \mathbf{p}^*\mathrm{d}\beta \big\rangle \, \mathrm{d}\|T_t\| \, \mathrm{d}t \\
&= \int_0^1 \int_{\mathbb{R}^d} \alpha(t) \, \big\langle v \wedge \vec{T}_t, \mathrm{d}\beta \big\rangle \, \mathrm{d}\|T_t\| \, \mathrm{d}t \\
&= \int_0^1 \alpha(t) \, \big\langle \partial(v \wedge T_t), \beta \big\rangle \, \mathrm{d}t.
\end{aligned}$$

Using the weak formulation (8.7) we deduce (8.19) in case (I).

For case (II) let $\alpha \in \mathcal{D}^1((0,1))$ and $\beta \in \mathcal{D}^{k-1}(\mathbb{R}^d)$. Applying the Leibniz rule again,

$$\mathrm{d}\eta = \mathbf{t}^*\mathrm{d}\alpha \wedge \mathbf{p}^*\beta - \mathbf{t}^*\alpha \wedge \mathbf{p}^*\mathrm{d}\beta = -\mathbf{t}^*\alpha \wedge \mathbf{p}^*\mathrm{d}\beta$$

since $\mathrm{d}\alpha = 0$. Then,

$$\begin{aligned}
\big\langle U, \mathbf{t}^*\alpha \wedge \mathbf{p}^*\mathrm{d}\beta \big\rangle &= -\int_0^1 \int_{\mathbb{R}^d} \big\langle (1,v) \wedge \vec{T}, \mathbf{t}^*\alpha \wedge \mathbf{p}^*\mathrm{d}\beta \big\rangle \, \mathrm{d}\|T_t\| \, \mathrm{d}t \\
&= -\int_0^1 \int_{\mathbb{R}^d} \big\langle (1,v), \mathbf{t}^*\alpha \big\rangle \big\langle \vec{T}, \mathbf{p}^*\mathrm{d}\beta \big\rangle \, \mathrm{d}\|T_t\| \, \mathrm{d}t \\
&= -\int_0^1 \int_{\mathbb{R}^d} \alpha(t) \, \big\langle \vec{T}_t, \mathrm{d}\beta \big\rangle \, \mathrm{d}\|T_t\| \, \mathrm{d}t \\
&= -\int_0^1 \alpha(t) \, \big\langle \partial T_t, \beta \big\rangle \, \mathrm{d}t \\
&= 0.
\end{aligned}$$

Passing from the first to the second line we have used Lemma 3.8 (i), while the last equality follows from $\partial T_t = 0$. So, (8.19) also follows in case (II).

Step 2. Notice that $\Psi := \overline{\Phi}^{-1}$ is Lipschitz, so via Proposition 4.30 we can define

$$W := \Psi_* U.$$

By Theorem 4.31 we can write this explicitly as

$$W = \Psi_*[u] \, \Psi_\# \mu$$

with the pushforward $\Psi_*[u]$ of the multi-vector field u, which is given by

$$\Psi_*[u](t,x) = \mathrm{D}\overline{\Phi}^{-1}(\overline{\Phi}(t,x))[u(\overline{\Phi}(t,x))]$$

for $\|U\|$-almost every (t,x). In fact, $\mu \ll \|U\| \ll \mu$ because $1 \leq |u| \leq (1 + \|v\|_\infty)$, whereby this formula for $\Psi_*[u]$ holds μ-almost everywhere.

By definition of $\overline{\Phi}^{-1}$ and Lemma 3.17 (applied with $s = t$ and $y = \Phi^t(x)$), we obtain

$$\Psi_*[u] = (1,0) \wedge \Psi_*[\vec{T}] \qquad \mu\text{-a.e.}, \tag{8.20}$$

which is still well-defined since $\mathrm{D}\overline{\Phi}^{-1}(\overline{\Phi}(t,x))$ exists on $\operatorname{span} \vec{T} \subset \operatorname{span} \vec{U}$.

Step 2. Since the boundary operator commutes with the pushforward, see Lemma 4.26, it holds that $\partial W = 0$ in $(0,1) \times \mathbb{R}^d$. Therefore, using again the Leibniz formula recalled above, for every $\alpha \in \mathcal{D}^0(\mathbb{R})$ and $\beta \in \mathcal{D}^k(\mathbb{R}^d)$,

$$0 = \big\langle \partial W, \mathbf{t}^*\alpha \wedge \mathbf{p}^*\beta \big\rangle = \big\langle W, \mathbf{t}^*\mathrm{d}\alpha \wedge \mathbf{p}^*\beta \big\rangle + \big\langle W, \mathbf{t}^*\alpha \wedge \mathbf{p}^*\mathrm{d}\beta \big\rangle.$$

The second term vanishes by Lemma 3.8 (ii) since $\mathbf{p}^*\mathrm{d}\beta$ contains only terms not involving $\mathrm{d}t$, so that the vector $(1,0)$ is orthogonal to $\operatorname{span} \mathbf{p}^*\mathrm{d}\beta$, and also taking into account (8.20).

We will show that

$$0 = \langle W, \mathbf{t}^*\mathrm{d}\alpha \wedge \mathbf{p}^*\beta \rangle = \int_0^1 \dot{\alpha}(t) \langle \Phi_*^{-t} T_t, \beta \rangle \,\mathrm{d}t. \tag{8.21}$$

First, by Lemma 3.8 (i) we have

$$\langle (1,0) \wedge \Psi_*[\vec{T}], \mathbf{t}^*\mathrm{d}\alpha \wedge \mathbf{p}^*\beta \rangle = \langle (1,0), \mathbf{t}^*\mathrm{d}\alpha \rangle \cdot \langle \Psi_*[\vec{T}], \mathbf{p}^*\beta \rangle = \dot{\alpha}(t) \langle \Psi_*[\vec{T}], \mathbf{p}^*\beta \rangle.$$

Next observe that the maps $\mathbf{p} \circ \Psi$ and $\Phi^{-t} \circ \mathbf{p}$ coincide on the plane $\{t\} \times \mathbb{R}^d$. Accordingly, given any k-vector $v \in \bigwedge_k(\{0\} \times \mathbb{R}^d)$, we have

$$(\mathbf{p} \circ \Psi)_* v = (\Phi^{-t} \circ \mathbf{p})_* v.$$

In particular, we can apply the previous equality with $v = \vec{T}$ to obtain

$$\begin{aligned}
(\mathbf{p} \circ \Psi)_*[\vec{T}](t,x) &= (\Phi^{-t} \circ \mathbf{p})_*[\vec{T}](t,x) \\
&= \Phi_*^{-t} \mathbf{p}_*[\vec{T}](t,x) \\
&= \mathrm{D}\Phi^{-t}(\Phi^t(x))[\vec{T}_t(\Phi^t(x))]
\end{aligned}$$

for almost every (t,x) with respect to the measure $\Psi_\# \mu = \mathrm{d}t \otimes \Phi_\#^{-t} \|T_t\|$. Integrating,

$$\begin{aligned}
\langle W, \mathbf{t}^*\mathrm{d}\alpha \wedge \mathbf{p}^*\beta \rangle &= \int_{(0,1)\times\mathbb{R}^d} \langle w, \mathbf{t}^*\mathrm{d}\alpha \wedge \mathbf{p}^*\beta \rangle \,\mathrm{d}\Psi_\#\mu \\
&= \int_0^1 \dot{\alpha}(t) \int_{\mathbb{R}^d} \langle \Psi_*[\vec{T}](t,x), \mathbf{p}^*\beta(t,x) \rangle \,\mathrm{d}\Phi_\#^{-t}\|T_t\|(x) \,\mathrm{d}t \\
&= \int_0^1 \dot{\alpha}(t) \int_{\mathbb{R}^d} \langle \mathrm{D}\Phi^{-t}(\Phi^t(x))[\vec{T}_t(\Phi^t(x))], \beta(x) \rangle \,\mathrm{d}\Phi_\#^{-t}\|T_t\|(x) \,\mathrm{d}t \\
&= \int_0^1 \dot{\alpha}(t) \langle \Phi_*^{-t} T_t, \beta \rangle \,\mathrm{d}t,
\end{aligned}$$

where we used again the explicit formula for the pushforward $\Phi_*^{-t} T_t$ given by Theorem 4.31. Thus, (8.21) holds.

Step 4. In the previous step we have shown that

$$\int_0^1 \dot{\alpha}(t) \langle \Phi_*^{-t} T_t, \beta \rangle \,\mathrm{d}t = 0$$

for every $\alpha \in \mathcal{D}^0((0,1))$ and $\beta \in \mathcal{D}^k(\mathbb{R}^d)$.

Via an integration-by-parts, thus the function $t \mapsto \langle \Phi_*^{-t} T_t, \beta \rangle$ is constant for any fixed $\beta \in \mathcal{D}^k(\mathbb{R}^d)$ and thus equal to its value at $t = 0$ (recall that we have chosen a weakly* continuous representative of $t \mapsto T_t$). Consequently,

$$\Phi_*^{-t} T_t = \overline{T},$$

or equivalently, $T_t = \Phi_*^t \overline{T}$. Thus the theorem holds in the case $\partial T_t = 0$ for $\mathcal{L}^1$-almost every $t \in (0, 1)$.

Step 5. Finally, if $(T_t)_t$ is a solution of (8.15) with the T_t's not necessarily boundaryless. Testing (8.7) with forms $\omega = \mathrm{d}\eta$ for $\eta \in \mathcal{D}^{k-1}(\mathbb{R}^d)$, one observes that boundary family $(\partial T_t)_t$ is a weak solution to the Geometric Transport Equation with the same velocity vector field v and initial value ∂T_0.

By Proposition 3.11, we have that $\partial(\partial T_t) = \partial^2 T_t = 0$, so Step 4 becomes applicable, so that

$$\partial T_t = \Phi_*^t(\partial T_0) = \partial(\Phi_*^t T_0).$$

The currents

$$S_t := T_t - \Phi_*^t T_0$$

are normal and $\partial S_t = 0$, By the linearity of the Geometric Transport Equation, the family $(S_t)_t$ are a weak solution for the initial value $S_0 = 0$. But by Step 4 the unique solution in this case is the family of zero currents, whereby we conclude that $S_t = 0$ for every $t \in (0, 1)$. Hence, $T_t = \Phi_*^t T_0$ for every $t \in (0, 1)$ and the proof is complete in the general case. $\square$

8.5 Advection theorem

The theme of the final two sections in this chapter is the relationship between paths of currents, that is, mappings $t \mapsto T_t \in \mathrm{I}_k(\mathbb{R}^d)$ and the space-time current $S \in \mathrm{I}_{1+k}(\mathbb{R}^{1+d})$ that is traced out by the path, so that $\mathbf{p}_* S|_t = T_t$. In particular, one should ask when we may "glue together" the T_t into such an S, which chiefly a question about rectifiability, as well as when we can find a vector field that agrees with the geometric derivative of the resulting glued space-time current S. Since this is a question about "differentiability" of S (in the geometric sense) and since a Lipschitz-condition on the path $t \mapsto T_t$ will play a decisive role, such results may be considered to be in the spirit of Rademacher's Differentiability Theorem 1.18.

We first consider the situation of a given (integral) space-time current $S \in \mathrm{I}_{1+k}(\mathbb{R}^{1+d})$, for which we recall that

$$\frac{\mathrm{D}}{\mathrm{D}t} S \in \mathrm{L}^1(\mathrm{d}t \otimes \|S|_t\|; \mathbb{R}^d)$$

by Proposition 6.10. It is a natural question to ask whether the slices are **advected** by $\frac{\mathrm{D}}{\mathrm{D}t}$, that is, if the family of currents

$$S(t) := \mathbf{p}_* S|_t, \tag{8.22}$$

with the slices $S|_t$ defined in Proposition 6.1, solve the Geometric Transport Equation for the velocity field $v := \frac{\mathrm{D}}{\mathrm{D}t}$. Let us first consider the question

Example 8.17. For $k = 0$ and $u \in \mathrm{BV}([0,1])$ consider the associated graph current $S_u := \tau\, \mathcal{H}^1 \llcorner \mathrm{graph}(u) \in \mathrm{I}_1([0,1] \times \mathbb{R})$ as in Example 6.13. From Example 6.13 we know that in this case

$$\frac{\mathrm{D}}{\mathrm{D}t} S = \dot{u},$$

with $\dot{u}$ only the *absolutely continuous* part of the (distributional or measure-theoretic) derivative Du of u, which in general also includes jump and Cantor parts. Now, assume that u is not constant, but $\dot{u} \equiv 0$. In BV there are in fact many such functions, namely u could be a pure jump function (such as $u := \mathbb{1}_{(1/2,1)}$) or a Cantor-type function (see Example 4.10), for both of which it is well-known that $\dot{u} \equiv 0$ (see [4] for the details). If S_u were to be a weak solution of the Geometric Transport Equation with the velocity field $v\colon [0,1] \times \mathbb{R} \to \mathbb{R}$, then (8.7) must hold, that is,

$$\int_0^1 \big\langle T_t, \omega \big\rangle\, \dot{\psi}(t) + \big\langle \mathcal{L}_{v_t} T_t, \omega \big\rangle\, \psi(t)\; \mathrm{d}t = 0$$

for all $\psi \in \mathrm{C}_c^\infty((0,1))$ and every $\omega \in \mathcal{D}^0(\mathbb{R}) \simeq \mathrm{C}_c^\infty(\mathbb{R})$. As $T_t = \delta_{u(t)}$ for $\mathcal{L}^1$-almost every $t \in [0,1]$ and $\mathcal{L}_{v_t} = \partial(\dot{u}\delta_{u(t)}) \equiv 0$, we obtain the necessary condition

$$\int_0^1 \omega(u(t))\, \dot{\psi}(t)\; \mathrm{d}t = 0.$$

However, this would imply that the distributional (time-)derivative Du is zero, whereby u would be constant, a contradiction. Note that in this case the currents S_u are not AC by Lemma 7.8 since we have seen in Example 6.13 that $\mathrm{Var}(S_u, \cdot) = Du$ and in both both the pure jump and the Cantor-type example $Du \not\ll \mathcal{L}^1$.

We conclude from the previous example that our question about the validity of the Geometric Transport Equation only makes sense if S is at least AC, which we henceforth assume. However, this turns out to be not enough. Indeed, the Flat Mountain S_{FM} constructed in Section 7.3 has zero geometric derivative everywhere and $\mathrm{Var}(S_{\mathrm{FM}}; \cdot) \ll \mathcal{L}^1$, so S_{FM} is AC by Lemma 7.8, but S_{FM} is non-constant in time. Thus, S_{FM} is not advected by any velocity field.

On the other hand, the Flat Mountain is not NC and this turns out to be the correct condition to ensure the advection by a space-time current's own geometric derivative:

Theorem 8.18 (Advection [10]). *Let $S \in \mathrm{I}_{1+k}([0,1] \times \mathbb{R}^d)$ be NC and $(\partial S) \llcorner (0,1) \times \mathbb{R}^d = 0$. Then,*

$$\frac{\mathrm{d}}{\mathrm{d}t} S(t) - \mathcal{L}_{v_t} S(t) = 0 \qquad \textit{for} \qquad v := \frac{\mathrm{D}}{\mathrm{D}t} S. \tag{8.23}$$

Proof. The integrability condition (8.6) in the definition of a weak solution to the Geometric Transport equation holds by Proposition 6.10.

Combining the NC-property of S with Lemma 7.3, we have the disintegration

$$S = S \llcorner \mathrm{Reg}(S) = \mathrm{d}t \otimes (|\nabla^S \mathbf{t}|^{-1}\, \xi \wedge S|_t)$$

or, equivalently,

$$\int_0^1 \frac{\xi}{|\nabla^S \mathbf{t}|} \wedge S|_t\, \psi(t)\, \mathrm{d}t = (\psi \circ \mathbf{t})S \tag{8.24}$$

for every $\psi \in \mathrm{C}_c^\infty((0,1))$, with ξ is given by (6.18). Here and in the following, the integral on the left-hand side should be understood as for measures, that is,

$$\left\langle \int_0^1 \frac{\xi}{|\nabla^S \mathbf{t}|} \wedge S|_t\, \psi(t)\, \mathrm{d}t, \eta \right\rangle = \int_0^1 \left\langle \frac{\xi}{|\nabla^S \mathbf{t}|} \wedge S|_t\, \psi(t), \eta \right\rangle \mathrm{d}t$$

for any $\eta \in \mathcal{D}_{k+1}(\mathbb{R}^d)$.

Recall the definition of the geometric derivative from (6.23), namely

$$\frac{\mathrm{D}}{\mathrm{D}t} S = \frac{\mathbf{p}\xi}{\mathbf{t}\xi} = \frac{\mathbf{p}\xi}{|\nabla^S \mathbf{t}|}, \qquad (\mathrm{d}t \otimes \|S|_t\|)\text{-a.e.},$$

with the second equality following from (6.22). Then, we compute via (8.24) as follows:

$$\begin{aligned}
\int_0^1 v_t \wedge S(t)\, \psi(t)\, \mathrm{d}t &= \int_0^1 \frac{\mathbf{p}(\xi)}{|\nabla^S \mathbf{t}|} \wedge \mathbf{p}_* S|_t\, \psi(t)\, \mathrm{d}t \\
&= \int_0^1 \mathbf{p}_* \left(\frac{\xi}{|\nabla^S \mathbf{t}|} \wedge S|_t \right) \psi(t)\, \mathrm{d}t \\
&= \mathbf{p}_* \left(\int_0^1 \frac{\xi}{|\nabla^S \mathbf{t}|} \wedge S|_t\, \psi(t)\, \mathrm{d}t \right) \\
&= \mathbf{p}_*[(\psi \circ \mathbf{t})S].
\end{aligned}$$

Then use the definition of the Lie derivative to compute for every $\omega \in \mathcal{D}_k(\mathbb{R}^d)$ that

$$\begin{aligned}
\int_0^1 \langle \mathcal{L}_{v_t} S(t), \omega \rangle\, \psi(t)\, \mathrm{d}t &= \int_0^1 \langle v_t \wedge S(t)\, \psi(t), \mathrm{d}\omega \rangle\, \psi(t)\, \mathrm{d}t \\
&= \langle \mathbf{p}_*[(\psi \circ \mathbf{t})S], \mathrm{d}\omega \rangle \\
&= \langle \partial \mathbf{p}_*[(\psi \circ \mathbf{t})S], \omega \rangle \\
&= \langle \mathbf{p}_* \partial[(\psi \circ \mathbf{t})S], \omega \rangle
\end{aligned}$$

For any k-current T, for any $f \in \mathrm{C}_c^\infty(\mathbb{R})$, and any $\omega \in \mathcal{D}_k(\mathbb{R}^d)$, by Lemma 4.11 it holds that

$$\partial[(\psi \circ \mathbf{t})S] = (\psi \circ \mathbf{t})\partial S - S \llcorner \mathrm{d}(\psi \circ \mathbf{t}) = (\psi \circ \mathbf{t})\partial S - S \llcorner (\dot{\psi} \circ \mathbf{t})\, \mathrm{d}t.$$

Also taking into account (6.2) in the form

$$\int S|_t \, \dot{\psi}(t) \, \mathrm{d}t = S \llcorner (\dot{\psi} \circ \mathbf{t}) \, \mathrm{d}t,$$

we obtain

$$\begin{aligned}\int_0^1 \langle \mathcal{L}_{v_t} S(t), \omega \rangle \, \psi(t) \, \mathrm{d}t &= \langle \partial[(\psi \circ \mathbf{t})S], \mathbf{p}^*\omega \rangle \\ &= \langle (\psi \circ \mathbf{t})\partial S, \mathbf{p}^*\omega \rangle - \langle S \llcorner (\dot{\psi} \circ \mathbf{t}) \, \mathrm{d}t, \mathbf{p}^*\omega \rangle \\ &= \langle (\psi \circ \mathbf{t})\partial S, \mathbf{p}^*\omega \rangle - \int_0^1 \langle S|_t, \mathbf{p}^*\omega \rangle \, \dot{\psi}(t) \, \mathrm{d}t.\end{aligned}$$

Since $\psi \in \mathrm{C}_c^\infty((0,1))$ and $\partial S \llcorner (0,1) \times \mathbb{R}^d = 0$, the first term vanishes. Recalling that, by definition, $S(t) = \mathbf{p}_* S|_t$, we thus conclude

$$\int_0^1 \langle S(t), \omega \rangle \, \dot{\psi}(t) + \langle \mathcal{L}_{v_t} S(t), \omega \rangle \, \psi(t) \, \mathrm{d}t = 0,$$

that is, (8.7). Hence, the family $(S(t))_t$ is indeed a weak solution to the Geometric Transport Equation with $v = \frac{\mathrm{D}}{\mathrm{D}t}$. □

Remark 8.19. In fact, the following converse of the theorem is true: For a space-time current $S \in \mathrm{I}_{1+k}([0,1] \times \mathbb{R}^d)$ such that the Geometric Transport Equation holds for any vector field $v \in \mathrm{L}^1(\mathrm{d}t \otimes \|S|_t\|; \mathbb{R}^d)$, the S satisfies NC automatically; see Section 7 of [10].

8.6 Rectifiability and differentiability

In this and the following sections we investigate the relationship between the space-time approach and the more basic treatment of moving currents via a time-indexed family of slices. Such results all concern the "advection" of the slices by the geometric derivative and we consider a number of theorems relating these objects.

We first turn our attention to understanding which paths of integral currents are "space-time rectifiable", meaning that the path consists of the slices of a space-time *integral* current. For this we first need to consider several notions of *pointwise* variation, as opposed to the previously introduced *space-time* variation.

Let (X, d) be a locally compact metric space and let $[a,b] \ni t \mapsto \gamma(t) \in X$, Define the **pointwise** d**-variation** of γ as

$$\mathsf{d}\text{-pVar}(\gamma; [a,b]) := \sup\left\{ \sum_{i=1}^{N-1} \mathsf{d}(\gamma(t_i), \gamma(t_{i+1})) \;:\; a \le t_1 \le \ldots \le t_N \le b \right\} \quad (8.25)$$

and the **essential** d**-variation** of the path γ as

$$\mathsf{d}\text{-eVar}((\gamma; [a,b]) := \inf\{ \mathsf{d}\text{-pVar}(\widetilde{\gamma}; [a,b]) \;:\; t \mapsto \widetilde{\gamma}(t) \in X \text{ with } \gamma = \widetilde{\gamma} \text{ a.e.} \}.$$

We extend these definitions to paths γ which are only defined for $\mathcal{L}^1$-almost everywhere in $[a,b]$. In this case, the supremum in (8.25) is taken over families of partitions such that γ is defined at t_i for every i.

Thus, we may define the right-continuous **good representative** $\widetilde{S}\colon [\sigma,\tau) \to \mathrm{I}_k(\overline{\Omega})$ of S for any $t \in [\sigma,\tau)$ as

$$\widetilde{S}(t) := S(t+) = \text{w*-}\lim_{s \downarrow t} S(s) \quad \text{in } \mathrm{I}_k(\overline{\Omega}),$$

which satisfies $\widetilde{S}(t) = S(t)$ for $\mathcal{L}^1$-almost every $t \in [\sigma,\tau)$. In the following we will drop the tilde and just refer to $\widetilde{S}(t)$ as $S(t)$.

For $S \in \mathrm{I}_{1+k}([0,1] \times \mathbb{R}^d)$ and $U \in \mathrm{N}_{k+1}([0,1] \times \mathbb{R}^d)$ with $\partial S \llcorner (0,1) \times \mathbb{R}^d = \partial U \llcorner (0,1) \times \mathbb{R}^d = 0$ we also set

$$\begin{aligned}
\mathbf{FI}^\circ\text{-eVar}(S;[a,b]) &:= \mathbf{FI}^\circ\text{-eVar}(t \mapsto S(t);[a,b]),\\
\mathbf{F}^\circ\text{-eVar}(U;[a,b]) &:= \mathbf{F}^\circ\text{-eVar}(t \mapsto U(t);[a,b])
\end{aligned}$$

for every closed interval $[a,b] \subset [0,1]$. Here, as usual, we denote by $S(t) := \mathbf{p}_* S|_t$ the pushforward of the slice $S|_t$ onto $\mathbb{R}^d$ (cf. (8.22)). Likewise we define the slices $U|_t$ and the pushforwards $U(t) := \mathbf{p}_* U|_t$ for $\mathcal{L}^1$-a.e. t. Moreover, we have used the metric $\mathsf{d}(T_1,T_2) := \mathbf{FI}^\circ(T_2 - T_1)$ and $\mathsf{d}(T_1,T_2) := \mathbf{F}^\circ(T_2 - T_1)$, respectively, in the definitions for d-eVar above.

Proposition 8.20. *Let $S \in \mathrm{I}_{1+k}([0,1] \times \mathbb{R}^d)$ with $\partial S \llcorner (0,1) \times \mathbb{R}^d = 0$. Then,*

$$\mathbf{FI}^\circ\text{-eVar}(S;[0,1]) < \infty \qquad \textit{and} \qquad \mathbf{FI}^\circ\text{-eVar}(S;\bullet) \le \mathrm{Var}(S;\bullet).$$

Proof. Fix an arbitrary partition of times $a \le t_1 \le \ldots \le t_N \le b$ such that the slices $S|_{t_i}$ are defined at t_i for $i = 1,\ldots,N$. By the cylinder formula (6.13),

$$S|_{t_{i+1}} - S|_{t_i} = \partial[S \llcorner (t_i,t_{i+1}) \times \mathbb{R}^d].$$

Thus,

$$S(t_{i+1}) - S(t_i) = \mathbf{p}_* \partial(S \llcorner (t_i,t_{i+1}) \times \mathbb{R}^d) = \partial \mathbf{p}_*(S \llcorner (t_i,t_{i+1}) \times \mathbb{R}^d),$$

Hence, $S \llcorner \{t_i < t \le t_{i+1}\}$ is a competitor in the definition of $\mathbf{FI}^\circ(S|_{t_{i+1}} - S|_{t_i})$, whereby

$$\begin{aligned}
\mathbf{FI}^\circ(S(t_{i+1}) - S(t_i)) &\le \mathbf{M}(\mathbf{p}_*(S \llcorner (t_i,t_{i+1}) \times \mathbb{R}^d))\\
&\le \int_{(t_i,t_{i+1}) \times \mathbb{R}^d} \|\mathbf{p}\vec{S}\| \, \mathrm{d}\|S\|\\
&= \mathrm{Var}(S;(t_i,t_{i+1})).
\end{aligned}$$

Summing over i, we obtain

$$\begin{aligned}\sum_{i=0}^{N-1} \mathbf{FI}^\circ(S(t_{i+1}) - S(t_i)) &\le \sum_{i=0}^{N-1} \mathrm{Var}(S;(t_i,t_{i+1}))\\ &\le \mathrm{Var}(S,(a,b))\\ &\le \mathbf{M}(S \llcorner (a,b)\times\mathbb{R}^d)\\ &< \infty,\end{aligned}$$

where we have used (6.26). Since the subdivision $\{t_i\}_{i=0}^N$ was arbitrary (with the constraint of all slices being defined), we conclude that indeed $t \mapsto S(t)$ has finite pointwise $\mathbf{FI}^\circ$-variation and then also essential $\mathbf{FI}^\circ$-variation.

By the same argument,

$$\mathbf{FI}^\circ\text{-eVar}(S;I) \le \mathrm{Var}(S;I)$$

for any closed subinterval $I \subset [0,1]$. □

Remark 8.21. The opposite inequality namely $\mathbf{FI}^\circ\text{-eVar}(S;\bullet) \ge \mathrm{Var}(S;\bullet)$, is in general false: Whenever a jump occurs at a time t_0, $\mathrm{Var}(S;\bullet)$ depends on the particular current that connects $S|_{t_0^-}$ and $S|_{t_0^+}$, while $\mathbf{FI}^\circ$-eVar always measures the jump with the mass of the minimal surface between $S|_{t_0^-}$ and $S|_{t_0^+}$.

Proposition 8.22. *Let $[0,1] \ni t \mapsto T_t \in \mathrm{I}_k(\mathbb{R}^d)$ with $\partial T_t = 0$ for every $t \in [0,1]$, and such that*

$$\sup_{t\in[0,1]} \mathbf{M}(T_t) < \infty, \qquad \mathbf{FI}^\circ\text{-eVar}(t \mapsto T_t;[0,1]) < \infty.$$

Then, for every $t \in [0,1]$ the left and right limits

$$T(t+) = \text{w*-}\lim_{s\downarrow 0} T_s, \qquad T(t-) = \text{w*-}\lim_{s\uparrow 0} T_s$$

exist (only one-sided limits at the endpoints $\{0,1\}$). They differ at most at countably many t's.

Proof. We only show this for the right limit at $t \in [0,1)$. Using the Federer–Fleming Compactness Theorem 5.6 (which uses the uniform mass bound), for every sequence $t_j \downarrow t$ there is a further subsequence (not relabeled) such that the weak*-limit

$$\text{w*-}\lim_{j\to\infty} T_{t_j}$$

exists in $\mathrm{I}_k(\mathbb{R}^d)$. This limit is also unique: If there were sequences $t_j \downarrow t$ and $\widetilde{t}_j \downarrow t$ with

$$0 < \delta < \mathbf{F}(S(t_j) - S(\widetilde{t}_j)) \qquad \text{for all } j,$$

then, up to selecting a subsequence,

$$\infty = \sum_j \mathbf{F}(S(t_j) - S(\widetilde{t}_j)) \le \mathbf{F}\text{-pVar}(S; [t,\tau]) < \infty,$$

which is a contradiction. Hence, every sequence $t_j \to t$ must yield the same w*-$\lim_{j\to\infty} T_{t_j}$ and we may thus unambiguously define $T(t+)$.

By a similar argument based on the finiteness of the variation, the set of t such that $\mathbf{F}(T(t+) - T(t-)) > 1/n$ must be finite for every n. Hence there can be at most countably many t's with $T(t+) \neq T(t-)$. $\square$

We now state a fundamental space-time rectifiability result, which entails that we may "glue" together boundaryless integral currents with bounded $\mathbf{FI}^\circ$-variation into a space-time current with these currents as slices.

Theorem 8.23 (Space-time rectifiability [10]). *Let* $[0,1] \ni t \mapsto T_t \in \mathrm{I}_k(\mathbb{R}^d)$ *with* $\partial T_t = 0$ *for every* $t \in [0,1]$*, and such that*

$$\sup_{t\in[0,1]} \mathbf{M}(T_t) < \infty, \qquad \mathbf{FI}^\circ\text{-eVar}(t \mapsto T_t; [0,1]) < \infty.$$

Then, there exists $S \in \mathrm{I}_{1+k}([0,1] \times \mathbb{R}^d)$ *with* $\partial S \llcorner (0,1) \times \mathbb{R}^d = 0$ *such that:*

(i) $S(t) := \mathbf{p}_* S|_t = T_t$ *for* $\mathcal{L}^1$*-a.e.* $t \in [0,1]$*;*

(ii) $\mathrm{Var}(S; \bullet) = \mathbf{FI}^\circ\text{-eVar}(t \mapsto T_t; \bullet)$.

Proof. We will construct a sequence of piecewise-constant approximations to the sought S and then pass to the limit.

Step 1. Fix $\varepsilon > 0$. By modifying $t \mapsto T_t$ on a null set we may assume without loss of generality that

$$\mathbf{FI}^\circ\text{-pVar}(t \mapsto T_t; [0,1]) \le \mathbf{FI}^\circ\text{-eVar}(t \mapsto T_t; [0,1]) + \varepsilon. \tag{8.26}$$

Furthermore, we assume without loss of generality that

$$T_0 = \text{w*-}\lim_{s\downarrow 0} T_s, \qquad T_1 = \text{w*-}\lim_{s\uparrow 0} T_s$$

are well-defined. This is possibly Proposition 8.22 and this also does not change the mass and variation bounds assumed on the path.

For $N \in \mathbb{N}$ define

$$t_i := \frac{i}{N}, \quad T_i := T_{t_i} \qquad i = 0, \dots, N.$$

Then, for every $i = 0, \dots, N-1$ take an integral current $W_i \in \mathrm{I}_{1+k}(\mathbb{R}^d)$ with

$$T_{i+1} - T_i = \partial W_i \qquad \text{and} \qquad \mathbf{M}(W_i) \le \mathbf{FI}^\circ(T_{i+1} - T_i) + \frac{\varepsilon}{N}$$

and define

$$S_N := -\sum_{i=0}^{N-1} \big([\![t_i, t_{i+1}]\!] \times T_i + \delta_{t_{i+1}} \times W_i \big). \tag{8.27}$$

Using the formula for the boundary of a product current from Lemma 4.19, we compute

$$\begin{aligned}
\partial S_N &= -\sum_{i=0}^{N-1} \big(\delta_{t_{i+1}} \times T_i - \delta_{t_i} \times T_i + \delta_{t_{i+1}} \times \partial W_i \big) \\
&= -\sum_{i=0}^{N-1} \big(-\delta_{t_i} \times T_i + \delta_{t_{i+1}} \times T_{i+1} \big) \\
&= -\delta_1 \times T_1 + \delta_0 \times T_0.
\end{aligned} \tag{8.28}$$

So,

$$\begin{aligned}
\mathbf{M}(S_N) &\le \frac{1}{N} \sum_{i=0}^{N-1} \mathbf{M}(T_i) + \sum_{i=0}^{N-1} \mathbf{M}(W_i), \\
\mathbf{M}(\partial S_N) &\le \mathbf{M}(T_1) + \mathbf{M}(T_0).
\end{aligned}$$

By assumption,

$$\frac{1}{N} \sum_{i=0}^{N-1} \mathbf{M}(T_i) \le C < \infty.$$

and

$$\sum_{i=0}^{N-1} \mathbf{M}(W_i) = \sum_{i=0}^{N-1} \left(\mathbf{FI}^\circ(T_{i+1} - T_i) + \frac{\varepsilon}{N} \right) \le \mathbf{FI}^\circ\text{-pVar}(t \mapsto T_t; [0,1]) + \varepsilon < \infty.$$

Likewise,

$$\mathrm{Var}(S_N; [0,1]) = \sum_{i=0}^{N-1} \mathbf{M}(W_i) \le \mathbf{FI}^\circ\text{-pVar}(t \mapsto T_t; [0,1]) + \varepsilon. \tag{8.29}$$

Thus, by the compactness principle for space-time integral currents in Theorem 6.23 there exists a subsequence N_j and $S \in \mathrm{I}_{1+k}([0,1] \times \mathbb{R}^d)$ such that

$$\begin{cases} S_{N_j} \overset{*}{\rightharpoonup} S \\ S_{N_j}(t) \overset{*}{\rightharpoonup} S(t) \quad \text{for all but countably many } t \in [0,1]. \end{cases}$$

Step 2. Next we will show that $S(t) = T_t$ for $\mathcal{L}^1$-almost every $t \in [0,1]$. So, fix $t \in [0,1]$ with $S_{N_j}(t) \overset{*}{\rightharpoonup} S(t)$ and such that our path is left-continuous at t, that is,

$$T(t) = T(t-) = \operatorname*{w^*\text{-}lim}_{s \uparrow t} T_s.$$

By Proposition 8.22, this is the case at all but countably many t's.

Now observe

$$S_{N_j}|_t = (\partial S_{N_j}) \llcorner \{\mathbf{t} < t\} - \partial(S_{N_j} \llcorner \{\mathbf{t} < t\})$$

by the cylinder formula (6.13) for slices. Observe that $\lfloor N_j t \rfloor$ (the largest integer below or equal to Nt) is the index with the property

$$t_{\lfloor N_j t \rfloor} < t < t_{\lfloor N_j t \rfloor + 1}.$$

So, from (8.27) we obtain

$$S_{N_j} \llcorner \{\mathbf{t} < t\} = - \sum_{i=0}^{\lfloor N_j t \rfloor - 1} \big([\![t_i, t_{i+1}]\!] \times T_i + \delta_{t_{i+1}} \times W_i \big) - [\![t_{\lfloor N_j t \rfloor}, t]\!] \times T_{\lfloor N_j t \rfloor},$$

and thus

$$\partial(S_{N_j} \llcorner \{\mathbf{t} < t\}) = -\partial \left(\sum_{i=0}^{\lfloor N_j t \rfloor - 1} \big([\![t_i, t_{i+1}]\!] \times T_i + \delta_{t_{i+1}} \times W_i \big) \right) - (\delta_t - \delta_{t_{\lfloor N_j t \rfloor}}) \times T_{\lfloor N_j t \rfloor}.$$

On the other hand, (8.28) entails

$$(\partial S_{N_j}) \llcorner \{\mathbf{t} < t\} = -\partial \left(\sum_{i=0}^{\lfloor N_j t \rfloor - 1} \big([\![t_i, t_{i+1}]\!] \times T_i + \delta_{t_{i+1}} \times W_i \big) \right) + \delta_{t_{\lfloor N_j t \rfloor}} \times T_{\lfloor N_j t \rfloor}.$$

Thus,

$$S_{N_j}|_t = \delta_t \times T_{\lfloor N_j t \rfloor} \qquad \text{and} \qquad S_{N_j}(t) = T_{\lfloor N_j t \rfloor / N_j}.$$

Since our path is left-continuous at t by assumption, we conclude

$$S(t) = \operatorname*{w^*\text{-}lim}_{j \to \infty} S_{N_j}(t) = \operatorname*{w^*\text{-}lim}_{j \to \infty} T_{\lfloor N_j t \rfloor / N_j} = T_t.$$

We conclude that (i) holds at such t.

Step 3. To establish also (ii) we first notice that the inequality $\mathbf{FI}^\circ\text{-eVar}(t \mapsto T_t; \bullet) \leq \operatorname{Var}(S; \bullet)$ follows directly from Proposition 8.20. For the other direction, we just need to use that the variation is lower semicontinuous with respect to the BV-weak* convergence, see Theorem 6.23, together with (8.29) and (8.26) to see

$$\operatorname{Var}(S; [0,1]) \leq \liminf_{j \to \infty} \operatorname{Var}(S_{N_j}; [0,1]) \leq \mathbf{FI}^\circ\text{-eVar}(t \mapsto T_t; [0,1]) + 2\varepsilon.$$

As $\varepsilon > 0$ was arbitrary, the claim (ii) follows. $\square$

Remark 8.24. The current S constructed in Theorem 8.23 has minimal variation among all those satisfying the conclusion. Also, one can dispense with the assumption $\partial T_t = 0$. See Remarks 6.2, 6.3 in [10] for the details. Finally, also a version for normal currents holds and is exposed as Proposition 6.4 in [10].

Assume that a path $[0,1] \ni t \mapsto T_t \in \mathrm{I}_k(\mathbb{R}^d)$ of currents is $\mathbf{FI}^\circ$**-absolutely continuous**, which is defined in analogy to (8.12), but with $\mathbf{FI}^\circ$ in place of $\mathbf{F}$. Then, $t \mapsto T_t$ also has finite $\mathbf{F}$-variation. Consequently, by Theorem 8.23 the path is space-time rectifiable and moreover $\mathrm{Var}(S;\,\boldsymbol{\cdot})$ is absolutely continuous with respect to $\mathcal{L}^1$ by (ii). Then, via Lemma 7.8 we have that S has the AC property and so Corollary 7.10 implies the *uniqueness* of the glued current S. Now, if we add the NC condition, then we even get geometric differentiability:

Theorem 8.25 (Geometric differentiability [10]). *Let* $[0,1] \ni t \mapsto T_t \in \mathrm{I}_k(\mathbb{R}^d)$ *be* $\mathbf{FI}^\circ$*-absolute continuous, satisfy* $\partial T_t = 0$ *for every* $t \in [0,1]$*, and be such that*

$$\sup_{t\in[0,1]} \mathbf{M}(T_t) < \infty.$$

Let S be the unique space-time rectified current given by Theorem 8.23. If S is NC, then S is advected by its own geometric derivative $\frac{\mathrm{D}}{\mathrm{D}S}$*, i.e.,* (8.23) *holds.*

Proof. The existence and uniqueness of S has already been discussed. By assumptions S is NC, so we can apply the Advection Theorem 8.18 to conclude. $\square$

Remark 8.26. In the case $k = 0$, corresponding to BV-functions as in Example 6.13, the NC-condition is automatically satisfied by the area formula, see Remark 7.15. Thus, for $k = 0$, the geometric differentiability holds without any further assumption. Of course, this is well-known and is a restatement of the "Fundamental Theorem of Lebesgue Calculus", namely that absolutely continuous functions can be written as the integral of their almost-everywhere existing derivative.

The rectifiability proof relies crucially on the fact that we have absolute continuity with respect to the (homogeneous) *integral* flat norm and so also the geometric differentiability theorem requires this notion of continuity. For some dimensions, however, it is known that $\mathbf{FI}^\circ$ coincides with $\mathbf{F}^\circ$ by Proposition 5.15 and we obtain the following stronger statement:

Corollary 8.27. *Suppose* $k \in \{0, d-2, d-1, d\}$*. Let* $[0,1] \ni t \mapsto T_t \in \mathrm{I}_k(\mathbb{R}^d)$ *be* $\mathbf{F}^\circ$*-absolute continuous, satisfy* $\partial T_t = 0$ *for every* $t \in [0,1]$*, and be such that*

$$\sup_{t\in[0,1]} \mathbf{M}(T_t) < \infty.$$

Let S be the unique space-time rectified current given by Theorem 8.23. If S is NC, then S is advected by its own geometric derivative $\frac{\mathrm{D}}{\mathrm{D}S}$*, i.e.,* (8.23) *holds.*

For general k it is an unsolved problem of Geometric Measure Theory whether $\mathbf{F}^\circ \sim \mathbf{FI}^\circ$ (for integral currents) and hence whether $\mathbf{F}^\circ$-absolute continuity is the same as $\mathbf{FI}^\circ$-absolute continuity. As a result, for $k \notin \{0, d-2, d-1, d\}$ only a form of "weak differentiability" holds; see Theorem 8.1 in [10].

Example 8.28. The Flat Mountain defined in Section 7.3 does not have the NC property. Thus, as a consequence of Remark 8.19, there is no vector field $v \in \mathrm{L}^1(\mathrm{d}t \otimes \|S_{\mathrm{FM}}|_t\|)$ such that S_{FM} solves the Geometric Transport Equation with velocity field v, as would be the case if Theorem 8.25 were applicable.

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