Directional oscillations and concentrations and weak↔strong compactness via microlocal compactness forms

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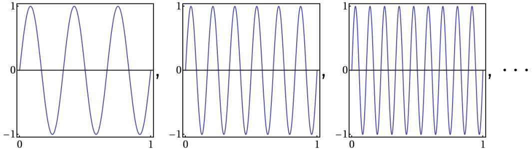
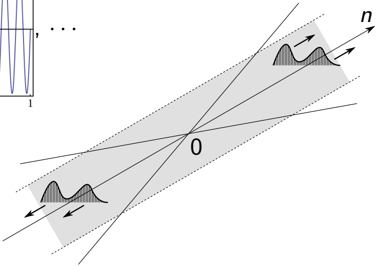
Oscillations & concentrations

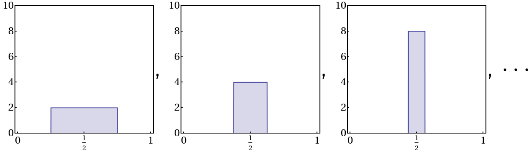
Prototypical oscillation: $u_j(x) = \sin(jx \cdot n)$



In Fourier space: mass wanders out to ∞:



Prototypical concentration in L^p**:** $u_j(x) = j^{d/p} h(jx)$



Why study oscillations / concentrations?

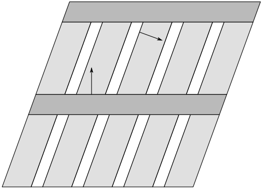
- **Weak↔strong compactness:**

$$\left\{\begin{array}{l}(u_j) \subset \mathrm{L}^p \text{ bounded},\\ u_j \to u \text{ in measure} \Leftrightarrow \text{no oscillations},\\ (u_j)\ p\text{-equiintegrable} \Leftrightarrow \text{no concentrations}\end{array}\right\} \overset{\text{Vitali's Convergence Thm}}{\Longrightarrow} u_j \to u$$

- **Compensated compactness** (Tartar's framework):

$$\left\{\begin{array}{l} u_j \rightharpoonup u \quad \text{in } \mathrm{L}^p,\\ \mathcal{A}u_j := \displaystyle\sum_{k=1}^{d} A^{(k)} \frac{\partial u_j}{\partial x_k} = 0 \quad \text{in } \mathrm{W}^{-1,p},\\ u_j(x) \in Z(x) \text{ a.e.}\end{array}\right\} \overset{???}{\Longrightarrow} u_j \to u$$

- **Efficient description of microstructure** (e.g. for minimizing sequences)



Physical meaning of weak/strong convergence:

Assume: $(u_j) \subset \mathrm{L}^2$ sequence of "measurements".

Weak convergence $u_j \rightharpoonup u$:

$$\int \varphi \cdot (u_j - u)\, \mathrm{d}x \overset{\text{Parseval}}{=} \int \hat{\varphi} \cdot \mathcal{F}[u_j - u]\, \mathrm{d}\xi, \qquad \forall \varphi \in \mathcal{S}.$$

Weak convergence is band-limited $\rightsquigarrow$ Measure with "finite precision".

Strong convergence $u_j \to u$:

$$\int |u_j - u|^2\, \mathrm{d}x \overset{\text{Parseval}}{=} \int |\mathcal{F}[u_j - u]|^2\, \mathrm{d}\xi.$$

Strong convergence is not band-limited: $\rightsquigarrow$ Measure with "infinite precision".

Conclusion:

- Different types of convergence have different physical meanings.
- The study of compactness is related to the "physical" properties of oscillations & concentrations.

Tools to systematically study oscillations & concentrations

Measure compactness / oscillations / concentrations
for a weakly compact generating sequence $(u_j) \subset \mathrm{L}^p$:

- Location:
  Scalar defect measures w*-$\lim_{j\to\infty} |u_j|^p \mathcal{L}^d$
- Location & value-distribution (oscillations):
  Young measures (Young '37,'42)
- Location & value-distribution (oscillations & concentrations):
  Generalized Young (DiPerna–Majda) measures (DiPerna & Majda '87)
- Location & direction:
  H-measures (Tartar '90, Gérard '91)
- Location & value-distribution & direction:
  Microlocal compactness forms (MCFs) $\longleftarrow$ in this work (2012).

Features of MCFs

- Represent **limits of functionals** (essentially contains generalized Young measure).
- Preserve **directional information** of oscillations/concentrations preserved (contains H-measure).
- Easy to "read off" **pointwise constraints** ($u(x) \in Z(x)$) and **differential constraints** ($\mathcal{A}u_j = 0$) on generating sequence.
- Allow an analogue of the **wavefront set** from microlocal analysis, but with respect to weak$\leftrightarrow$strong convergence (not C^∞-regularity).
- **Hierarchy of microstructure** (e.g. laminates) reflected in MCF.
- Allow **relaxation** of **anisotropic** functionals.
- Can be defined for all L^p-spaces (H-measure only for L^2).
- Have the same **compactness** as weak convergence.

Definition of MCFs

- For $h \in \mathrm{C}(\overline{\Omega} \times \mathbb{C}^N; \mathbb{C}^N)$:

$$(S^{p-1}h)(x,w) := (1-|w|)^{p-1} h\left(x, \frac{w}{1-|w|}\right), \qquad (x,w) \in \overline{\Omega} \times \mathbb{B}^N.$$

- Space $\mathbf{F}^p$ of **test functions**:

$$\mathbf{F}^p(\Omega;\mathbb{C}^N) := \left\{ f \in \mathrm{C}(\overline{\Omega} \times \mathbb{C}^N \times \mathbb{C}^N; \mathbb{C}) \;:\; f(x,z,q) = h(x,z)\cdot q \text{ and } S^{p-1}h \in \mathrm{C}(\overline{\Omega \times \mathbb{B}^N}; \mathbb{C}^N) \right\}.$$

- **Fourier multiplier** with symbol $\Psi \in \mathcal{M} := \mathrm{C}^\infty(\mathbb{S}^{d-1}; \mathbb{C}^{N\times N})$:

$$T_\psi \colon \mathrm{L}^p(\Omega;\mathbb{C}^N) \to \mathrm{L}^p(\Omega;\mathbb{C}^N), \qquad T_\psi[u] := \mathcal{F}^{-1}\left[\Psi\left(\frac{\xi}{|\xi|}\right)\hat{u}(\xi)\right]$$

- **Cut-off function** $\eta \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ with $\eta \equiv 1$ on $B(0,1)$, $\operatorname{supp}\eta \subset B(0,2)$. $\eta_R(\xi) := \eta(\xi/R)$ for $\xi \in \mathbb{R}^d$ and any $R > 1$.

- **Microlocal compactness form** $\omega \in \mathbf{MCF}^p(\Omega;\mathbb{C}^N)$ is a sesquilinear form

$$\langle\!\langle f \otimes \Psi, \omega \rangle\!\rangle = \omega(f,\Psi) = \lim_{R\to\infty}\lim_{j\to\infty} \int_\Omega h(\bullet, u_j) \cdot \overline{T_{(1-\eta_R)\Psi}[u_j]}\,\mathrm{d}x$$

for all $f \in \mathbf{F}^p(\Omega;\mathbb{C}^N)$, $\Psi \in \mathcal{M}$. (Well-defined?)

Existence Theorem

Theorem (R. 2012)

Let $(u_j) \subset \mathrm{L}^p(\Omega;\mathbb{C}^N)$ be norm-bounded. Then, after selecting a subsequence, there exist an MCF $\omega \in \mathbf{MCF}^p(\Omega;\mathbb{C}^N)$ such that

$$\langle\!\langle f \otimes \Psi, \omega \rangle\!\rangle = \lim_{R\to\infty} \lim_{j\to\infty} \int_\Omega h(\bullet, u_j) \cdot \overline{T_{(1-\eta_R)\Psi}[u_j]} \,\mathrm{d}x$$

for all $f \in \mathbf{F}^p(\Omega;\mathbb{C}^N)$ with $f(x,z,q) = h(x,z)\cdot q$ and $\Psi \in \mathcal{M}$. Moreover:

(A) **Independence.** *The above limit (and hence ω) is independent of the choice of η and the sequence $R \to \infty$.*

(B) **Basic estimate.** *For all f, Ψ as above it holds that*

$$\big|\langle\!\langle f \otimes \Psi, \omega \rangle\!\rangle\big| \le C \cdot \big(\sup_j \|1 + |u_j|\|_p^p\big) \cdot \|f\|_{\mathbf{F}^p} \cdot \|\Psi\|_{\mathrm{C}^d},$$

where $C = C_d \max\{p, (p-1)^{-1}\}$ with a dimensional constant C_d.

Parts of the main definition formula

$$\langle\!\langle f \otimes \Psi, \omega \rangle\!\rangle = \lim_{R \to \infty} \lim_{j \to \infty} \int_\Omega h(\bullet, u_j) \cdot \overline{T_{(1-\eta_R)\Psi}[u_j]} \, \mathrm{d}x$$

localization in x and target space

localization in ξ

"forget" strongly converging part

- $h(\bullet, u_j) \in \mathrm{L}^{p/(p-1)}$, $\overline{T_{(1-\eta_R)\Psi}[u_j]} \in \mathrm{L}^p$.
- Product in L^1 $\rightsquigarrow$ concentrations expressed.
- In L^2, Parseval yields:

$$\int_\Omega h(\bullet, u_j) \cdot \overline{T_{(1-\eta_R)\Psi}[u_j]} \, \mathrm{d}x = \int_\Omega \mathcal{F}\left[h(\bullet, u_j)\right](\xi) \cdot \overline{(1 - \eta_R(\xi))\Psi\left(\frac{\xi}{|\xi|}\right) \hat{u}_j(\xi)} \, \mathrm{d}\xi.$$

Representation Theorem

Theorem (R. 2012)

The MCF $\omega \in \mathbf{MCF}^p(\Omega; \mathbb{C}^N)$ *from the preceding Existence Theorem can be considered as a triple* $\omega = (\omega_x, \lambda_\omega, \omega_x^\infty)$ *consisting of*

(i) *a parametrized family* $(\omega_x)_{x \in \Omega}$ *of continuous (complex-valued) sesquilinear forms*

$$\omega_x \colon \mathrm{C}(\mathbb{C}^N; \mathbb{C}^N) \times \mathcal{M} \to \mathbb{C},$$

(ii) *a positive and finite measure on* $\overline{\Omega}$, *and*

(iii) *a parametrized family* $(\omega_x^\infty)_{x \in \overline{\Omega}}$ *of continuous (complex-valued) sesquilinear forms*

$$\omega_x^\infty \colon \mathrm{C}(\partial\mathbb{B}^N; \mathbb{C}^N) \times \mathcal{M} \to \mathbb{C}.$$

Then it holds that

$$\langle\!\langle f \otimes \Psi, \omega \rangle\!\rangle = \int_\Omega \underbrace{\langle h(x, \bullet) \otimes \Psi, \omega_x \rangle}_{\substack{= \omega_x(h(x,\bullet), \Psi) \\ \text{“oscillations”}}} \, \mathrm{d}x + \int_{\overline{\Omega}} \underbrace{\langle h^\infty(x, \bullet) \otimes \Psi, \omega_x^\infty \rangle}_{\substack{= \omega_x^\infty(h^\infty(x,\bullet), \Psi) \\ \text{“concentrations”}}} \, \mathrm{d}\lambda_\omega(x).$$

Recession function: $h^\infty(x, z) := \lim_{\substack{x' \to x \\ z' \to z \\ t \to \infty}} \dfrac{h(x', tz')}{t^{p-1}} \quad (x \in \overline{\Omega},\ z \in \mathbb{C}^N).$

Oscillations

Consider: $u_j(x) = A\mathbb{1}_{(0,\theta)}(jx \cdot n_0 - \lfloor jx \cdot n_0 \rfloor) + B\mathbb{1}_{(\theta,1)}(jx \cdot n_0 - \lfloor jx \cdot n_0 \rfloor)$, $x \in \Omega$, where $A, B \in \mathbb{C}^N$, $n_0 \in \mathbb{S}^{d-1}$ and $\mathbb{1}_{(0,\theta)}$ is the indicator function of $(0, \theta)$.

What MCF does (u_j) generate?

Lemma (Oscillation Lemma)

Let $w \in \mathrm{L}^\infty(\mathbb{R}; \mathbb{C}^N)$ be 1-periodic (the "profile function") and assume that $w_j(x) := w(js)$ generates the homogeneous Young measure $\nu \in \mathbf{M}^1(\mathbb{C}^N)$. Then, the simple oscillation in direction $n_0 \in \mathbb{S}^{d-1}$,

$$u_j(x) := w(jx \cdot n_0), \qquad x \in \Omega,$$

generates an microlocal compactness form

$$\omega = \mathcal{L}^d \llcorner \Omega \otimes \left[\overline{(z - Z_0)}\, \nu(\mathrm{d}z)\right] \otimes \overline{\delta}_{\pm n_0} \quad \in \mathbf{MCF}^2(\Omega; \mathbb{C}^N),$$

where $Z_0 := \fint_0^1 w \, \mathrm{d}s$ is the average of w over one period cell and $\overline{\delta}_{\pm n_0} := \overline{\delta}_{-n_0} + \overline{\delta}_{+n_0}$. That is, for all $f \in \mathbf{F}^p(\Omega; \mathbb{C}^N)$, $\Psi \in \mathcal{M}$,

$$\langle\!\langle f \otimes \Psi, \omega \rangle\!\rangle = \int_\Omega \int h(x, z) \cdot \overline{\left[\Psi(+n_0) + \Psi(-n_0)\right](z - Z_0)} \, \mathrm{d}\nu(z) \, \mathrm{d}x.$$

Oscillations II

Example: Simple oscillation

$$u_j(x) = A\mathbb{1}_{(0,\theta)}(\lfloor jx \cdot n_0 \rfloor) + B\mathbb{1}_{(\theta,1)}(\lfloor jx \cdot n_0 \rfloor), \qquad x \in \Omega,$$

where $A, B \in \mathbb{C}^N$, $n_0 \in \mathbb{S}^{d-1}$ and $\mathbb{1}_{(0,\theta)}$ is the indicator function of $(0,\theta)$.

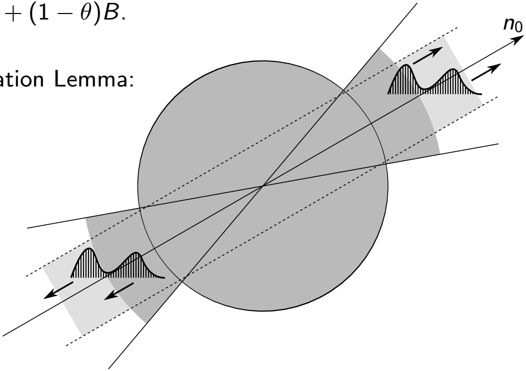
Young measure: $\nu = \theta\delta_A + (1-\theta)\delta_B$.

MCF: By the Oscillation Lemma the sequence (u_j) generates

$$\omega = \mathcal{L}^d \llcorner \Omega \otimes \left[\theta\overline{(A-M)}\delta_A + (1-\theta)\overline{(B-M)}\delta_B\right] \otimes \overline{\delta}_{\pm n_0} \quad \in \mathbf{MCF}^2(\Omega;\mathbb{C}^N),$$

where $M := \theta A + (1-\theta)B$.

Proof of Oscillation Lemma:



Oscillations III

$$\omega = \mathcal{L}^d \llcorner \Omega \otimes \left[\theta\overline{(A-M)}\delta_A + (1-\theta)\overline{(B-M)}\delta_B\right] \otimes \overline{\delta}_{\pm n_0}$$

homogeneous
oscillation on Ω

sequence takes
values $\{A, B\}$…

…with volume fractions
$\theta, 1-\theta$, respectively

oscillation in direction $\pm n_0 \in \mathbb{S}^{d-1}$

Concentrations

Let $w \in \mathrm{L}^p(\mathbb{R}^d)$ have compact support and $w \geq 0$. Further, let $Z_0 \in \mathbb{C}^N$ with $|Z_0| = 1$. Define

$$u_j(x) := j^{d/p} Z_0 w(jx), \qquad x \in \mathbb{R}^d.$$

Then, u_j generates the MCF

$$\omega = \delta_0 \otimes \overline{Z_0} \delta_{\infty Z_0} \otimes \bar{\mu} \quad \in \mathbf{MCF}^p(\mathbb{R}^d; \mathbb{C}^N),$$

where $\bar{\mu}$ is the surface measure (acting on $\overline{\Psi}$)

$$\bar{\mu} = \left(\int_0^\infty \mathcal{F}\big[|w|^{p-1}\big](t\eta)\, \overline{\hat{w}(t\eta)}\, t^{d-1} \,\mathrm{d}t \right) (\mathcal{H}^{d-1} \llcorner \mathbb{S}^{d-1})(\mathrm{d}\eta).$$

The above is a shorthand notation for the MCF ω acting on $f(x, z, q) = h(x, z) \cdot q \in \mathbf{F}^p(\Omega; \mathbb{C}^N)$, $\Psi \in \mathcal{M}$, as

$$\langle\!\langle f \otimes \Psi \rangle\!\rangle = h^\infty(0, Z_0) \cdot \int_{\mathbb{S}^{d-1}} \overline{\Psi(\xi) Z_0} \,\mathrm{d}\bar{\mu}(\xi).$$

Remarks

- In the Existence Theorem, we can choose for example ($\leftrightarrow$ scalar defect measure)

$$\lambda_\omega = \operatorname*{w^*\text{-}lim}_{j\to\infty} |u_j|^p \, \mathcal{L}^d. \qquad (\star)$$

- The families $(\omega_x)_x$, $(\omega_x^\infty)_x$ are uniquely determined *if* we use the canonical choice $(\star)$.

- **Proofs** of existence/representation theorems use techniques from harmonic analysis, measure theory (disintegration/slicing of measures), and the theory of generalized Young measures (we use an argument similar to Kristensen & R. 2010, ARMA).

- The representation as before allows us to (in some sense) localize in both the x- and ξ-variable and so "circumvent the uncertainty principle"! Reason: We only look at *infinite* frequencies.

Some properties of MCF

Lemma

Let $(u_j) \subset \mathrm{L}^p(\Omega; \mathbb{C}^N)$ *generate the MCF* $\omega \in \mathbf{MCF}^p(\Omega; \mathbb{C}^N)$ *and* $u_j \rightharpoonup u$ *in* $\mathrm{L}^p(\Omega; \mathbb{C}^N)$. *Then* $u_j \to u$ *strongly if and only if* $\omega = 0$

$\rightsquigarrow$ The MCF ω measures the difference between weak and strong convergence.

Young measures:

Proposition

Let $(u_j) \subset \mathrm{L}^p(\Omega; \mathbb{C}^N)$ *generate the MCF* $\omega \in \mathbf{MCF}^p(\Omega; \mathbb{C}^N)$ *and the (generalized) Young measure* $\nu \in \mathbf{Y}(\Omega; \mathbb{C}^N)$. *Assume further that* $u_j \rightharpoonup u = [\nu]$. *Then, the knowledge of* ω *and of the limit* u *completely determine* ν.

H-measures: They are trivially contained (in the case $u_j \rightharpoonup 0$ in L^2).

Differential constraints

Differential operator $\mathcal{A}$:

$$\mathcal{A} := \sum_{k=1}^{d} A^{(k)} \frac{\partial}{\partial x_k}$$

Symbol: $\mathbb{A}(\xi) := \sum_{k=1}^{d} A^{(k)} \xi_k.$ $\quad (\mathcal{A}u = 0 \Leftrightarrow (2\pi\mathrm{i})\mathbb{A}(\xi)\hat{u}(\xi) = 0)$

Homogeneous symbol: $\mathbb{A}_0(\xi) := \sum_{k=1}^{d} A^{(k)} \xi_k / |\xi|.$

Murat's constant-rank property: rank ker $\mathbb{A}(\xi) = \mathrm{const}$ for all $\xi \in \mathbb{S}^{d-1}$.

Theorem (R. 2012)

Let $(u_j) \subset \mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^N)$ *with* $u_j \rightharpoonup u$ *in* $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^N)$ *generate the MCF* $\omega \in \mathbf{MCF}^2(\mathbb{R}^d; \mathbb{C}^N)$*. Then,*

$$\mathcal{A}u_j \to 0 \qquad \textit{in } \mathrm{W}^{-1,2}(\mathbb{R}^d; \mathbb{C}^l).$$

if and only if

$$\left\langle\!\left\langle f \otimes \Psi \mathbb{A}_0, \omega \right\rangle\!\right\rangle = 0$$

for all $f \in \mathbf{F}^2(\Omega; \mathbb{C}^N)$ *and all* $\Psi \in \mathrm{C}^{\lfloor d/2 \rfloor + 1}(\mathbb{S}^{d-1}; \mathbb{C}^{N \times l})$*.*

Higher-order laminates

Homogeneous MCFs: $\omega \in \mathbf{MCF}^p(\Omega;\mathbb{C}^N)$ is called **homogeneous** if $\omega_x, \omega_x^\infty$ are constant in x and $\lambda_\omega = \alpha\mathcal{L}^d \llcorner \Omega$ for a constant $\alpha > 0$.

Proposition (Laminations)

Let $n_0 \in \mathbb{S}^{d-1}$. Assume that

(i) $u_j \rightharpoonup A = \mathrm{const}$ *and* $v_j \rightharpoonup B = \mathrm{const}$ *in* $\mathrm{L}^p(\mathbb{R}^d;\mathbb{C}^N)$,

(ii) (u_j), (v_j) *are p-equiintegrable,*

(iii) (u_j), (v_j) *generate the homogeneous MCFs* $\omega_1, \omega_2 \in \mathbf{MCF}^p_{\mathrm{hom}}(\mathbb{C}^N)$ *and the homogeneous Young measures* $\nu_1, \nu_2 \in \mathbf{M}^1(\mathbb{C}^N)$*, respectively.*

Then, for any $\theta \in (0,1)$ there exists a homogeneous microlocal compactness form $\bar\omega \in \mathbf{MCF}^p_{\mathrm{hom}}(\mathbb{C}^N)$ with

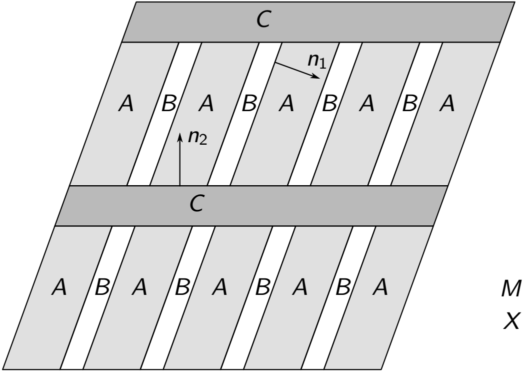
$$\bar\omega = \underbrace{\theta\omega_1 + (1-\theta)\omega_2}_{\text{faster scales}} + \underbrace{\mathcal{L}^d \otimes \left[\theta\overline{(A-X)}\nu_1 + (1-\theta)\overline{(B-X)}\nu_2\right] \otimes \bar\delta_{\pm n_0}}_{\text{slower scale}},$$

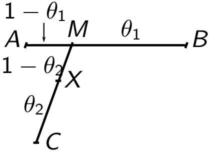
where $X := \theta A + (1-\theta)B$.

Proof: Similar argument as in Oscillation Lemma, and averaging.

$\rightsquigarrow$ Hierarchy of scales reflected.

Lamination example





$$M = \theta_1 A + (1-\theta_1)B$$
$$X = \theta_2 M + (1-\theta_2)C$$

Generated MCF:

$$\begin{aligned}\omega &= \theta_2\omega_{A/B} + (1-\theta_2)\omega_C + \mathcal{L}^d \otimes \left[\theta_2\overline{(M-X)}\nu_1 + (1-\theta_2)\overline{(C-X)}\nu_2\right] \otimes \overline{\delta}_{\pm n_2} \\ &= \mathcal{L}^d \llcorner \Omega \otimes \Big\{ \left[\theta_2\theta_1\overline{(A-M)}\delta_A + \theta_2(1-\theta_1)\overline{(B-M)}\delta_B\right] \otimes \overline{\delta}_{\pm n_1} \\ &\qquad + \left[\theta_2\overline{(M-X)}(\theta_1\delta_A + (1-\theta_1)\delta_B)\right] \otimes \overline{\delta}_{\pm n_2} \\ &\qquad + \left[(1-\theta_2)\overline{(C-X)}\delta_C\right] \otimes \overline{\delta}_{\pm n_2} \Big\}.\end{aligned}$$

Wavefront set

Sphere compactification: $\sigma\mathbb{C}^N := \mathbb{C}^N \uplus \infty\mathbb{S}^{N-1}$.

Wavefront set for $\omega \in \mathbf{MCF}(\Omega; \mathbb{C}^N)$:
$\mathrm{WF}(\omega) \subset \overline{\Omega} \times \sigma\mathbb{C}^N \times \mathbb{S}^{d-1}$ is the smallest closed subset A of $\overline{\Omega} \times \sigma\mathbb{C}^N \times \mathbb{S}^{d-1}$ with the property that for any $f(x,z,q) = \varphi(x)g(z) \cdot q \in \mathbf{F}^p(\Omega; \mathbb{C}^N)$ and any multiplier $\Psi \in \mathcal{M}$,

$$\operatorname{supp} \varphi \otimes g \otimes \Psi \subset A^c \qquad \text{implies} \qquad \langle\!\langle f \otimes \Psi, \omega \rangle\!\rangle = 0.$$

Examples:

- Oscillations: $\omega = \mathcal{L}^d \llcorner \Omega \otimes \left[\theta\overline{(A-M)}\delta_A + (1-\theta)\overline{(B-M)}\delta_B\right] \otimes \overline{\delta}_{\pm n_0}$:

$$\mathrm{WF}(\omega) = \overline{\Omega} \times \{A, B\} \times \{+n_0, -n_0\}.$$

- Concentrations: $\omega = \delta_0 \otimes \overline{Z_0}\delta_{\infty Z_0} \otimes \bar{\mu}$:

$$\mathrm{WF}(\omega) = (0, \infty Z_0) \times \operatorname{supp} \bar{\mu},$$

$\rightsquigarrow$ "Concentrations lie in $\infty\mathbb{S}^{N-1}$".

Compensated compactness

Consider general framework (Tartar):

$$\begin{cases} u_j \rightharpoonup u \quad \text{in } \mathrm{L}^p, \\ \mathcal{A}u_j := \sum_{k=1}^{d} A^{(k)} \dfrac{\partial u_j}{\partial x_k} = 0 \quad \text{in } \mathrm{W}^{-1,p}, \\ u_j(x) \in Z \text{ a.e.}, \qquad Z \subset \mathbb{C}^N \text{ closed.} \end{cases} \tag{CC}$$

Symbol: $\mathbb{A}(\xi) := \sum_{k=1}^{d} A^{(k)} \xi_k. \quad (\mathcal{A}u = 0 \Leftrightarrow (2\pi \mathrm{i})\mathbb{A}(\xi)\hat{u}(\xi) = 0)$

Theorem (R. 2012)

Let $(u_j) \subset \mathrm{L}^p(\mathbb{R}^d; \mathbb{C}^N)$ *generate* $\omega \in$ $\mathbf{MCF}^p(\mathbb{R}^d; \mathbb{C}^N)$ *and let* $\mathcal{A}$ *be a linear PDE operator satisfying Murat's* **constant-rank property**, *i.e.*

$$\text{rank ker } \mathbb{A}(\xi) = \text{const} \qquad \text{for all } \xi \in \mathbb{S}^{d-1}.$$

Moreover, assume (CC). *Then,*

$$\mathrm{WF}(\omega) \subset \Xi := \left\{ (x, z, \xi) \in \mathbb{R}^d \times \sigma\mathbb{C}^N \times \mathbb{S}^{d-1} \;:\; \operatorname{span}_{\mathbb{C}} Z \cap \ker \mathbb{A}(\xi) \neq \{0\} \right\}.$$

In particular, if $\Xi = \emptyset$, *then* $\omega = 0$ *and* (u_j) *is strongly compact.*

Compensated compactness example I: Gradients

Let $(u_j) \subset \mathrm{L}^p(\Omega;\mathbb{R}^{m\times d})$ generate the MCF $\omega \in \mathbf{MCF}^p(\Omega;\mathbb{C}^{m\times d})$ and

$$\begin{cases} \operatorname{curl} u_j = 0 & \text{in } \mathrm{W}^{-1,p}, \\ u_j(x) \in \operatorname{span}\{M\} & \text{a.e.,} \qquad M \in \mathbb{R}^{m\times d} \text{ a fixed matrix.} \end{cases}$$

Theorem $\Rightarrow$

$$\mathrm{WF}(\omega) \subset \big\{\, (x,z,\xi) \in \overline{\Omega} \times \sigma\mathbb{R}^{m\times d} \times \mathbb{S}^{d-1} \;:\; \operatorname{span}_{\mathbb{C}}\{M\} \cap \ker \mathbb{A}(\xi) \neq \{0\} \,\big\},$$

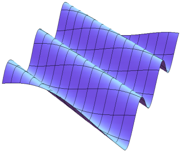
where $\ker \mathbb{A}(\xi) = \{\, a \otimes \xi \;:\; a \in \mathbb{C}^m \,\}$.

(i) If rank $M \geq 2$, then $\omega = 0$ and (u_j) is strongly compact.

(ii) If $M = a \otimes n_0$ for $a \in \mathbb{R}^m$, $n_0 \in \mathbb{S}^{d-1}$, then

$$\mathrm{WF}(\omega) \subset \big\{\, (x,z,\xi) \in \overline{\Omega} \times \sigma\mathbb{R}^{m\times d} \times \mathbb{S}^{d-1} \;:\; \xi = \pm n_0 \,\big\},$$

$\rightsquigarrow$ Oscillations / concentrations have "direction $\pm n_0$":



Compensated compactness example II: Linear elasticity

There exists a (2nd order) $\mathcal{A}$ acting on $\mathbb{R}^{d\times d}_{\mathrm{sym}}$-valued vector fields with

$$u = \frac{1}{2}\left(\nabla v + \nabla v^T\right) \qquad \text{if and only if} \qquad \mathcal{A}u = 0,$$

Let $(u_j) \subset \mathrm{L}^p(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ generate the MCF $\omega \in \mathbf{MCF}^p(\mathbb{R}^d; \mathbb{C}^{m\times d}_{\mathrm{sym}})$ with

$$\begin{cases} \mathcal{A}u_j = 0 & \text{in } \mathrm{W}^{-1,p}, \\ u_j(x) \in \operatorname{span}\{M\} & \text{a.e.,} \qquad M \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \text{ a fixed matrix.} \end{cases}$$

Theorem $\Rightarrow$

$$\mathrm{WF}(\omega) \subset \left\{ (x,z,\xi) \in \overline{\Omega} \times \sigma\mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{S}^{d-1} \;:\; \operatorname{span}_{\mathbb{C}}\{M\} \cap \ker \mathbb{A}(\xi) \neq \{0\} \right\},$$

where $\ker \mathbb{A}(\xi) = \left\{ a \odot \xi = \frac{1}{2}\left(a \otimes \xi + \xi \otimes a\right) \;:\; a \in \mathbb{C}^d \right\}$.

(i) If $M \notin \{a \odot b\}$, then $\omega = 0$ and (u_j) is strongly compact.

(ii) If $M = \gamma(a \odot b)$ for some $a, b \in \mathbb{R}^d$ with $|a| = |b| = 1$, $\gamma \neq 0$, then

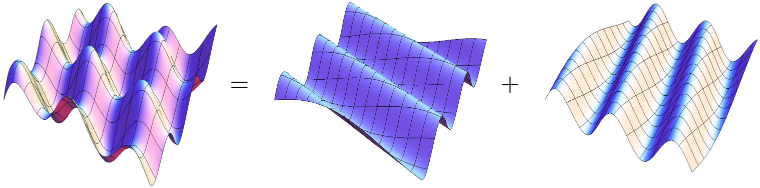
$$\mathrm{WF}(\omega) \subset \left\{ (x,z,\xi) \in \overline{\Omega} \times \sigma\mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{S}^{d-1} \;:\; \xi = \pm a \text{ or } \xi = \pm b \right\}.$$

Compensated compactness example II: Linear elasticity (continued)

The case $M = a \odot b$ (continued):

$$\mathrm{WF}(\omega) \subset \left\{ (x, z, \xi) \in \overline{\Omega} \times \sigma\mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{S}^{d-1} \ : \ \xi = \pm a \text{ or } \xi = \pm b \right\}.$$

$\rightsquigarrow$ Oscillations and concentrations have direction $\pm a$ or $\pm b$:



$\rightsquigarrow$ **Weak* lower semicontinuity of integral functionals** depending on the symmetric gradient and in the space BD of functions of bounded deformation (R. 2011, ARMA)

Summary & Outlook

The theory of MCFs . . .

- . . . measures the **difference between weak and strong compactness** $\rightsquigarrow$ very weak regularity.
- . . . can be considered **microlocal analysis** for weak$\leftrightarrow$strong compactness $\rightsquigarrow$ much more adapted to nonlinear PDEs than classical microlocal analysis, which measures C^∞-regularity.
- . . . allows **localizing singularities** in x, ξ, and the target space ("circumvent the uncertainty principle").
- . . . represents **differential/pointwise constraints**.
- . . . enables "geometric" proofs of **compensated compactness**.
- . . . reflects the **hierarchy of microstructure**.

Outlook:

- **Relaxation** of integral functionals with anisotropy.
- **Propagation of regularity / singularities** for hyperbolic conservation laws.
- Finer investigations into **shape of microstructure**.

Thank you for your attention!