

# The positive Jacobian constraint in elasticity theory and orientation-preserving Young measures

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## Positive Jacobian constraints in Calculus of Variations

- Let  $\Omega \subset \mathbb{R}^d$ .
- Given: **Minimization problem** (e.g. in nonlinear elasticity):

$$\int_{\Omega} f(x, \nabla u(x)) \, dx \rightarrow \min \quad \text{over } u \in W^{1,p}(\Omega; \mathbb{R}^d).$$

- Often: **Differential inclusions** as side constraint:

$$\nabla u(x) \in K(x) \quad \text{for a.e. } x \in \Omega.$$

- Here we consider the **orientation-preserving constraints**:

$$\det \nabla u > 0 \quad \text{a.e.} \quad \text{or}$$

$$\det \nabla u \geq r > 0 \quad \text{a.e.} \quad \text{or}$$

$$\det \nabla u = r > 0 \quad \text{a.e.}$$

Meaning:

- 1 No compression to zero volume.
  - 2 No orientation reversal.
- Applications: Elasticity theory, fluid dynamics, ...
  - *Main difficulty*: Constraint is heavily nonlinear and non-convex!

- Consider the constraint

$$\det \nabla u > 0 \quad \text{a.e.}$$

- Assume we satisfy the constraint “approximately”, i.e.

$$(u_j) \subset W^{1,p}(\Omega; \mathbb{R}^d) \quad \text{with} \quad u_j|_{\partial\Omega} = u|_{\partial\Omega}$$

and

$$\int_{\{\det \nabla u_j < 0\}} \underbrace{|\det \nabla u_j(x)|^{p/d}}_{\in L^1} dx \rightarrow 0.$$

- **Question 1:** Can we find new sequence  $(v_j) \subset W^{1,p}_u(\Omega; \mathbb{R}^d)$  such that

$$\det \nabla v_j > 0 \quad \text{a.e.} \quad \text{and} \quad \|u_j - v_j\|_{W^{1,p}} \rightarrow 0 \quad ?$$

- **Question 2:** Given  $g \in W^{1-1/p,p}(\partial\Omega; \mathbb{R}^d)$  (trace space to  $W^{1,p}(\Omega; \mathbb{R}^d)$ ), does there exist

$$u \in W^{1,p}(\Omega; \mathbb{R}^d) \quad \text{with} \quad u|_{\partial\Omega} = g \quad \text{and} \quad \det \nabla u > 0 \quad ?$$

(Note: No “compatibility” assumption on  $g$ !)

## The case $p \geq d$

- Say  $u_j = u \in W^{1,p}(\Omega; \mathbb{R}^d)$  with

$$\det \nabla u = 0 \quad \text{a.e.,}$$

this satisfies the constraint  $\int_{\{\det \nabla u < 0\}} |\det \nabla u(x)|^{p/d} dx = 0$ .

- Can find  $v_j \rightarrow u$  in  $W^{1,p}(\Omega; \mathbb{R}^d)$  such that  $v_j|_{\partial\Omega} = u|_{\partial\Omega}$  and

$$\det \nabla v_j > 0 \quad \text{a.e. ?}$$

- **Observation:** Question 1 is unsolvable if  $p \geq d$ !

In this case,  $\det \nabla v_j \in L^1(\Omega)$  and by Stokes' Theorem,

$$\begin{aligned} \int_{\Omega} \det \nabla v_j dx &= \int_{\Omega} dv_j^1 \wedge \cdots \wedge dv_j^d = \int_{\partial\Omega} v_j^1 \wedge dv_j^2 \wedge \cdots \wedge dv_j^d \\ &= \int_{\partial\Omega} u^1 \wedge du^2 \wedge \cdots \wedge du^d = \cdots = \int_{\Omega} \det \nabla u dx = 0. \end{aligned}$$

So  $\det \nabla v_j > 0$  a.e. is impossible. □

- Similar argument shows: Also *Question 2* unsolvable if  $p \geq d$ !
- Underlying reason:  $\det$  is a Null-Lagrangian.

Let

- $f: \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$  be continuous,
- $|f(x, A)| \leq C(1 + |A|^p)$  for a  $p \in [1, \infty)$ ,
- $(v_j) \subset L^p(\Omega; \mathbb{R}^N)$  norm-bounded.

For many applications it is important to answer:

**What is**  $w\text{-}\lim_{j \rightarrow \infty} f(x, v_j(x))$  ?

**It is not**  $f(\cdot, (w\text{-}\lim v_j))$ ! Because of:

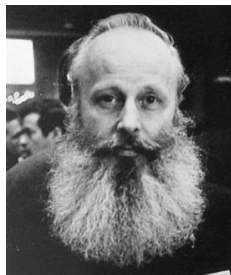
- **oscillation** effects (e.g.  $v_j(x) = \sin(jx)$ )
- **concentration** effects (e.g.  $v_j(x) = j^{1/p} \mathbb{1}_{(0, 1/j)}$ )

If  $(v_j)$  **equiintegrable** (prevents concentrations): **Young measures** ('37, '42)

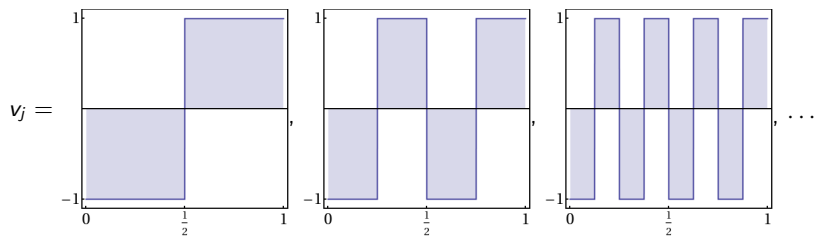
**Fundamental Theorem of YM theory:** Up to a subsequence, there exists a family of probability measures  $(\nu_x)_{x \in \Omega}$  (with some measurability properties) on  $\mathbb{R}^N$  and

$$\int_{\Omega} f(x, v_j(x)) \, dx \rightarrow \int_{\Omega} \int_{\mathbb{R}^N} f(x, \cdot) \, d\nu_x \, dx$$

for *all* continuous  $f: \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$  with  $p$ -growth.



## Example: Oscillation 1

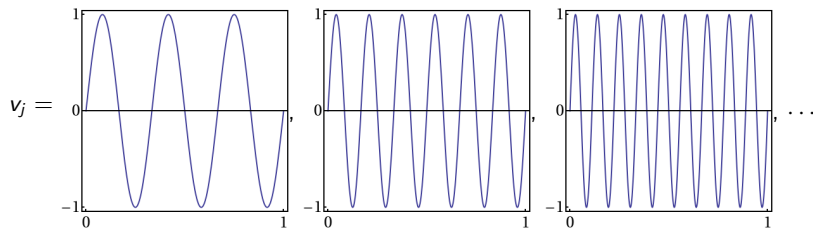


Then:

$$\nu_x = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1} \text{ a.e.}$$

*Interpretation:*  $v_j$  asymptotically is in  $+1$  and  $-1$  each with volume fraction  $\frac{1}{2}$ .

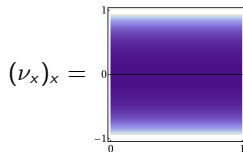
## Example: Oscillation 2



$$v_j := \sin(2j\pi x) \xrightarrow{Y} (\nu_x)_x$$

Then:

$$\nu_x = \frac{1}{\pi\sqrt{1-y^2}} \mathcal{L}_y^1 \lfloor (-1, 1) \text{ a.e.}$$



## Characterization of gradient Young measures

For several purposes (e.g. relaxation theorems), one is interested in the following:

*Can one characterize the class of Young measures which can be generated by a sequence of **gradients** of  $W^{1,p}(\Omega; \mathbb{R}^m)$ -functions?*

### Theorem (Kinderlehrer–Pedregal '91/'94)

A Young measure  $\nu = (\nu_x)$  is a **gradient  $p$ -Young measure**, that is there exists a sequence  $(\nabla u_j) \subset L^p(\Omega; \mathbb{R}^{d \times d})$  generating  $\nu = (\nu_x)$ , if and only if the following conditions hold:

(KP1)  $\int |A|^p d\nu_x(A) < \infty.$

(KP2) The barycenter  $\int A d\nu_x(A)$  is a gradient, i.e. there exists  $\nabla u \in L^p(\Omega; \mathbb{R}^{d \times d})$  with  $\int A d\nu_x(A) = \nabla u(x)$  a.e.

(KP3) For every quasiconvex function  $h: \mathbb{R}^{d \times d} \rightarrow \mathbb{R}$  with  $|h(A)| \leq c(1 + |A|^p)$ , the **Jensen-type inequality**

$$h(\nabla u(x)) \leq \int h(A) d\nu_x(A) \quad \text{holds for a.e. } x \in \Omega.$$



## Characterization of gradient Young measures II

Here: A locally bounded mapping  $h: \mathbb{R}^{d \times d} \rightarrow \mathbb{R}$  is called **quasiconvex** if

$$h(M) \leq \int_{B(0,1)} h(M + \nabla \psi(x)) \, dx$$

for all  $M \in \mathbb{R}^{d \times d}$  and all  $\psi \in C_c^\infty(B(0,1); \mathbb{R}^d)$  (compactly supported).

### Applications of Kinderlehrer–Pedregal Theorem:

- Relaxation theorems (no minimizer for original problem  $\rightsquigarrow$  extend to minimization problem on Young measures).
- Theoretical investigations.
- Properties of quasiconvexity.

### Proof ideas:

- **Necessity** of (KP1)–(KP3): essentially equivalent to a lower semicontinuity theorem.
- **Sufficiency** needs the Hahn–Banach Theorem and several sophisticated techniques.

## Question 3:

Can one characterize the class of Young measures which can be generated by a sequence  $(\nabla u_j)$  of gradients of  $W^{1,p}(\Omega; \mathbb{R}^m)$ -functions satisfying the side constraint

$$\det \nabla u_j > 0 \quad \text{a.e.} \quad ? \quad (\star)$$

### Theorem (Koumatos & R. & Wiedemann '13)

Let  $\mathbf{p} < \mathbf{d}$ . A Young measure  $\nu = (\nu_x)$  is an **orientation-preserving gradient  $p$ -Young measure**, that is there exists a sequence  $(\nabla u_j) \subset L^p(\Omega; \mathbb{R}^{d \times d})$  generating  $\nu = (\nu_x)$  and satisfying  $(\star)$ , if and only if the following conditions hold:

(KP1)–(KP3) from the Kinderlehrer–Pedregal Theorem.

$$(\text{SUPP}) \quad \text{supp } \nu_x \in \{ M \in \mathbb{R}^{d \times d} : \det M \geq 0 \}$$

**Main difficulty:** Constraint on Jacobian is highly non-linear, non-convex.

(KP1)–(KP3): Same as in Kinderlehrer–Pedregal Theorem.

(SUPP): Let  $\nabla u_j$  generate the Young measure  $(\nu_x)_x$  and  $\det \nabla u_j > 0$ .  
Let  $\varphi \in C_c(\Omega)$  and  $h \in C_c(\mathbb{R}^{d \times d})$  such that

$$\text{supp } h \subset \subset \{M \in \mathbb{R}^{d \times d} : \det M < 0\}.$$

Then, by Young measure representation of limits,

$$\int_{\Omega} \varphi(x) \int h \, d\nu_x \, dx = \lim_{j \rightarrow \infty} \int_{\Omega} \varphi h(\nabla u_j) \, dx \stackrel{(*)}{=} 0.$$

( $\star$ ): by the assumptions on  $\nabla u_j$ . Varying  $\varphi$ ,

$$\int h \, d\nu_x = 0 \quad \text{for a.e. } x \in \Omega.$$

Since this holds for all  $h$  as above,

$$\text{supp } \nu_x \subset \{M \in \mathbb{R}^{d \times d} : \det M \geq 0\} \quad \text{for a.e. } x \in \Omega.$$

- (KP1)–(KP3) yield, via the Kinderlehrer–Pedregal Theorem, that there exists  $(\nabla u_j) \subset W^{1,p}(\Omega; \mathbb{R}^d)$  with

- 1  $\nabla u_j \xrightarrow{Y} \nu$ , i.e.  $(\nabla u_j)$  generating  $\nu$ ,
- 2  $u_j = u$  on  $\partial\Omega$ ,
- 3  $(\nabla u_j)$   $p$ -equiintegrable.

- **But:** Also need to satisfy constraint  $\det \nabla u_j > 0$  a.e.!
- By definition of Young measure generation:

$$\int_{\Omega} f(x, \nabla u_j(x)) \, dx \quad \rightarrow \quad \int_{\Omega} \int_{\mathbb{R}^N} f(x, \cdot) \, d\nu_x \, dx$$

for all continuous  $f: \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$  with  $p$ -growth.

- Set  $f(x, A) := \mathbb{1}_{\{\det A < 0\}} |\det A|^{p/d} \dots$  continuous,  $p$ -growth.  
Thus, by (SUPP), we satisfy the differential inclusions *approximately*:

$$\int_{\{\det \nabla u_j < 0\}} |\det \nabla u_j(x)|^{p/d} \, dx \quad \rightarrow \quad \int_{\Omega} \int_{\{\det A < 0\}} |\det A|^{p/d} \, d\nu_x \, dx = 0.$$

- Recall **Question 1:** Can find new sequence  $(v_j) \subset W_u^{1,p}(\Omega; \mathbb{R}^d)$  such that

$$\det \nabla v_j > 0 \quad \text{a.e.} \quad \text{and} \quad \|u_j - v_j\|_{W^{1,p}} \rightarrow 0 \quad ?$$

# “Geometry” of the set $\{ M \in \mathbb{R}^{d \times d} : \det M = 0 \}$ in matrix space I

- **Observation:** Let  $M_0 \in \mathbb{R}^{d \times d}$  with  $\det M_0 < 0$ . Then,  $M$  can be written as the barycenter of a probability measure  $\mu \in \mathbf{M}^1(\mathbb{R}^{d \times d})$  with

$$\text{supp } \mu \subset \{ M \in \mathbb{R}^{d \times d} : \det M = 0 \}.$$

- Indeed, if (modified singular value decomposition)

$$M_0 = \begin{pmatrix} -\sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_d \end{pmatrix} \quad \text{with} \quad 0 \leq \sigma_1 \leq \sigma_2 \leq \dots \leq \sigma_d,$$

then (trivially)

$$M_0 = \frac{1}{2} \begin{pmatrix} 0 & & & \\ & 2\sigma_2 & & \\ & & \sigma_3 & \\ & & & \ddots \\ & & & & \sigma_d \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -2\sigma_1 & & & \\ & 0 & & \\ & & \sigma_3 & \\ & & & \ddots \\ & & & & \sigma_d \end{pmatrix} =: \frac{1}{2} M_1 + \frac{1}{2} M_2.$$

- $\det M_1 = \det M_2 = 0$  and  $\mu := \frac{1}{2} \delta_{M_1} + \frac{1}{2} \delta_{M_2}$ .

## “Geometry” of the set $\{ M \in \mathbb{R}^{d \times d} : \det M = 0 \}$ in matrix space II

- **More intricate question:** Can choose  $\mu$  as a **gradient** Young measure or even a  **$p$ -laminar** (a special type of gradient  $p$ -Young measure)?
- A probability measure  $\nu$  is a **finite-order laminar** if either

$$\nu = \delta_A, \quad A \in \mathbb{R}^{d \times d}, \quad \text{or} \quad \nu = \theta \nu_1 + (1 - \theta) \nu_2,$$

where  $\nu_1, \nu_2$  are  $p$ -laminars (recursively),  $\theta \in (0, 1)$ , and

$$\text{rank} \left( \int A \, d\nu_1(A) - \int A \, d\nu_2(A) \right) = 1$$

- A  **$p$ -laminar** is a probability measure  $\nu$  with  $\int |\cdot|^p \, d\nu < \infty$  and such that there exist finite-order laminars  $\nu_j$  such that

$$\nu_j \xrightarrow{*} \nu \quad \text{as measures.}$$

- By the Kinderlehrer–Pedregal Theorem:  $p$ -laminars  $\nu$  can be generated by a sequence of **gradients**! That is, there exists  $(\nabla w_j) \subset L^p(\Omega; \mathbb{R}^d)$  with

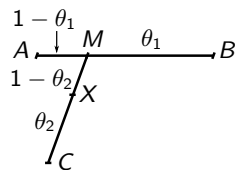
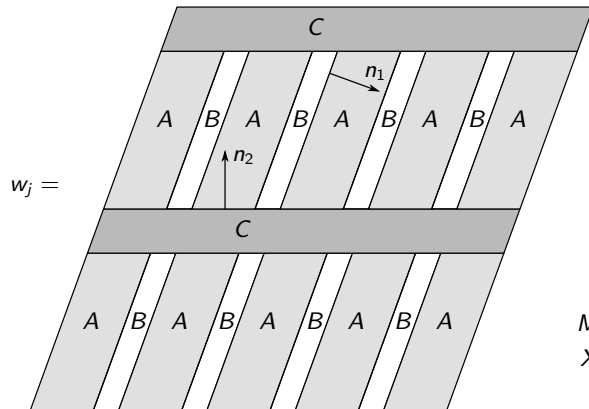
$$\int_{\Omega} f(x, \nabla w_j(x)) \, dx \quad \rightarrow \quad \int_{\Omega} \int_{\mathbb{R}^N} f(x, \cdot) \, d\nu \, dx$$

for all continuous  $f: \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$  with  $p$ -growth.

## Lamination example

**2<sup>nd</sup> order laminate:**  $\nu = \theta_2(\theta_1\delta_A + (1 - \theta_1)\delta_B) + (1 - \theta_2)\delta_C$

where  $\text{rank}(A - B) = 1$ ,  $\text{rank}[(\theta_1 A + (1 - \theta_1)B) - C] = 1$ .



$$M = \theta_1 A + (1 - \theta_1) B$$

$$X = \theta_2 M + (1 - \theta_2) C$$

# “Geometry” of the set $\{ M \in \mathbb{R}^{d \times d} : \det M = 0 \}$ in matrix space III

## Proposition

Let  $M_0 \in \mathbb{R}^{d \times d}$  with  $\det M_0 < 0$ . Then, there exists a homogeneous Young measure  $\nu \in \mathbf{M}^1(\mathbb{R}^{d \times d})$  that is a  $p$ -laminate of infinite order for every  $p \in [1, d)$  and such that the following assertions hold:

$$(i) \int A \, d\nu(A) = M_0,$$

$$(ii) \operatorname{supp} \nu \subset \{ M \in \mathbb{R}^{d \times d} : \det M = 0 \},$$

$$(iii) \int |\cdot|^p \, d\nu \leq C_p |M_0|^p,$$

$$(iv) \int |A - M_0|^p \, d\nu(A) \leq C_p |\det M_0|^{p/d},$$

where  $C_p = C(d, p)$ .

**Note:** This cannot hold for  $p > d$ : In this case we would have

$$0 > \det M_0 \stackrel{\det \text{ Null-Lagrangian}}{=} \int \det \, d\nu = 0.$$



## Proof of Proposition 1

- Without loss of generality  $M_0 = \begin{pmatrix} -\sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_d \end{pmatrix}$  with  $\sigma_i > 0$   
(modified singular value decomposition).

- $\gamma := \sqrt{\sigma_1 \sigma_2}$ .

- Decompose  $M_0$  twice along rank-one lines as follows:

$$\begin{aligned} M_0 &= \frac{1}{2} [M_0 + \underbrace{\gamma(e_1 \otimes e_2)}_{\text{rank}=1}] + \frac{1}{2} [M_0 - \gamma(e_1 \otimes e_2)] \\ &= \frac{1}{4} [M_0 + \gamma(e_1 \otimes e_2) + \gamma(e_2 \otimes e_1)] + \frac{1}{4} [M_0 + \gamma(e_1 \otimes e_2) - \gamma(e_2 \otimes e_1)] \\ &\quad + \frac{1}{4} [M_0 - \gamma(e_1 \otimes e_2) + \gamma(e_2 \otimes e_1)] + \frac{1}{4} [M_0 - \gamma(e_1 \otimes e_2) - \gamma(e_2 \otimes e_1)] \\ &=: \frac{1}{4} M_{1,B1} + \frac{1}{4} M_{1,G1} + \frac{1}{4} M_{1,G2} + \frac{1}{4} M_{1,B2}. \end{aligned}$$

- Compute:  $\det M_{1,G1} = \det M_{1,G2} = 0$  and  $\det M_{1,B1} = \det M_{1,B2} < 0$ .
- Set  $r := 2^{\frac{p}{d}-1}$ , hence  $r < 1$ . Then:

$$|\det M_{1,B1}| = |\det M_{1,B2}| = 2 |\det M_0| = (2r)^{d/p} |\det M_0|.$$

## Proof of Proposition II

- So far:  $M_0 = \frac{1}{4}M_{1,B1} + \frac{1}{4}M_{1,G1} + \frac{1}{4}M_{1,G2} + \frac{1}{4}M_{1,B2}$ .
- Apply the same scheme recursively to the “bad” matrices  $M_{1,B1}, M_{1,B2}$ .  
↪ sequence  $\nu_j, j = 1, 2, \dots$ , of finite-order laminates with  $\int A \, d\nu_j(A) = M_0$ .
- $|\nu_j|(\{M \in \mathbb{R}^{d \times d} : \det M < 0\}) = \frac{2^j}{4^j} \rightarrow 0$ .
- Exemplary estimate:

$$\begin{aligned} \int |A - M_0|^p \, d\nu_j(A) &= \sum_{i=1}^j \sum_{k=1}^{2^i} \frac{1}{4^i} |M_{i,Gk} - M_0|^p + \sum_{k=1}^{2^j} \frac{1}{4^j} |M_{j,Bk} - M_0|^p \\ &\leq \dots \\ &\leq \left[ \frac{2^{1/2}}{(2r)^{1/p} - 1} \right]^p \cdot |\det M_0|^{p/d} \cdot \left[ \sum_{i=1}^j \frac{2^i (2r)^i}{4^i} + \frac{2^j (2r)^j}{4^j} \right] \\ &\leq \left[ \frac{2^{1/2}}{(2r)^{1/p} - 1} \right]^p \cdot |\det M_0|^{p/d} \cdot \left[ \frac{1}{1-r} + r^j \right] \\ &\leq C_p |\det M_0|^{p/d} \quad \text{since } r < 1! \end{aligned}$$

- Sought  $p$ -laminate is  $\nu := w^* \text{-}\lim_{j \rightarrow \infty} \nu_j$ .

- **Situation so far:** Have  $w = u_j \in W^{1,p}(\Omega; \mathbb{R}^d)$  with  $w = u$  on  $\partial\Omega$  and

$$\int_{\{\det \nabla w < 0\}} |\det \nabla w(x)|^{p/d} dx < \varepsilon.$$

- Need to find  $v \in W^{1,p}(\Omega; \mathbb{R}^d)$  with  $v = u$  on  $\partial\Omega$

$$\det \nabla v > 0 \quad \text{a.e.} \quad \text{and} \quad \|v - w\|_{W^{1,p}} = O(\varepsilon).$$

- Here: We will only show  $\det \nabla v \geq 0$  a.e..
- Fix  $x_0 \in \Omega$ .
- **Case (1):** If  $\det \nabla w(x_0) < 0$  use Proposition to get  $p$ -laminates (of infinite order)  $\nu_{x_0}$  such that

$$\nabla w(x_0) = \int A d\nu_{x_0}(A)$$

and

$$\text{supp } \nu_{x_0} \subset \{M \in \mathbb{R}^{d \times d} : \det M \geq 0\}.$$

- **Case (2):** If  $\det \nabla w(x_0) \geq 0$ : Set  $\nu_{x_0} := \delta_{\nabla w(x_0)}$ .

## Return to proof of Theorem: Convex Integration

- Can show:  $\nu = \nu_x$  satisfies the Kinderlehrer–Pedregal condition (KP1)–(KP3).
- Kinderlehrer–Pedregal Theorem  $\Rightarrow$  There exists a generating sequence  $(w_j^1) \subset W^{1,p}(\Omega; \mathbb{R}^d)$ , pick  $w^1$  “sufficiently far along” this sequence.
- Iterate, in each step  $w^k$  gets closer to satisfying  $\det \nabla v = 0$  a.e.:

$$\int_{\{\det \nabla w^k < 0\}} |\det \nabla w^k|^{p/d} dx \leq C 2^{-kp} \int_{\{\det \nabla u < 0\}} |\det \nabla u|^{p/d} dx.$$

- Can show:  $(w^k)$  is Cauchy in  $W^{1,p}$ .
- Limit  $v_j := w$  indeed satisfies  $\det \nabla w = 0$  a.e.!
- Now want also  $\det \nabla v_j > 0$ : Perturbation argument & more careful matrix geometry argument (similar organization as before).
- **Thus:** Have found sequence  $(\nabla v_j) \subset L^p(\Omega; \mathbb{R}^{d \times d})$  generating  $\nu = (\nu_x)$  and with

$$\det \nabla v_j > 0 \quad \text{a.e.} \quad \text{and} \quad \|v_j - u_j\|_{W^{1,p}} \rightarrow 0$$

$\rightsquigarrow$  Our Theorem is proved.

### Corollary

Let  $\Omega \subset \mathbb{R}^d$  be open and bounded and  $1 < p < d$ . Let  $u \in W^{1,p}(\Omega; \mathbb{R}^d)$  be weakly orientation-preserving,

$$\det \nabla u \geq 0 \quad \text{a.e.}$$

Then, there exists a sequence  $(v_j) \subset W^{1,p}(\Omega; \mathbb{R}^d)$  that is strictly orientation-preserving,

$$\det \nabla v_j > 0 \quad \text{a.e.} \quad \text{for all } j \in \mathbb{N},$$

and such that  $\|v_j - u\|_{1,p} \rightarrow 0$  as  $j \rightarrow \infty$ .

**Recently:** We also considered the more general constraint

$$J_1(x) \leq \det \nabla u_j(x) \leq J_2(x) \quad \text{for all } j \in \mathbb{N} \text{ and a.e. } x \in \Omega.$$

where

- 1  $J_1: \Omega \rightarrow [-\infty, +\infty)$ ,  $J_2: \Omega \rightarrow (-\infty, +\infty]$  measurable,
- 2  $J_1(x) \leq J_2(x)$  for a.e.  $x \in \Omega$ , and
- 3  $\int_{\Omega} |J_1^+(x)|^{p/d} dx < \infty$  and  $\int_{\Omega} |J_2^-(x)|^{p/d} dx < \infty$ .

**Examples:**  $\det \nabla u_j(x) = r > 0$ ,  $\det \nabla u_j(x) = 1$ , ...

**Question 4:** Can we carry out our program also for this more general case?

## Theorem (Koumatos & R. & Wiedemann '13 #2)

Let  $1 < p < d$ . Suppose that  $\Omega \subset \mathbb{R}^d$  is an open bounded Lipschitz domain, and let  $\nu = (\nu_x)_{x \in \Omega} \subset \mathbf{M}^1(\mathbb{R}^{d \times d})$  be a  $p$ -Young measure. Moreover let  $J_1: \Omega \rightarrow [-\infty, +\infty)$ ,  $J_2: \Omega \rightarrow (-\infty, +\infty]$  are as before. Then, the following statements are equivalent:

- (i) There exists a sequence of gradients  $(\nabla u_j) \subset L^p(\Omega; \mathbb{R}^{d \times d})$  that generates  $\nu$ , such that

$$J_1(x) \leq \det \nabla u_j(x) \leq J_2(x) \quad \text{for all } j \in \mathbb{N} \text{ and a.e. } x \in \Omega.$$

- (ii) The conditions (KP1)–(KP3) from the Kinderlehrer–Pedregal Theorem hold and

$$(\text{SUPP}) \quad \text{supp } \nu_x \subset \{ A \in \mathbb{R}^{d \times d} : J_1(x) \leq \det A \leq J_2(x) \} \quad \text{for a.e. } x \in \Omega.$$

Furthermore, in this case the sequence  $(u_j)$  can be chosen so that  $(\nabla u_j)$  is  $p$ -equiintegrable and  $u_j - u \in W_0^{1,p}(\Omega, \mathbb{R}^d)$ .

**Proof:** A “refinement” of the proof for the case  $\det \nabla u > 0$ .

## Corollary

Let  $\Omega \subset \mathbb{R}^d$  be a bounded Lipschitz domain,  $1 < p < d$ ,  $J : \Omega \rightarrow \mathbb{R}$  be measurable with

$$\int_{\Omega} |J(x)|^{p/d} dx < \infty,$$

and let  $g \in W^{1-1/p,p}(\partial\Omega; \mathbb{R}^d)$ . Then, there exists  $v \in W^{1,p}(\Omega; \mathbb{R}^d)$  such that

$$\begin{cases} \det \nabla v(x) = J(x) & \text{for a.e. } x \in \Omega, \\ v|_{\partial\Omega} = g & \text{in the sense of trace.} \end{cases}$$

## Corollary

Let  $\Omega \subset \mathbb{R}^d$  be a bounded Lipschitz domain,  $1 < p < d$  and  $g \in W^{1-1/p,p}(\partial\Omega; \mathbb{R}^d)$ . Then, there exists an incompressible map  $v \in W^{1,p}(\Omega; \mathbb{R}^d)$  with the given boundary data (in the sense of trace), i.e.

$$\begin{aligned} \det \nabla v(x) &= 1 \text{ almost everywhere,} \\ v|_{\partial\Omega} &= g. \end{aligned}$$



- In  $W^{1,p}$  with  $p \geq d$ , the Jacobian determinant is “rigid”.
- In  $W^{1,p}$  with  $p < d$ , the Jacobian determinant is “soft”.
- One can write *any* matrix as the barycenter of a  $p$ -laminate (of infinite order) with support in zero-determinant matrices (new work: also strictly positive/negative determinant possible).
- Good estimates hold.
- This can be used to prove a Kinderlehrer–Pedregal-type theorem under the side-constraint of positive Jacobian.
- In the spirit of convex integration (recently very successful theory).

### Outlook:

- Applications to nonlinear elasticity, fluid dynamics, ...

**Thank you for your attention!**

**References:**

K. Koumatos, F. R., and E. Wiedemann: *Orientation-preserving Young measures*, submitted (August 2013), arXiv:1307.1007.

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