Technische Universität Dresden • Fakultät Mathematik

Galois and Monodromy Groups in Algebraic Geometry

Bachelorarbeit

zur Erlangung des ersten Hochschulgrades

***Bachelor of Science* (*B.Sc.*)**

vorgelegt von

Yorick Lennard FUHRMANN

(geboren am 14.01.2001 in Berlin)

Tag der Einreichung: 27. 06. 2022

Jun.-Prof. Dr. Mario Kummer (Institut für Geometrie)

Erklärung

Hiermit erkläre ich, dass ich die am 27. 06. 2022 eingereichte Bachelorarbeit zum Thema *Galois and Monodromy Groups in Algebraic Geometry* unter Betreuung von Jun.-Prof. Dr. Mario Kummer selbstständig erarbeitet, verfasst und Zitate kenntlich gemacht habe. Andere als die angegebenen Hilfsmittel wurden von mir nicht benutzt.

Dresden, 27. 06. 2022 Unterschrift

Contents

1 Introduction

In the geometric context of a dominant morphism between complex affine varieties we will consider two subgroups of the automorphism group of a generic fibre of the morphism. We will assume this fibre to be finite of cardinality d to obtain subgroups of the permutation group on d elements. The first of these groups discussed in Section 3.1 will be a monodromy group arising in the topological context of covering maps which we will obtain by assuming the affine varieties to be of the same dimension. The second one of these groups discussed in Section 3.2 will be a Galois group obtained by a normalization of the extension of function fields of the affine varieties. In Section 3.3 we show in Theorem 3.13 from [Har79] that the mentioned monodromy group and the Galois group are equal. We then consider examples with concrete computations of both groups. The established theory is used in Section 4.1 by showing that every finite group is a monodromy group of a geometric covering map, that is a covering map of complex affine varieties obtained from a dominant morphism. This result can be understood as an extension of the fact that every finite group is a Galois group of some Galois extension. In Section 4.2 we consider automorphisms of the corresponding function fields and show that they induce deck transformations of such coverings.

This text gives an introduction to the theory of covering maps and then applies these concepts of algebraic topology in a geometric setting. Knowledge of general topology and the fundamental group as well as basic algebraic geometry and Galois theory will be assumed. We mainly use [Bre93] for topological and [Har92], [Har77] for foundational geometric results.

Throughout all paths are continuous. The symbol I stands for the unit interval $[0, 1]$ with standard topology and k is an algebraically closed field.

2 Topological Foundations

Before considering a geometric context we will set up the topological foundations needed for Section 3 by defining the monodromy group in Section 2.2. For that we prove the Path Lifting Property for coverings (Theorem 2.3) and the Covering Homotopy Theorem (Theorem 2.4) in Section 2.1. The consideration of deck transformations in Section 2.3 will lead to the classification of covering spaces in Section 2.4, as well as the question when topological spaces admit coverings by simply connected spaces (Theorem 2.39).

2.1 Covering Maps

Definition 2.1 (Covering Map)**.** [Bre93, Definition 3.1, p. 139] Let X and Y be topological spaces. A continuous map $p\colon X \to Y$ is called a *covering map* (and X is called a *covering space* of Y) if X and Y are Hausdorff, path-connected and locally path-connected, and if each point $y \in Y$ has a path-connected neighbourhood $U \subseteq Y$ such that $p^{-1}(U)$ is a nonempty disjoint union of sets U_α (which are the path components of $p^{-1}(U)$) on which $p|_{U_\alpha}\colon U_\alpha \xrightarrow{\sim} U$ is a homeomorphism. Such sets U are called *elementary* or *evenly covered*.

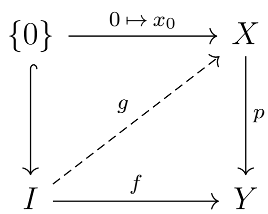
As every point in Y has a neighbourhood homeomorphic to an elementary set in X, a covering map is surjective. Furthermore, the number of points in the preimage of a point in Y under a covering map is locally constant. Since the base space X is connected, it is constant on all of Y. This number is called the *number of sheets* of the covering.

Example 2.2.

- The map $p\colon \mathbb{R} \to S^1,\ t \mapsto (\cos(2\pi t), \sin(2\pi t))$ which projects the real line seen as a helix onto the unit circle is a covering map with infinitely many sheets.
- The map $p\colon S^1 \to S^1,\ z \mapsto z^n$ where we view z as a complex number with $|z| = 1$ and $n \in \mathbb{Z}_{>0}$ is a covering map with n sheets.

The following theorem shows that for a covering map $p\colon X \to Y$ paths in Y can be lifted to the covering space X if a starting point is chosen.

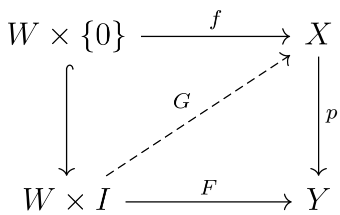
Theorem 2.3 (Path Lifting Property)**.** *[Bre93, Theorem 3.3, p. 140] Let $p\colon X \to Y$ be a covering map and let $f\colon I \to Y$ be a path. Let $x_0 \in X$ such that $p(x_0) = f(0)$. Then there exists a unique path $g\colon I \to X$ such that $p \circ g = f$ and $g(0) = x_0$, i.e. making the diagram below commute. The path g is called* lifting *of f.*



Proof. By the Lebesgue Lemma [Bre93, Lemma 9.11, p. 28] there exists $n \in \mathbb{N}$ such that $f([\frac{i}{n}, \frac{i+1}{n}])$ lies in an elementary set for every $i \in \{0, \dots, n-1\}$. We can now lift by induction on i via the local homeomorphisms of elementary sets: at every step $g\left(\frac{i}{n}\right)$ is defined through $g\left(\frac{(i-1)+1}{n}\right)$ for $i \in \{1, \dots, n-1\}$ or $f(0)$ for $i = 0$. This determines the subset $U_i \subseteq X$ over the elementary set containing $f([\frac{i}{n}, \frac{i+1}{n}])$ uniquely. Then define g on $[\frac{i}{n}, \frac{i+1}{n}]$ through $p|_{U_i}^{-1} \circ f$, where $p|_{U_i}$ is the local homeomorphism from U_i to the elementary set containing $f([\frac{i}{n}, \frac{i+1}{n}])$. $\square$

If $F\colon X \times I \to Y$ is a homotopy and $X' \subseteq X$ a subspace then F is a homotopy relative to X' (i.e. rel X') if the function $F(x', \cdot)\colon I \to Y$ is constant for every $x' \in X'$. So if $f, g\colon X \to Y$ are continuous functions, then f and g are homotopic rel X', i.e. $f \simeq g$ rel X', if there exists a homotopy $F\colon X \times I \to Y$ between f and g such that $F(x', t) = f(x') = g(x')$ for all $x' \in X'$ and $t \in I$.
We can use the Path Lifting Property to show that homotopies can be lifted to covering spaces as well. The definition of the lifted homotopy is natural as we know how to lift paths, but continuity of the lifting will turn out to be nontrivial.

Theorem 2.4 (Covering Homotopy Theorem)**.** *[Bre93, Theorem 3.4, p. 140] Let W be a locally connected topological space and let $p\colon X \to Y$ be a covering map. Let $F\colon W \times I \to Y$ be a homotopy and let $f\colon W \times \{0\} \to X$ be a lifting of the restriction $F|_{W\times\{0\}}$. Then there is a unique homotopy $G\colon W \times I \to X$ with $p \circ G = F$ and $G|_{W\times\{0\}} = f$, i.e. making the diagram below commute. Furthermore, if F is a homotopy* rel W' *for $W' \subseteq W$, then so is G.*



Proof. Define G uniquely on each $\{w\} \times I$ using Theorem 2.3. Let $w \in W$. By [Bre93, Lemma 3.2, p. 140] we can find a connected neighbourhood $w \in N \subseteq W$ and $n \in \mathbb{N}$ such that $F(N \times [\frac{i}{n}, \frac{i+1}{n}])$ is in some elementary set U_i, $i \in \{0, \ldots, n-1\}$. By induction on i using the induction hypothesis assume that G is continuous on $N \times \{\frac{i}{n}\}$, so as $N \times \{\frac{i}{n}\}$ is connected, the image $G(N \times \{\frac{i}{n}\})$ is also connected and contained in a single component V of $p^{-1}(U_i)$. So on $N \times [\frac{i}{n}, \frac{i+1}{n}]$ we have $G = (p|_V)^{-1} \circ F$, where $p|_V\colon V \xrightarrow{\sim} U_i$ is homeomorphic. So G is continuous on $N \times [\frac{i}{n}, \frac{i+1}{n}]$ and on $N \times I$ by induction, which makes G continuous on $W \times I$. The last statement follows from the construction of G: If $F(w', \cdot)$ is constant for each $w' \in W'$, then the lifting G on $\{w'\} \times I$ is also constant, because for $a, b \in I$ we have $p(G(w', a)) = F(w', a) = F(w', b) = p(G(w', b))$, so $G(w', a)$ and $G(w', b)$ are in the same fibre of p, which is discrete. Continuity of G implies $G(w', a) = G(w', b)$. □

Corollary 2.5. *[Bre93, Corollary 3.5, p. 141] Let $p\colon X \to Y$ be a covering map. Let f_0 and f_1 be paths in Y with $f_0 \simeq f_1$* rel ∂I*. Let $\tilde{f}_0$ and $\tilde{f}_1$ be liftings of f_0 and f_1 such that $\tilde{f}_0(0) = \tilde{f}_1(0)$. Then $\tilde{f}_0(1) = \tilde{f}_1(1)$ and $\tilde{f}_0 \simeq \tilde{f}_1$* rel ∂I*.*

Proof. By using Theorem 2.4 a homotopy $F\colon I \times I \to Y$ rel ∂I with $F(t, 0) = f_0(t)$, $F(t, 1) = f_1(t)$ for all $t \in I$ lifts to a homotopy $\tilde{F}\colon I \times I \to X$ rel ∂I with

$\tilde{F}(t,0) = \tilde{f}_0(t)$, $\tilde{F}(t,1) = \tilde{f}_1(t)$ for all $t \in I$. In particular $\tilde{f}_0(1) = \tilde{F}(1,0) = \tilde{F}(1,1) = \tilde{f}_1(1)$. □

Corollary 2.6. *[Bre93, Corollary 3.7, p. 141] Let $p\colon X \to Y$ be a covering map and $p(x_0) = y_0$. Then the group homomorphism*

$$p_\#\colon \pi_1(X, x_0) \to \pi_1(Y, y_0),\ [f] \mapsto [p \circ f]$$

induced by p on the fundamental groups is injective, and its image consists of the equivalence classes of loops at y_0 in Y which lift to loops at x_0 in X.

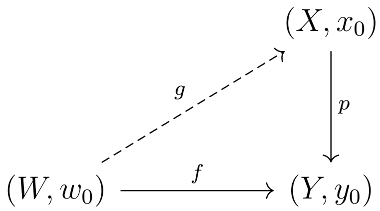
Proof. For $\alpha, \beta \in \pi_1(X, x_0)$ represented by loops f and g, the images $p_\#(\alpha)$ and $p_\#(\beta)$ are represented by loops $p \circ f$ and $p \circ g$. If $p_\#(\alpha) = p_\#(\beta)$ then $p \circ f \simeq p \circ g$ rel ∂I, and by Corollary 2.5 $f \simeq g$ rel $\partial I \Rightarrow \alpha = \beta$, so $p_\#$ is injective. Clearly the loop $p \circ f$ at y_0 lifts to the loop f at x_0 in X, and every loop at y_0 in Y which lifts to a loop at x_0 in X is a projection of its lifting, so in the image of $p_\#$. □

The following two statements will be essential for the consideration of deck transformations in Section 2.3. The Lifting theorem answers the question when maps from general topological spaces into covered spaces can be lifted along the covering map.

Theorem 2.7 (The Lifting Theorem)**.** *[Bre93, Theorem 4.1, p. 143] Let $p\colon X \to Y$ be a covering map with $p(x_0) = y_0$. Let W be a path-connected and locally path-connected topological space and $f\colon W \to Y$ a continuous map with $f(w_0) = y_0$, inducing a group homomorphism*

$$f_\#\colon \pi_1(W, w_0) \to \pi_1(Y, y_0),\ [\alpha] \mapsto [f \circ \alpha].$$

Then there exists a continuous map $g\colon W \to X$ with $g(w_0) = x_0$ such that $p \circ g = f$ if and only if $f_\#(\pi_1(W, w_0)) \subseteq p_\#(\pi_1(X, x_0))$.



Proof. Assuming $f_\#(\pi_1(W, w_0)) \subseteq p_\#(\pi_1(X, x_0))$ we define the function g: For $w \in W$ let $\lambda\colon I \to W$ be a path in W from w_0 to w. Then $f \circ \lambda$ is a path in Y which can be lifted to a path $\mu\colon (I, 0) \to (X, x_0)$ by Theorem 2.3, set $g(w) = \mu(1)$. Then $(p \circ g)(w) = p(\mu(1)) = f(\lambda(1)) = f(w)$.
g is well-defined: Suppose λ' is another path in W from w_0 to w and let $(\lambda')^{-1}$ be its reverse parameterization $t \mapsto \lambda'(1-t)$. Then $\lambda * (\lambda')^{-1}$ is a loop at $w_0 \in W$ and

$(f \circ \lambda) * (f \circ (\lambda')^{-1})$ is a loop at y_0 in Y. Since $[(f \circ \lambda) * (f \circ (\lambda')^{-1})] = f_\#[\lambda * (\lambda')^{-1}] \in \operatorname{im}(f_\#) \subseteq \operatorname{im}(p_\#)$, $(f \circ \lambda) * (f \circ (\lambda')^{-1})$ lifts to a loop in X at x_0 by Corollary 2.6. The reverse of the part of this lift corresponding to $(\lambda')^{-1}$ then is a lift μ' of λ', and $\mu'(1) = \mu(1)$.
g is continuous: Let $w \in W$ and set $y = f(w)$. Let $U \subseteq Y$ be an elementary neighbourhood of y and let $V \subseteq W$ be a path-connected neighbourhood of w such that $f(V) \subseteq U$. For $w' \in V$ we can find a path from w_0 to w' by concatenating a given path λ from w_0 to w with a path σ from w to w'. Since $f(V)$ is contained in an elementary set, the lift of $f \circ \sigma$ is $p|_U^{-1} \circ f \circ \sigma$, where $p|_U^{-1}$ is the inverse of p mapping U to the component of $p^{-1}(U)$ containing $g(w)$. This same component is used for all $w' \in V$, so g is continuous at w.
Conversely, $p \circ g = f$ implies $\operatorname{im}(f) \subseteq \operatorname{im}(p)$, so $f_\#(\pi_1(W, w_0)) \subseteq p_\#(\pi_1(X, x_0))$. ☐

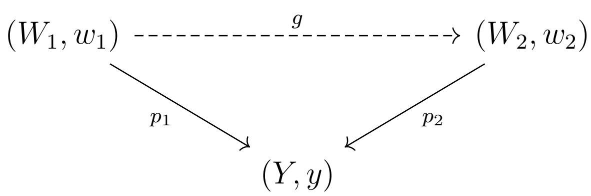
The following lemma in particular shows that in the case of the equivalence in Theorem 2.7, the lifting $g \colon W \to X$ is unique.

Lemma 2.8. *[Bre93, Lemma 4.4, p. 145] Let W be a connected topological space, $p \colon X \to Y$ a covering map and $f \colon W \to Y$ continuous. Let $g_1, g_2 \colon W \to X$ be continuous liftings of f. If $g_1(w) = g_2(w)$ for a point $w \in W$, then $g_1 = g_2$.*

Proof. Let $x = g_1(w) = g_2(w)$ for some $w \in W$. Let $U \subseteq Y$ be an open elementary neighbourhood of $f(w)$. Let $V \subseteq X$ be the component of $p^{-1}(U)$ containing x. Then $A := g_1^{-1}(V) \cap g_2^{-1}(V) \subseteq W$ is open and nonempty since $w \in A$. For $a \in A$ we have $p(g_1(a)) = f(a) = p(g_2(a))$, so $g_1(a) = g_2(a)$. Thus, $\{w \in W \colon g_1(w) = g_2(w)\}$ is a union of such open sets A and itself open. But it is also closed, since it is the preimage of the diagonal under the map $g_1 \times g_2 \colon W \to X \times X$, and the diagonal is closed since X is Hausdorff. Because W is connected it is all of W. ☐

This also enables us to consider the situation of two covering spaces of the same topological space, where one of the covering spaces is simply connected.

Corollary 2.9. *[Bre93, Corollary 4.5, p. 145] Let $p_i \colon W_i \to Y$, $i = 1, 2$, be covering maps such that W_1 is simply connected, and let $w_i \in W_i$ and $y \in Y$ such that $p_i(w_i) = y$. Then there is a unique continuous map $g \colon W_1 \to W_2$ satisfying $g(w_1) = w_2$ and $p_2 \circ g = p_1$. Furthermore, g is a covering map.*

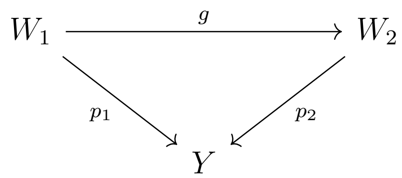


Proof. As W_1 is simply connected the continuous map $g\colon W_1 \to W_2$ with $g(w_1) = w_2$ and $p_2 \circ g = p_1$ exists by Theorem 2.7. The uniqueness of g follows from Lemma 2.8.
g is surjective: Let $w \in W_2$ and let γ be a path in W_2 from w_2 to w. Then $p_2 \circ \gamma$ is a path in Y from $p_2(w_2)$ to $p_2(w)$. Lifting to W_1 yields a path $\tilde{\gamma}$ starting in some $\tilde{w} \in p_1^{-1}(p_2(w_2))$. Then $g \circ \tilde{\gamma}$ again starts in w_2. Uniqueness of path lifting (Theorem 2.3) implies $g \circ \tilde{\gamma} \simeq \gamma$ rel ∂I, so the endpoint w is in the image of g.
g is a covering map: Let $w \in W_2$ and let $y := p_2(w)$. We can find a path-connected neighbourhood U of y which is evenly covered by both p_1 and p_2 (obtained by intersecting evenly covered neighbourhoods and taking the path-component of y). Let V be the sheet of $p_2^{-1}(U)$ containing w. We show that V is evenly covered by g. Let $\{U_\alpha\}$ be the sheets of $p_1^{-1}(U)$. Then g maps each U_α into $p_2^{-1}(U)$, and as U_α is connected the image lies in a single sheet of $p_2^{-1}(U)$. Then $g_1^{-1}(V)$ is nonempty since g is surjective and $g_1^{-1}(V)$ is the union of the sheets U_α that are mapped into V by g. This mapping is homeomorphic: For such a U_α let $p_1|_{U_\alpha}\colon U_\alpha \xrightarrow{\sim} U$ and $p_2|_V\colon V \xrightarrow{\sim} U$ be the homeomorphisms obtained by restriction, let $g|_{U_\alpha}\colon U_\alpha \to V$. Then $g|_{U_\alpha} = p_2|_V^{-1} \circ p_1|_{U_\alpha}$ is also homeomorphic. □

So a simply connected covering space also covers every other covering space.

Definition 2.10 (Universal cover)**.** A simply connected covering space X of a topological space Y is called *universal cover* of Y.

Definition 2.11 (Equivalent covers)**.** Two covering spaces W_1 and W_2 of a given space Y with covering maps $p_1\colon W_1 \to Y$ and $p_2\colon W_2 \to Y$ are called *equivalent*, if there exists a homeomorphism $g\colon W_1 \xrightarrow{\sim} W_2$ such that $p_2 \circ g = p_1$.



Given a fixed space Y the defined equivalence of covers defines an equivalence relation on the set of covering spaces of Y. The following statement justifies to speak of *the* universal cover, as different universal covers are equivalent.

Corollary 2.12. *[Bre93, Corollary 4.6, p. 145] Let $p_i\colon W_i \to Y$, $i = 1, 2$, be covering maps such that W_1 and W_2 are both simply connected. If $w_i \in W_i$ such that $p_1(w_1) = p_2(w_2)$ then there is a unique equivalence $g\colon W_1 \xrightarrow{\sim} W_2$ satisfying $g(w_1) = w_2$ and $p_2 \circ g = p_1$. Furthermore, g is a covering map.*

Proof. With Corollary 2.9 we obtain the continuous maps $g\colon W_1 \to W_2$ with $g(w_1) = w_2$ and $p_2 \circ g = p_1$, $k\colon W_2 \to W_1$ with $k(w_2) = w_1$ and $p_1 \circ k = p_2$. Then $p_1 \circ k \circ g =$

$p_2 \circ g = p_1$ and $(k(g(w_1)) = k(w_2) = w_1$, so $k \circ g = \mathrm{id}_{W_1}$ by Lemma 2.8. Analogously $g \circ k = \mathrm{id}_{W_2}$, so $g = k^{-1}$. □

In Section 2.4 we will give a necessary and sufficient condition for the existence of universal covers.

2.2 The Monodromy Group

For a covering map $p\colon X \to Y$ we will construct an action of the fundamental group of the base space $\pi_1(Y, y_0)$ on the fibre $p^{-1}(y_0) \subseteq X$. First we observe that the fundamental group of a topological space does not depend on the base point if the base points are connected by a path.

Theorem 2.13. *[Bre93, Theorem 2.3, p. 135] Let X be a topological space and $\gamma\colon I \to X$ be a path in X from $\gamma(0) = x_0$ to $\gamma(1) = x_1$. Then the map*

$$h_\gamma\colon \pi_1(X, x_1) \to \pi_1(X, x_0),\ [f] \mapsto [\gamma * f * \gamma^{-1}]$$

is a group isomorphism of the fundamental groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ with inverse $h_{\gamma^{-1}}$.

Proof. The image of h_γ only depends on the homotopy class of f, so it is well-defined. It is a homomorphism since $(\gamma * f * \gamma^{-1}) * (\gamma * g * \gamma^{-1}) \simeq \gamma * f * (\gamma^{-1} * \gamma) * g * \gamma^{-1} \simeq \gamma * (f * g) * \gamma^{-1}$ rel ∂I. If δ is another path in X extending γ then we have $h_\gamma \circ h_\delta = h_{\gamma * \delta}$, and for $\gamma \simeq \delta$ rel ∂I we have $h_\gamma = h_\delta$. If c_x is the constant path in $x \in X$ we get $h_{c_x} = 1$. This implies $h_\gamma \circ h_{\gamma^{-1}} = h_{\gamma * \gamma^{-1}} = h_{c_{x_0}} = 1$ as $\gamma * \gamma^{-1} \simeq c_{x_0}$ rel ∂I and similarly $h_{\gamma^{-1}} \circ h_\gamma = 1$, so h_γ is a group isomorphism. □

Corollary 2.14. *The fundamental group $\pi_1(X, x_0)$ of a topological space X only depends, up to (nontrivial) isomorphism, on the path component of the base point x_0. In particular if X is path-connected, its fundamental group $\pi_1(X)$ does not depend on the choice of the base point.*

We can now construct the action of the fundamental group on the fibre which gives rise to the definition of the monodromy group in the following.

Construction 2.15. [Bre93, p. 146] Let $p\colon X \to Y$ be a covering map, $y_0 \in Y$ a fixed base point and define $J := \pi_1(Y, y_0)$ as the fundamental group of Y in y_0, $F := p^{-1}(y_0)$ as the fibre of y_0 under p. Then J acts on F via a right action $F \times J \to F$:
Let $x \in F$ and $\alpha \in J$. Represent α by a loop $f\colon I \to Y$. By Theorem 2.3 we can lift f to get a path $g\colon I \to X$ with $g(0) = x$. Then define $x \cdot \alpha := g(1)$. By Corollary 2.5 this does not depend on the choice of f: for any other $f' \in \alpha$ we have $f \simeq f'$ rel ∂I

and for any lifting g' of f' with $g'(0) = x = g(0)$ it follows $g'(1) = g(1)$. So we have a well-defined function $F \times J \to F$. We now show that it is a group action.

- $x \cdot 1 = x$, because lifting the constant loop c_{y_0} in y_0 gives a constant path in X with $g(0) = x$, so $g(1) = x$.
- $(x \cdot \alpha) \cdot \beta = x \cdot (\alpha\beta)$. Lift a loop representing α to a path f starting at x. This path goes from x to $x \cdot \alpha$. Then lift a loop representing β to a path g starting at $x \cdot \alpha$. This path goes from $x \cdot \alpha$ to $(x \cdot \alpha) \cdot \beta$. Then $f * g$ is a lift of a loop representing $\alpha\beta$ which starts at x and ends at $x \cdot (\alpha\beta)$.

Definition 2.16 (Monodromy Action)**.** [Bre93, p. 146] The right action

$$p^{-1}(y_0) = F \curvearrowleft J = \pi_1(Y, y_0)$$

constructed in Construction 2.15 is called *monodromy action* of π_1 on the fibre. Letting $\mathrm{Aut}(F)$ be the group of bijections $F \to F$, the homomorphism $J \to \mathrm{Aut}(F)$ induced by the action is called *monodromy*, its image M is called *monodromy group*.

A first observation is the behaviour of the monodromy group under a change of the base point of the covered space.

Remark 2.17. Consider a covering map $p\colon X \to Y$. By Definition 2.1, the covered space Y is path-connected. So using Corollary 2.14, the fundamental group $\pi_1(Y, y_0)$ does not (up to isomorphism) depend on the choice of the base point y_0. We will show that the monodromy group of the covering does not depend on this choice either.
For that, let $y_0, y_1 \in Y$ and let $\gamma\colon I \to Y$ be a path in Y from $\gamma(0) = y_0$ to $\gamma(1) = y_1$. From Theorem 2.13 we have an isomorphism

$$h_\gamma\colon \pi_1(Y, y_1) \xrightarrow{\sim} \pi_1(Y, y_0),\ [f] \mapsto [\gamma * f * \gamma^{-1}].$$

Let $F_i := p^{-1}(y_i)$ be the fibres and $\mu_i\colon \pi_1(Y, y_i) \to \mathrm{Aut}(F_i)$ the monodromies for $i = 0, 1$. Let $M_i := \mathrm{im}(\mu_i)$, $i = 0, 1$, be the monodromy groups. We construct an isomorphism $\bar{g}\colon \mathrm{Aut}(F_1) \xrightarrow{\sim} \mathrm{Aut}(F_0)$ so that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(Y, y_1) & \xrightarrow{\mu_1} & \mathrm{Aut}(F_1) \\ \downarrow{\scriptstyle h_\gamma} & & \downarrow{\scriptstyle \bar{g}} \\ \pi_1(Y, y_0) & \xrightarrow{\mu_0} & \mathrm{Aut}(F_0) \end{array}$$

For $x \in F_0$ let $\tilde{\gamma}$ be the lifting of γ starting in x. Then $p(\tilde{\gamma}(1)) = \gamma(1) = y_1$, so $x_\gamma := \tilde{\gamma}(1) \in F_1$. Now taking x_γ as a starting point the reversed path $\tilde{\gamma}^{-1}$ is the lifting

of γ^{-1}, ending again in x. So $x_{\gamma\gamma^{-1}} = x$ for $x \in F_0$ and symmetrically $x_{\gamma^{-1}\gamma} = x$ for $x \in F_1$. Therefore, we have a bijection $g\colon F_0 \to F_1,\ x \mapsto x_\gamma$ inducing an isomorphism $\bar{g}\colon \mathrm{Aut}(F_1) \xrightarrow{\sim} \mathrm{Aut}(F_0),\ \sigma \mapsto \bar{\sigma}$, where $\bar{\sigma}(x) = g^{-1}(\sigma(g(x))) = (\sigma(x_\gamma))_{\gamma^{-1}}$.
Now let $\alpha = [f] \in \pi_1(Y, y_1)$ and $x \in F_0$. For $h_\gamma(\alpha) = [\gamma * f * \gamma^{-1}]$ let $\tilde{\gamma}$ be a lifting of γ starting in x and ending in x_γ, lift f to $\tilde{f}$ starting in x_γ and ending in $x_{\gamma f}$, lift γ^{-1} to $\widetilde{\gamma^{-1}}$ starting in $x_{\gamma f}$ and ending in $x_{\gamma f \gamma^{-1}}$. Then $\tilde{\gamma} * \tilde{f} * \widetilde{\gamma^{-1}}$ is a lifting of $h_\gamma(\alpha)$, ending in $x_{\gamma f \gamma^{-1}} \in F_0$. Hence, $x \cdot h_\gamma(\alpha) = x_{\gamma f \gamma^{-1}}$.
For $g(x) = x_\gamma \in F_1$ the lifting $\tilde{f}$ of f with starting point x_γ again yields the endpoint $x_{\gamma f}$, so $g(x) \cdot \alpha = x_\gamma \cdot \alpha = x_{\gamma f} = g(x_{\gamma f \gamma^{-1}}) = g(x \cdot h_\gamma(\alpha)) \Rightarrow \bar{g}(\mu_1(\alpha))(x) = g^{-1}(\mu_1(\alpha)(x_\gamma)) = g^{-1}(x_\gamma \cdot \alpha) = g^{-1}(g(x \cdot h_\gamma(\alpha))) = x \cdot h_\gamma(\alpha) = \mu_0(h_\gamma(\alpha))(x)$. This shows $\bar{g} \circ \mu_1 = \mu_0 \circ h_\gamma$.
It follows $M_0 = \mu_0(\pi_1(Y, y_0)) = \mu_0(h_\gamma(\pi_1(Y, y_1))) = \bar{g}(\mu_1(\pi_1(Y, y_1))) = \bar{g}(M_1) \cong M_1$.

We note some properties of the monodromy action from which two statements about the number of sheets of a covering map will be an immediate consequence.

Lemma 2.18. *[Bre93, p. 146 (3)] The monodromy action $F \curvearrowleft J$ from Definition 2.16 is transitive.*

Proof. For $x_1, x_2 \in F$ take a path $\tilde{\alpha}\colon I \to X$ from x_1 to x_2. This path exists since X is path-connected. It projects to a loop $\alpha := p \circ \tilde{\alpha}$ in Y with base point y_0. We then have $x_1 \cdot [\alpha] = x_2$. □

Lemma 2.19. *[Bre93, p. 146 (4)] For the monodromy action $F \curvearrowleft J$ from Definition 2.16, put $J_{x_0} = \{\alpha \in J\colon x_0 \cdot \alpha = x_0\}$, the stabilizer of x_0 in J. Then*

$$J_{x_0} = \mathrm{im}(p_\#\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)),$$

where $p_\#$ is the group homomorphism induced by p on the fundamental groups.

Proof. We have $\alpha \in J_{x_0} \Leftrightarrow (\alpha = [f]$ and f lifts to a loop in $x_0) \overset{2.6}{\Leftrightarrow} \alpha \in \mathrm{im}(p_\#)$. □

Theorem 2.20. *[Bre93, Theorem 5.1, p. 147] Let $p\colon X \to Y$ be a covering map with $p(x_0) = y_0$. Then there is a one-to-one correspondence between the right cosets $p_\#(\pi_1(X, x_0))\backslash\pi_1(Y, y_0)$ and the fibre $p^{-1}(y_0)$.*

Proof. With the notation from above consider the map

$$\phi\colon J_{x_0}\backslash J \to F,\ \ J_{x_0}\alpha \mapsto x_0 \cdot (J_{x_0}\alpha) := x_0 \cdot \alpha.$$

ϕ is bijective: As the action $F \curvearrowleft J$ is transitive by Lemma 2.18, for each fibre element $x \in F$ there exists an $\alpha \in J$ such that $\phi(J_{x_0}\alpha) = x_0 \cdot (J_{x_0}\alpha) = x_0 \cdot \alpha = x$. If $x_0 \cdot \alpha = x_0 \cdot \beta$ for $\alpha, \beta \in J$, then $x_0 \cdot \alpha\beta^{-1} = x_0 \Rightarrow \alpha\beta^{-1} \in J_{x_0} \Rightarrow J_{x_0}\alpha = J_{x_0}\beta$. The statement now follows from Lemma 2.19. □

Corollary 2.21. *[Bre93, Corollary 5.2, p. 147] Let $p\colon X \to Y$ be a covering map with $p(x_0) = y_0$. Then the number of covering sheets equals the index of $p_\#(\pi_1(X, x_0))$ in $\pi_1(Y, y_0)$.*

Corollary 2.22. *[Bre93, Corollary 5.3, p. 147] Let $p\colon X \to Y$ be a covering map with $p(x_0) = y_0$ and let X be simply connected, i.e. $\pi_1(X, x_0) = 1$. Then the number of covering sheets equals the order of $\pi_1(Y, y_0)$.*

Example 2.23. The real sphere $S^n \subseteq \mathbb{R}^{n+1}$ is a double covering of the real projective space $\mathbb{P}^n(\mathbb{R})$ via the map $x \mapsto [x]$, where $[x]$ is the one-dimensional subspace spanned by x in $\mathbb{R}^{n+1}$. As S^n is simply connected for $n \geq 2$ it follows $\pi_1(\mathbb{P}^n(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$ for $n \geq 2$ with Corollary 2.22.

2.3 Deck Transformations

We now consider automorphisms $D\colon X \xrightarrow{\sim} X$ of the covering space which are compatible with the covering map. We will see that they form a group which is closely linked to the fundamental groups of X and Y.

Definition 2.24 (Deck Transformation)**.** [Bre93, Definition 6.1, p. 147] Let $p\colon X \to Y$ be a covering map. A homeomorphism $D\colon X \xrightarrow{\sim} X$ is called a *deck transformation*, if $p \circ D = p$.

If D is a deck transformation, then $p \circ D^{-1} = p \circ D \circ D^{-1} = p$, hence D^{-1} is also a deck transformation. If D and D' are deck transformations, then $p \circ D \circ D' = p \circ D' = p$, so $D \circ D'$ is one as well. So for a fixed covering map, the deck transformations form a group $\Delta = \Delta(p)$ with respect to the composition.

Example 2.25. [Hat02, p. 70] We consider the examples from Example 2.2:

- For the projection $p\colon \mathbb{R} \to S^1,\ t \mapsto (\cos(2\pi t), \sin(2\pi t))$ of the helix onto the unit circle the deck transformations are the vertical translations taking the helix onto itself, so $\Delta \cong \mathbb{Z}$.
- For the n-sheeted covering space $p\colon S^1 \to S^1,\ z \mapsto z^n$, the deck transformations are the rotations of S^1 by angles that are multiples of $\frac{2\pi}{n}$, so $\Delta \cong \mathbb{Z}/n\mathbb{Z}$.

For the remainder of this section, fix a covering map $p\colon X \to Y$ and set $J = \pi_1(Y, y_0)$, $F = p^{-1}(y_0)$ as before. Note that by Lemma 2.8 with $f = p$ that if two deck transformations $D, D' \in \Delta$ agree in one point, i.e. $D(x) = D'(x)$ for some $x \in X$, they are equal. In particular $D(x) = x$ for some $x \in X$ implies $D = 1$. So we obtain an injective group homomorphism

$$\delta\colon \Delta \hookrightarrow \mathrm{Aut}(F),\ D \mapsto D|_F,$$

which yields a natural action of Δ on the fibre F.

Lemma 2.26. *[Bre93, Proposition 6.2, p. 148] If $D \in \Delta$, $\alpha \in J$ and $x \in F$, then $D(x) \cdot \alpha = D(x \cdot \alpha)$.*

Proof. Let f be a loop at y_0 representing α and lift f to a path g starting at x. Then $g(1) = x \cdot \alpha$. Because of $p \circ D \circ g = p \circ g = f$, the path $D \circ g$ is a lifting of f starting in Dx and ending in $(Dx) \cdot \alpha$ by definition of the action. At the same time, $D \circ g$ ends at the image of the end of g under D, which is $D(g(1)) = D(x \cdot \alpha)$. □

We now prove a central equivalence for deck transformations.

Theorem 2.27. *[Bre93, Theorem 6.3, p. 148] Let $x_0 \in X$ such that $p(x_0) = y_0$ and $x \in F$. Then the following statements are equivalent:*

(i) There exists a deck transformation $D \in \Delta$ such that $D(x_0) = x$.

(ii) There exists an α in the normalizer $N_J(p_\#(\pi_1(X, x_0)))$ such that $x = x_0 \cdot \alpha$.

(iii) $p_\#(\pi_1(X, x_0)) = p_\#(\pi_1(X, x))$.

Proof. By Theorem 2.7 a continuous map $D\colon (X, x_0) \to (X, x)$ such that $p \circ D = p$ exists if and only if $p_\#(\pi_1(X, x_0)) \subseteq p_\#(\pi_1(X, x))$. Similarly, a continuous map $D'\colon (X, x) \to (X, x_0)$ such that $p \circ D' = p$ exists if and only if $p_\#(\pi_1(X, x)) \subseteq p_\#(\pi_1(X, x_0))$. If both of these maps exist, then $p \circ D \circ D' = p$ and $D(D'(x)) = x$, so $D \circ D' = 1$ by Lemma 2.8, which proves the equivalence $(i) \Leftrightarrow (iii)$.
We now compute $J_{x_0 \cdot \alpha} = \{\beta \in J\colon (x_0 \cdot \alpha) \cdot \beta = (x_0 \cdot \alpha)\} = \{\beta \in J\colon x_0 \cdot \alpha\beta\alpha^{-1} = x_0\} = \{\beta \in J\colon \alpha\beta\alpha^{-1} \in J_{x_0}\} = \alpha^{-1} J_{x_0} \alpha$.
$(ii) \Rightarrow (iii)$: If $\alpha \in N_J(p_\#(\pi_1(X, x_0))) \overset{2.19}{=} N_J(J_{x_0})$ such that $x = x_0 \cdot \alpha$, then $p_\#(\pi_1(X, x)) \overset{2.19}{=} J_x = J_{x_0 \cdot \alpha} = \alpha^{-1} J_{x_0} \alpha = J_{x_0} \overset{2.19}{=} p_\#(\pi_1(X, x_0))$.
$(iii) \Rightarrow (ii)$: Suppose $J_{x_0} \overset{2.19}{=} p_\#(\pi_1(X, x_0)) = p_\#(\pi_1(X, x)) \overset{2.19}{=} J_x$. There exists an $\alpha \in J$ such that $x = x_0 \cdot \alpha$ as J is transitive on F by Lemma 2.18. Then $J_{x_0} = J_x = J_{x_0 \cdot \alpha} = \alpha^{-1} J_{x_0} \alpha$, so $\alpha \in N_J(J_{x_0}) \overset{2.19}{=} N_J(p_\#(\pi_1(X, x_0)))$. □

We note consequences of the previous theorem which will justify the definition of regular covering maps afterwards.

Corollary 2.28. *[Bre93, Corollary 6.4 and 6.5, p. 148f]*

(i) $p_\#(\pi_1(X, x_0)) \trianglelefteq J \Leftrightarrow \Delta$ is transitive on F.

(ii) For $x_0 \in F$ the conjugates of $p_\#(\pi_1(X, x_0))$ are the groups $p_\#(\pi_1(X, x))$ for $x \in F$.

Proof. (i) (⇐) If Δ is transitive on F for every $x \in F$ there exists $D \in \Delta$ such that $D(x_0) = x$, so $J_{x_0} = J_x$ for every $x \in F$ by Theorem 2.27. Let $\alpha \in J$ and $x \coloneqq x_0 \cdot \alpha$. By the last part of Theorem 2.27 we have $\alpha^{-1} J_{x_0} \alpha = J_{x_0 \cdot \alpha} = J_x = J_{x_0}$, so $\alpha \in N_J(p_\#(\pi_1(X, x_0)))$ and $p_\#(\pi_1(X, x_0)) \trianglelefteq J$.
(⇒) If $N_J(p_\#(\pi_1(X, x_0))) = J$ and $x \in F$, let $\alpha \in J$ such that $x = x_0 \cdot \alpha$. Then $\alpha \in N_J(p_\#(\pi_1(X, x_0)))$ and there exists $D \in \Delta$ such that $D(x_0) = x$ by Theorem 2.27, so Δ is transitive.

(ii) This is an immediate consequence of the transitivity of the monodromy action from Lemma 2.18 and the formula $p_\#(\pi_1(X, x_0 \cdot \alpha)) \overset{2.19}{=} J_{x_0 \cdot \alpha} \overset{2.27}{=} \alpha^{-1} J_{x_0} \alpha \overset{2.19}{=} \alpha^{-1}(p_\#(\pi_1(X, x_0)))\alpha$. □

Definition 2.29 (Regular Covering Map)**.** [Bre93, Definition 6.6, p. 149] A covering map $p\colon X \to Y$ is called *regular* if Δ is transitive on the fibre $p^{-1}(y_0)$, i.e. if $p_\#(\pi_1(X, x_0))$ is normal in $\pi_1(Y, y_0)$.

Remark 2.30. Using the criterion $p_\#(\pi_1(X, x_0)) \trianglelefteq \pi_1(Y, y_0)$ every universal cover is regular.

Example 2.31. For $\mathbb{C}^\times \coloneqq \mathbb{C} \setminus \{0\}$ the map $p\colon \mathbb{C}^\times \to \mathbb{C}^\times,\ z \mapsto z^n$ for $n \in \mathbb{Z}_{>0}$ defines a regular cover. Analogously to Example 2.25 the deck transformations are multiplications with n-th roots of unity, and we have $\Delta \cong \mathbb{Z}/n\mathbb{Z}$. A universal cover of $\mathbb{C}^\times$ is given by $\exp\colon \mathbb{C} \to \mathbb{C}^\times,\ z \mapsto \exp(z)$.

Definition 2.32. [Bre93, Definition 6.7, p. 149] Define $\Theta\colon N_J(J_{x_0}) \to \Delta$ by $\Theta(\alpha) = D_\alpha$, where D_α is the unique deck transformation with $D_\alpha(x_0) = x_0 \cdot \alpha$.

This map allows us to describe the deck transformation group only by the fundamental groups and their homomorphism induced by the covering map.

Theorem 2.33 (Classification of Deck Transformations)**.** *[Bre93, Theorem 6.8, p. 149] The map $\Theta\colon N_J(J_{x_0}) \to \Delta$ defined in Definition 2.32 is surjective with kernel J_{x_0}, in particular*

$$\Delta \cong N_J(p_\#(\pi_1(X, x_0)))/\pi_1(X, x_0).$$

Proof. Θ is a homomorphism: $D_\beta D_\alpha(x_0) = D_\beta(x_0 \cdot \alpha) \overset{2.26}{=} (D_\beta(x_0)) \cdot \alpha = (x_0 \cdot \beta)\alpha = x_0 \cdot (\beta\alpha) = D_{\beta\alpha}(x_0) \Rightarrow D_\beta D_\alpha = D_{\beta\alpha}$.
Θ is surjective: If $D \in \Delta$, then by Theorem 2.27 there is an $\alpha \in N_J(J_{x_0})$ such that $D(x_0) = x_0 \cdot \alpha = D_\alpha(x_0)$, so $D = D_\alpha$.
$\ker(\Theta)$: $D_\alpha = 1 \Leftrightarrow x_0 \cdot \alpha = D_\alpha(x_0) = x_0 \Leftrightarrow \alpha \in J_{x_0}$, so $\ker(\Theta) = J_{x_0}$. □

In the two situations below, the deck transformation group has a particularly nice structure.

Corollary 2.34. *[Bre93, Corollary 6.9 and 6.10, p. 149]*

(i) If the covering map $p\colon X \to Y$ is regular, then

$$\Delta \cong \pi_1(Y, y_0)/\pi_1(X, x_0).$$

(ii) If the covering space X is simply connected (i.e. $\pi_1(X, x_0) = 1$), then

$$\Delta \cong \pi_1(Y, y_0).$$

So in the case of X being simply connected the fundamental group $\pi_1(Y, y_0)$ acts on the fibre $F = p^{-1}(y_0)$

- by deck transformations via the isomorphism $\Theta\colon \pi_1(Y, y_0) \xrightarrow{\sim} \Delta,\ \alpha \mapsto D_\alpha$, and the map $x \mapsto D_\alpha(x)$ for $x \in F$, and
- by the monodromy action (Definition 2.16).

The following proposition answers in which cases the actions coincide:

Proposition 2.35. *[Bre93, Problem 1, p. 150] In the above situation $\pi_1(X, x_0) = 1$ the actions of $J = \pi_1(Y, y_0)$ on F coincide if and only if $\pi_1(Y, y_0)$ is abelian.*

Proof. ($\Rightarrow$) If the actions coincide we have $D_\delta(x) = x \cdot \delta$ for all $\delta \in J,\ x \in F$. Then for $\alpha, \beta \in J$ and every $x \in F$ follows $x \cdot (\alpha\beta) = D_\alpha(x) \cdot \beta \overset{2.26}{=} D_\alpha(x \cdot \beta) = (x \cdot \beta)\alpha = x \cdot (\beta\alpha) \Rightarrow \alpha\beta = \beta\alpha$, so J is abelian.
($\Leftarrow$) Assume J is abelian. Let $\alpha \in J$ and $x \in F$. By transitivity (Lemma 2.18) there exists $\beta \in J$ such that $x = x_0 \cdot \beta$. Then $D_\alpha(x) = D_\alpha(x_0 \cdot \beta) \overset{2.26}{=} D_\alpha(x_0) \cdot \beta = (x_0 \cdot \alpha)\beta = x_0 \cdot (\alpha\beta) \overset{\text{J abelian}}{=} x_0 \cdot (\beta\alpha) = (x_0 \cdot \beta)\alpha = x \cdot \alpha$, so the actions coincide. ☐

Remark 2.36. An equivalent condition for a fundamental group to be abelian and therefore for Proposition 2.35 is that the base-point-change-homomorphisms $h_\gamma\colon \pi_1(Y, y_1) \to \pi_1(Y, y_0)$ from Theorem 2.13 only depend on the endpoint, i.e. for all paths γ, γ' from y_0 to y_1 we have $h_\gamma = h_{\gamma'}$.

Therefore, from the perspective of deck transformations and the considered actions on the fibre a simply connected covering space appears desirable, the existence of which we will investigate in the next section.

2.4 Classification of Covering Spaces

Studying coverings a natural question is if there are properties of topological spaces which determine the covering spaces they admit. In this section we see that if the topological space admits a simply connected covering space it's fundamental group already contains all information about the possible covering spaces, up to equivalence (as defined in Definition 2.11). We will now consider a stronger form of equivalence by specifying base points.

Theorem 2.37. *[Bre93, Theorem 8.1, p. 154] Let Y be a path-connected and locally path-connected Hausdorff space and suppose Y admits a simply connected covering space $\tilde{Y}$. We then have the following one-to-one correspondences:*

Equivalence classes of covering spaces of Y with base points mapping to $y_0 \in Y$	$\overset{\sim}{\longleftrightarrow}$	*Subgroups of $\pi_1(Y, y_0)$*
Equivalence classes of covering spaces of Y without base points	$\overset{\sim}{\longleftrightarrow}$	*Conjugacy classes of subgroups of $\pi_1(Y, y_0)$*

The correspondence is given by $X \longleftrightarrow p_\#(\pi_1(X))$ where $p\colon X \to Y$ is the respective covering map.

Proof. The second correspondence follows from the first one and Corollary 2.28: if x ranges over F then $p_\#(\pi_1(X, x))$ ranges over the conjugates of $p_\#(\pi_1(X, x_0))$.
It remains to show that the function mapping a covering map with base point $p\colon (X, x_0) \to (Y, y_0)$ to the subgroup $p_\#(\pi_1(X, x_0)) \leq \pi_1(Y, y_0)$ is one-to-one: The function is one-one by Theorem 2.7.
It is also onto: suppose $H \leq \pi_1(Y, y_0)$ is an arbitrary subgroup. Since $\tilde{Y}$ is simply connected, Theorem 2.33 gives the isomorphism $\Theta\colon \pi_1(Y, y_0) \xrightarrow{\sim} \Delta,\ \alpha \mapsto D_\alpha$. Let $\Delta_H = \Theta(H) \leq \Delta$ and put $X := \tilde{Y}/\Delta_H$, the quotient space of $\tilde{Y}$ in which orbits of Δ_H are identified. X maps to Y as $\tilde{Y}$ does. Let x_0 be the image of the base point $\tilde{y}_0$ of $\tilde{Y}$. We now identify $p_\#(\pi_1(X, x_0))$. Let f be a loop in X at x_0. Lifting this to $\tilde{Y}$ at $\tilde{y}_0$ gives the same path as a lifting of the projection of f to a loop at y_0 in Y. So the lift ends at $D_\alpha(\tilde{y}_0)$, where $\alpha \in \pi_1(Y, y_0)$ is the homotopy class of the projection of f to Y. But for f to be a loop in $X = \tilde{Y}/\Delta_H$ we must have that $\tilde{y}_0$ and $D_\alpha(\tilde{y}_0)$ are in the same orbit of Δ_H, which is equivalent to $D_\alpha \in \Delta_H \Leftrightarrow \alpha \in H$. As α is arbitrary in $p_\#(\pi_1(X, x_0))$ by the free choice of f it follows $H = p_\#(\pi_1(X, x_0))$. □

We now answer the question when a path-connected and locally path-connected Hausdorff space X admits a simply connected covering space. A necessary condition can be found easily: if we have a loop f in an elementary subspace U of X, the lifting $\tilde{f}$ to the covering space is also a loop. If the covering space $\tilde{X}$ is simply connected, the loop must be homotopically trivial (i.e. $\tilde{f} \simeq c_{\tilde{x}}$ rel ∂I) in the covering space. Composing the homotopy with the covering map yields $f \simeq c_x$ rel ∂I, so the original loop is homotopically trivial in X. This leads to the following property:

Definition 2.38 (Semilocally simply connected). [Bre93, Definition 8.3, p. 155] A topological space X is called *semilocally simply connected* if each point $x \in X$ has a neighbourhood $U \subseteq X$ such that all loops in U are homotopically trivial in X, i.e. the homomorphism $\pi_1(U, u) \to \pi_1(X, u)$ is trivial for every $u \in U$.

This property is also sufficient, as the following theorem shows.

Theorem 2.39. *[Bre93, Theorem 8.4, p. 155] Let Y be a path-connected and locally path-connected Hausdorff space. Then Y has a simply connected covering space (i.e. a universal cover) if and only if Y is semilocally simply connected.*

Proof. We already saw the direction ($\Rightarrow$). We prove ($\Leftarrow$) by constructing a simply connected covering space $\tilde{Y}$. Choose a fixed base point $y_0 \in Y$, define

$$\tilde{Y} := \{[f] \text{ rel } \partial I \colon f \text{ is a path in } Y \text{ with } f(0) = y_0\}$$

and $p \colon \tilde{Y} \to Y,\ p([f]) = f(1)$. We will define a topology on $\tilde{Y}$ and show that p is the desired covering map. Set

$$\mathcal{B} := \{U \subseteq Y : U \text{ open, path-connected, semilocally simply connected}\},$$

which is a basis of the topology on Y as Y is locally path-connected. For $f(1) \in U \in \mathcal{B}$ let

$$U_{[f]} := \{[g] \in p^{-1}(U) \colon g \simeq f * \alpha \text{ rel } \partial I \text{ for some path } \alpha \text{ in } U\},$$

which is a subset of $\tilde{Y}$. For the remainder of the proof, all homotopies of paths starting at y_0 are rel ∂I unless otherwise indicated.
(1) $[g] \in U_{[f]} \Rightarrow U_{[g]} = U_{[f]}$.

Let $[h] \in U_{[g]}$. Then $h \simeq g * \beta$ for a path β in U. Since $g \simeq f * \alpha$ it follows $h \simeq (f * \alpha) \simeq \beta \simeq f * (\alpha * \beta)$, so $[h] \in U_{[f]}$ and $U_{[g]} \subseteq U_{[f]}$. Conversely, $g \simeq f * \alpha \Rightarrow g * \alpha^{-1} \simeq f * \alpha * \alpha^{-1} \simeq f$, so $[f] \in U_{[g]}$ and as above $U_{[f]} \subseteq U_{[g]}$.
(2) $p|_{U_{[f]}} \to U$ is a bijection.

The mapping is surjective since Y and U are path-connected. Now let $[g], [g'] \in U_{[f]}$, which is $U_{[g]}$ and $U_{[g']}$. Suppose $g(1) = g'(1)$. Since $[g'] \in U_{[g]}$ we have $g' \simeq g * \alpha$ for a loop α in U. Then α is homotopically trivial as U is semilocally simply connected in Y. Thus, $g' \simeq g * \alpha \simeq g * c_{g(1)} \simeq g$ and $[g'] = [g]$, which shows the injectivity.
(3) $U, V \in \mathcal{B},\ U \subseteq V,\ f(1) \in U \Rightarrow U_{[f]} \subseteq V_{[f]}$.

This follows directly from the definition.
(4) The sets $U_{[f]}$ for $U \in \mathcal{B},\ f(1) \in U$ form a basis for a topology on $\tilde{Y}$.

The sets $U_{[f]}$ cover $\tilde{Y}$. Let $[f] \in U_{[g]} \cap V_{[g']} = U_{[f]} \cap V_{[f]}$. Let $W \subseteq U \cap V$ with $W \in \mathcal{B}$ and $f(1) \in W$. Then $[f] \in W_{[f]} \subseteq U_{[f]} \cap V_{[f]}$.

So we can consider the topology on $\tilde{Y}$ generated by the sets $U_{[f]}$ for $U \in \mathcal{B},\ f(1) \in U$.
(5) p is open and continuous.

We have $p(U_{[f]}) = U$ by (2), and the sets $U_{[f]}$ form a basis, so p is open. Furthermore, $p^{-1}(U) = \bigcup\{U_{[f]} \colon [f] \in p^{-1}(U)\}$ which is open for $U \in \mathcal{B}$, so p is continuous.
(6) $p \colon U_{[f]} \xrightarrow{\sim} U$.

This follows from (2) and (6).
We have now shown that p satisfies all requirements of a covering map except for showing that $\tilde{Y}$ is path-connected. For that we need the next claim.
(7) Let $F \colon I \times I \to Y$ be a homotopy with $F(0,t) = y_0$. Put $f_t(s) = F(s,t)$ which is a path starting in y_0. Let $\tilde{f}(t) = [f_t] \in \tilde{Y}$. Then $\tilde{f}$ is a path in $\tilde{Y}$ covering the path $f_t(1) = F(1,t)$ in Y, i.e. $p \circ \tilde{f} = f_{.}(1)$.

We show that $\tilde{f}$ is continuous, the rest is clear. Let $t_0 \in I$. We show continuity at t_0. Let $U \in \mathcal{B}$ be a neighbourhood of $f_{t_0}(1)$. For t near t_0 we have $f_t(1) \in U$. Thus, $\tilde{f}(t) = [f_t] \in U_{[f_{t_0}]}$ for t near t_0 because the portion of $F(\cdot, t)$ for t in a small interval near t_0 is a homotopy rel $\{0\}$ between f_{t_0} and f_t with the right end of the homotopy describing a path α in U, i.e. $f_t \simeq f_{t_0} * \alpha$. Since $U_{[f_{t_0}]}$ maps homeomorphically to U it follows that $\tilde{f}(t)$ is continuous at t_0 because it maps to the continuous function $F(1,t)$ in U, for t near t_0.
(8) $\tilde{Y}$ is path-connected. (And hence p is a covering map.)

For $[f] \in \tilde{Y}$ put $F(s,t) = f(st)$. By (7) we obtain a path in $\tilde{Y}$ from $\tilde{y}_0 = [c_{y_0}]$ to the arbitrary point $[f] \in \tilde{Y}$.
(9) $\tilde{Y}$ is simply connected.

Let $\alpha \in \pi_1(Y, y_0)$ and let f be a loop in Y representing α. Let $F(s,t) = f(st)$ and let $f_t(s) = F(s,t)$. Then with (7) we have a path $\tilde{f}$ where $\tilde{f}(t) = [f_t]$. This path covers f since $p(\tilde{f}(t)) = p([f_t]) = f_t(1) = f(t)$. Now $\tilde{f}(0) = [f_0] = \tilde{y}_0$ and by definition $\tilde{y}_0 \cdot \alpha = \tilde{f}(1) = [f_1] = [f]$. If $\tilde{y}_0 \cdot \alpha = \tilde{y}_0$ then

$$1 = [c_{y_0}] = \tilde{y_0} = \tilde{y}_0 \cdot \alpha = [f] = \alpha,$$

and $\alpha = 1 \in \pi_1(Y, y_0)$. So all loops $\alpha \in \pi_1(Y, y_0)$ lifting to loops at the base point $\tilde{y}_0$ are trivial. By Lemma 2.19 it follows

$$\{1\} = \{\alpha \colon \tilde{y}_0 \cdot \alpha = \tilde{y}_0\} = J_{\tilde{y}_0} = p_\#(\pi_1(Y, y_0)).$$

As $p_\#$ is injective by Corollary 2.6, $\tilde{Y}$ is simply connected. □

In particular, we could replace the assumption of admitting a simply connected covering space (i.e. a universal cover) in Theorem 2.37 with being semilocally simply connected. The next example illustrates this property:

Example 2.40 (Hawaiian Earring)**.** [Hat02, Example 1.25, p. 49 and p. 63] We give an example of a path-connected and locally path-connected Hausdorff space which is not semilocally simply connected. For that, consider

$$\mathcal{H} := \bigcup_{n\in\mathbb{N}} C_n,$$

$$\text{where } C_n := \{(x,y)\in\mathbb{R}^2 \colon (x-1/n)^2+y^2 = 1/n^2\},$$

with the subspace topology induced by $\mathbb{R}^2$, which makes $\mathcal{H}$ a Hausdorff space. The C_n have the common point $(0,0)$, so as every C_n is path-connected, $\mathcal{H}$ is path-connected as well. Let $p \in \mathcal{H}\setminus\{(0,0)\}$ and U_p be a neighbourhood of p. Then p lies in a unique C_m, and we can find a sufficiently small ball around p which is contained in $U_p \cap C_m \setminus \{(0,0)\}$. Any neighbourhood U_0 of $(0,0)$ is itself path-connected, as any $p \in U_0$ lies in a C_n which is path-connected. This shows that $\mathcal{H}$ is locally path-connected.

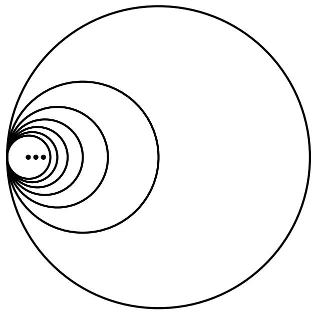


Figure 1: The Hawaiian earring $\mathcal{H}$.

For each n the fundamental group $\pi_1(C_n,(0,0))$ is infinitely cyclic by $C_n \cong S^1$, generated by the loop traversing C_n counterclockwise. As any nontrivial power of this loop is nontrivial in $\mathcal{H}$, the inclusion $C_n \hookrightarrow \mathcal{H}$ induces an inclusion of fundamental groups $\pi_1(C_n,(0,0)) \hookrightarrow \pi_1(\mathcal{H},(0,0))$. Every neighbourhood of $(0,0)$ contains all but finitely many C_n, but each C_n is not simply connected, so $\mathcal{H}$ is not semilocally simply connected. We illustrate this fact by showing that $\mathcal{H}$ has no universal cover with an argument from above:

Suppose $p\colon X \to \mathcal{H}$ is a universal cover of $\mathcal{H}$. Let $U \subseteq \mathcal{H}$ be the elementary open set containing $(0,0)$. As above, U contains a C_n for an n sufficiently large as U is a neighbourhood of $(0,0)$. Let f be the loop traversing C_n. Lifting f along p gives a homotopically trivial loop $\tilde{f}$ as X is simply connected, i.e. $\tilde{f} \simeq c_{\tilde{x}}$ rel ∂I with $\tilde{x} \in p^{-1}(0,0)$ via the homotopy $\tilde{F}\colon I\times I \to X$ satisfying $\tilde{F}(t,0) = \tilde{f}(t)$, $\tilde{F}(t,1) = \tilde{x}$ and $\tilde{F}(0,t) = \tilde{x} = \tilde{F}(1,t)$ for all $t \in I$. Then $F := p\circ\tilde{F}\colon I\times I \to \mathcal{H}$ is a homotopy satisfying $F(t,0) = p(\tilde{f}(t)) = f(t)$, $F(t,1) = p(\tilde{x}) = (0,0)$ and $F(0,t) = (0,0) = F(1,t)$ for all $t\in I$, so $f \simeq c_{(0,0)}$ rel ∂I, which implies that f is homotopically trivial in $\mathcal{H}$ and with the injectivity of $\pi_1(C_n,(0,0)) \hookrightarrow \pi_1(\mathcal{H},(0,0))$ it is trivial in C_n as well. As f generates $\pi_1(C_n,(0,0))$ it follows that C_n is simply connected, a contradiction. Thus, $\mathcal{H}$ has no universal cover.

Example 2.41 (Locally simply connected)**.** [Hat02, Example 1.25, p. 49 and p. 63] The property *semilocally simply connected* can not be replaced with *locally simply connected*, which is for every point p and every neighbourhood U of p there exists a smaller simply connected neighbourhood $V \subseteq U$ of p.
For that again consider the Hawaiian earring from Example 2.40 and the image $\mathcal{C}$ of the map

$$c\colon \mathcal{H} \times I \to \mathbb{R}^3, \quad c(x, y, t) = ((1-t)x,\, (1-t)y,\, t),$$

equipped with the subspace topology of $\mathbb{R}^3$.

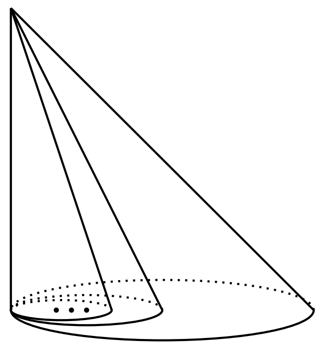


Figure 2: A cone $\mathcal{C}$ over the Hawaiian earring $\mathcal{H}$.

As every loop in $\mathcal{C}$ can be contracted to the tip of the cone $(0, 0, 1)$, the cone $\mathcal{C}$ is simply connected and hence semilocally simply connected, in particular $\mathcal{C}$ admits a universal cover. But every neighbourhood U of the point $(0, 0, 0)$ deformation retracts onto a neighbourhood of $(0, 0)$ in $\mathcal{H}$ which contains a circle C_n, making U not simply connected. Thus, $\mathcal{C}$ is not locally simply connected.

3 Galois Groups and Monodromy Groups

Having discussed purely topological definitions and statements we now change over to the geometric context of affine varieties, where we will apply the topological groundwork from Section 2.

3.1 Covering Maps of Affine Varieties

In Theorem 3.5 we will show that a dominant morphism of complex affine varieties of the same dimension restricts to a covering map. This will then be the foundation for considering the monodromy action on the fibre of the morphism in Construction 3.12. We will use affine varieties defined over the complex numbers $\mathbb{C}$ since the covering map will use the analytic topology of $\mathbb{C}^n$.

Proposition 3.1. *[Har92, Proposition 14.4, p. 176] Let $f\colon X \to Y$ be a surjective morphism of affine varieties defined over a field k with $\operatorname{char}(k) = 0$. Then there exists a nonempty open subset $U \subseteq Y$ such that for any smooth point $p \in f^{-1}(U) \cap X_{sm}$ the differential df_p is surjective.*

Theorem 3.2 (Inverse function theorem for manifolds)**.** *[Jän01, Inverse Function Theorem, p. 6] Let M and N be smooth manifolds. If $f\colon M \to N$ is smooth and the derivative $df_p\colon T_pM \to T_{f(p)}M$ for $p \in M$ is invertible, then there exists an open neighbourhood $U \subseteq M$ of p such that the restriction $f|_U\colon U \to f(U)$ is a diffeomorphism.*

Lemma 3.3. *[Har92, Exercise 14.1, p. 175] Let $X \subseteq \mathbb{A}^n(\mathbb{C})$ be a complex affine variety and $p \in X$. Then X is smooth at p if and only if X is a complex submanifold of $\mathbb{C}^n$ in p.*

Proof. We prove the direction ($\Rightarrow$) which we will use later.
Let $X = V(f_1, \dots, f_m) \subseteq \mathbb{A}^n(\mathbb{C})$ with $f_i \in \mathbb{C}[x_1, \dots, x_n]$. The tangent space of X at p is given by

$$T_pX = \bigcap_{i=1}^{m} V(J_{\mathbb{C}}(f_i)(p) \cdot (x-p)) \subseteq \mathbb{A}^n(\mathbb{C}),$$

where $J_{\mathbb{C}}(f_i)(p)$ denotes the complex Jacobian of f_i evaluated at the point p. Define $f = (f_1, \dots, f_m)\colon \mathbb{C}^n \to \mathbb{C}^m$. As T_pX is the intersection of m hyperplanes it is the solution of a system of m linear equations, and it follows $\dim_{\mathbb{C}}(T_pX) = n - \operatorname{rk}(J_{\mathbb{C}}(f))$. By assumption $\dim(X) = \dim_{\mathbb{C}}(T_pX)$, so we have $r \coloneqq \operatorname{codim}(X) = \operatorname{rk}(J_{\mathbb{C}}(f)) \leq n, m$. So we can choose the f_j such that the matrix

$$\left(\frac{\partial f_j}{\partial x_i}(p)\right)_{i,j=1,\dots,r}$$

is invertible. Using the complex implicit function theorem [FG02, 7.6 Implicit function theorem, p. 34] there exist open sets $U \subseteq \mathbb{C}^{n-r}$ and $V \subseteq \mathbb{C}^r$ such that $p \in V \times U$ and a holomorphic map $h\colon U \to V$ with

$$\{(h(y), y)\colon y \in U\} = \{(x, y) \in V \times U\colon f(x, y) = 0\} = \{(x, y) \in (V \times U) \cap X\}.$$

Hence, the graph $y \mapsto (h(y), y)$ is a chart of X around p and the open neighbourhood $(V \times U) \cap X$ is a complex manifold of dimension $n - r = \dim(X)$. □

Lemma 3.4. *Let $X \subseteq \mathbb{A}^n(\mathbb{C})$ be a complex affine variety and let $U \subseteq X$ be a Zariski-open nonempty subset. Then X and U are path-connected in the analytic topology of $\mathbb{C}^n$. Furthermore, if X is smooth, then X and U are locally path-connected.*

Proof. X is irreducible, hence connected in the Zariski topology. By [Sha88, Theorem 1, p. 126] X is connected in the analytic topology. By [BCR98, Theorem 2.4.5, p. 35] X is semi-algebraically connected by identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. Then with [BCR98, Proposition 2.5.13, p. 42] X is semi-algebraically path-connected and especially path-connected in the analytic topology.
For $U \subseteq X$ Zariski-open there exists a closed subset $Y \subseteq X$ with $U = X \setminus Y$. As Y is the common zero of finitely many $f_i \in \Gamma(X)$, the coordinate ring of X, the set U is the Union of the essentially open sets $D(f_i)$. As each of them is affine they are euclidean-homeomorphic to an affine variety and hence path-connected in the analytic topology. Because X is irreducible, each of the intersections $D(f_i) \cap D(f_j)$, $i \neq j$, is nonempty, so their union U is also path-connected in the analytic topology. By Lemma 3.3 X is a complex submanifold of $\mathbb{C}^n$ in every point and thus locally path-connected. As U is an open subspace of the locally path-connected space X, it is locally path-connected as well. □

We now did all preparatory work necessary for showing that surjective morphisms restrict to covering maps if we require smoothness, equal dimension and same cardinality of the fibre.

Theorem 3.5. *Let $X \subseteq \mathbb{A}^n(\mathbb{C})$ and $Y \subseteq \mathbb{A}^m(\mathbb{C})$ be complex affine varieties. Let X and Y be smooth, i.e. $X = X_{\text{sm}}$, $Y = Y_{\text{sm}}$, and of the same dimension, i.e. $\dim(T_pX) = \dim(X) = \dim(Y) = \dim(T_{p'}Y)$ for all $p \in X$, $p' \in Y$. Let $f\colon X \to Y$ be a surjective morphism with finite fibre, i.e. $|f^{-1}(p')| < \infty$ for all $p' \in Y$. Then there exists a nonempty (Zariski-)open subset $U \subseteq Y$ such that for $V := f^{-1}(U)$ the restriction $f|_V\colon V \to U$ is a covering map of topological spaces with respect to the induced analytic topology.*

Proof. By Proposition 3.1 there exists a nonempty Zariski-open subset $U \subseteq Y$ such that for all $p \in f^{-1}(U)$ the differential df_p is surjective. So for a $p \in V := f^{-1}(U)$ the map $df_p\colon T_pX \to T_{f(p)}Y$ is surjective and as a linear map of vector spaces of the same (finite) dimension bijective, so it is especially invertible.
The following construction uses the induced analytic topology. For $p' \in U \subseteq Y$ consider the preimage $\tilde{V}_{p'} := f^{-1}(p')$. For every $p \in \tilde{V}_{p'}$ using Lemma 3.3 f is a smooth map of complex submanifolds in p and p' with invertible differential df_p. So by Theorem 3.2 we obtain a neighbourhood $\tilde{V}_{p'}^p$ of p such that $f|_{\tilde{V}_{p'}^p}\colon \tilde{V}_{p'}^p \to f(\tilde{V}_{p'}^p)$ is a diffeomorphism. Now define $W_{p'} := U \cap (\bigcap_{p \in \tilde{V}_{p'}} f(\tilde{V}_{p'}^p))$, where the intersection of the $f(\tilde{V}_{p'}^p)$ is an intersection of finitely many open sets and thus open. So $W_{p'}$ is a neighbourhood of p' in U. Shrink $\tilde{V}_{p'}^p$ to a neighbourhood $V_{p'}^p$ of p such that $f(V_{p'}^p) = W_{p'}$, so that $f|V_{p'}^p\colon V_{p'}^p \to W_{p'}$ is homeomorphic. By shrinking $V_{p'}^p$ and $W_{p'}$ again it can be ensured that $W_{p'}$ is path-connected and the $V_{p'}^p$ are disjoint, the

latter is possible since $\mathbb{C}^n$ is Hausdorff.
As V and U are open subsets of X and Y they are Hausdorff and by Lemma 3.4 path-connected and locally path-connected with the induced analytic topology of $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively. So now for every $p' \in U$ there exists a path-connected neighbourhood $W_{p'} \subseteq U$ of p', such that $f^{-1}(W_{p'})$ is the nonempty disjoint union of open sets $V^p_{p'} \subseteq X$, on which $f|_{V^p_{p'}}$ is a homeomorphism $V^p_{p'} \xrightarrow{\sim} W_{p'}$. Hence, $f|_V \colon V \to U$ is a covering map by Definition 2.1. □

Remark 3.6.
- The above result also holds if we only require $f\colon X \to Y$ to be a dominant morphism, i.e. a morphism with dense image in Y. Then using [GW20, Theorem 10.19, p. 251] the image $f(X)$ already contains a dense Zariski-open subset.
- Using [Har77, Theorem 5.3, p. 33] the set of smooth points X_{sm} of a variety X is a Zariski-open subset and the assumption $X = X_{\text{sm}}$, $Y = Y_{\text{sm}}$ can be dropped.
- The assumption $|f^{-1}(p')| < \infty$ for all $p' \in Y$ can be relaxed to $p' \in U$ for a Zariski-open set $U \subseteq Y$. We will later require that $|f^{-1}(p')| = d \in \mathbb{Z}_{\geq 1}$ holds for all $p' \in U$. Such morphisms will be called *generically finite*, see Section 3.2, Definition 3.8.

3.2 Extensions of Function Fields

Below we show that for a dominant rational map between affine varieties which is generically finite (in the sense of Definition 3.8), we obtain a finite extension of corresponding function fields.
This holds for affine varieties over arbitrary algebraically closed fields k, but the complex case will be the interesting one for the following Section 3.3.

Definition 3.7 (Rational Map)**.** [Har92, Definition 7.3, p. 74 and p. 74f, 77] Let $X \subseteq \mathbb{A}^n(k)$ and $Y \subseteq \mathbb{A}^m(k)$ be affine varieties. A *rational map* $f\colon X \dashrightarrow Y$ is an equivalence class of pairs (U, g_U) with $U \subseteq X$ a nonempty open subset and g_U a regular map $U \to Y$, where two pairs (U, g_U) and (V, g_V) are said to be equivalent if $g_U|_{U\cap V} = g_V|_{U\cap V}$. This relation is clearly reflexive and symmetric. Transitivity is given as follows: if $g_U|_{U\cap V} = g_V|_{U\cap V}$ and $g_V|_{V\cap W} = g_W|_{V\cap W}$ then $g_U|_{U\cap V\cap W} = g_W|_{U\cap V\cap W}$ and with [Har77, Lemma 4.1, p. 24] it follows $g_U|_{U\cap W} = g_W|_{U\cap W}$ as $U \cap V \cap W$ is an open subvariety of $U \cap W$. If (U, g_U) represents f we will also write $f|_U$ for g_U.
The *graph* Γ_f of a rational map f is the closure of the graph of $f|_U$ in $X \times Y$, where f is defined on $U \subseteq X$. This is well-defined, since for another open set $V \subseteq X$ on

which f is defined we have $\Gamma_{f|_U} = \overline{\Gamma_{f|_{U\cap V}}}$ and $\Gamma_{f|_V} = \overline{\Gamma_{f|_{U\cap V}}}$, where the closures are taken in $U \times Y$ and $V \times Y$ respectively. Taking the closure in $X \times Y$ we have $\overline{\Gamma_{f|_U}} = \overline{\Gamma_{f|_{U\cap V}}} = \overline{\Gamma_{f|_V}}$. The *image* of f is the projection $\pi_2(\Gamma_f)$ of Γ_f onto $\mathbb{A}^m(k)$. f is called *dominant*, if its image is all of Y, or equivalently, if for a pair (U, g_U) (and hence for every pair) the image of g_U is dense in Y.
Let $f\colon X \dashrightarrow Y$ and $g\colon Y \dashrightarrow Z$ be rational maps. If there are pairs (U, α_U) representing f and (V, β_V) representing g such that $\alpha_U^{-1}(V) \neq \emptyset$ we define the *composition* $g \circ f$ as the rational map represented by $(\alpha_U^{-1}(V), \beta_V \circ \alpha_U|_{\alpha_U^{-1}(V)})$. A rational map $f\colon X \dashrightarrow Y$ is *birational* if there exists a rational map $g\colon Y \dashrightarrow X$ such that $f \circ g$ and $g \circ f$ are defined and equivalent to the identity. X and Y are called *birational* if there exists a birational map $f\colon X \dashrightarrow Y$.

Definition 3.8 (Generic Fibre)**.** Let $f\colon X \dashrightarrow Y$ be a dominant rational map of affine varieties X and Y. We say the *generic fibre of f has d points* if there exists a non-empty Zariski-open subset $U \subseteq Y$ such that the fibre $f^{-1}(p)$ consists of exactly d points for all $p \in U$. In this case the map f is called *generically finite.*

Lemma 3.9. *[Har77, p. 25] Let $f\colon X \dashrightarrow Y$ be a dominant rational map of affine varieties X and Y defined over k. Then f induces an inclusion $f^*\colon K(Y) \hookrightarrow K(X)$ of the function fields of Y and X.*

Proof. Let f be represented by (U, f_U) and let $g \in K(Y)$ be a rational function identified with the rational map represented by (V, g_V), where $\emptyset \neq V \subseteq Y$ is open and $g_V \in \mathcal{O}_Y(V)$. Since $f_U(U) \subseteq Y$ is dense, $f_U^{-1}(V)$ is a nonempty open subset of X, so $g_V \circ f_U$ is regular on $f_U^{-1}(V)$ and gives a rational function on X. This defines a ring homomorphism of function fields $f^*\colon K(Y) \hookrightarrow K(X)$. □

The field extension $K(X)/f^*(K(Y))$ will also be abbreviated as $K(X)/K(Y)$, but it will be clear from the context that the extension is given by f, or more specifically, by the inclusion f^*.
A first property of the inclusion $K(Y) \hookrightarrow K(X)$ could be interpreted as an algebro-geometric correspondence between the (finite) degree of the field extension and the rational map:

Theorem 3.10. *[Har92, Proposition 7.16, p. 80] Let $f\colon X \dashrightarrow Y$ be a dominant rational map of affine varieties X and Y defined over k and $f^*\colon K(Y) \hookrightarrow K(X)$ the induced inclusion of function fields. Then f is generically finite if and only if the extension $K(Y)/K(X)$ given by f^* is finite. In this case, if* $\mathrm{char}(K) = 0$ *and the extension has degree $d \in \mathbb{Z}_{\geq 1}$, then the generic fibre of f has exactly d points.*

Proof. With the projection $\pi_1\colon \Gamma_f \to X$ it follows that X is birational to the graph Γ_f, so we may replace f by the projection π_2 of Γ_f onto Y. So we can assume that

X is closed in $\mathbb{A}^n(k)$ and f is a linear projection $\mathbb{A}^n(k) \to \mathbb{A}^m(k)$. By induction on $m-n$, we may further assume that $m = n-1$, hence we reduced to the situation $f\colon X \to Y$, $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1})$. In this case the function field $K(X)$ is generated by x_n over $K(Y)$. Now there are two cases to consider:

- x_n is algebraic over $K(Y)$: Then let

$$G(x_1, \ldots, x_n) = a_d x_n^d + a_{d-1} x_n^{d-1} + \ldots + a_0$$

be the minimal polynomial of x_n over $K(Y)$, where the $a_i \in K(Y)$ are rational functions. After clearing denominators we may assume that $a_0, \ldots, a_d$ are regular functions on Y given by residue classes of polynomials in $x_1, \ldots, x_{n-1}$, i.e. $a_i \in \Gamma(Y)$, the coordinate ring of Y. Let $D(x_1, \ldots, x_{n-1})$ be the discriminant of G as a polynomial in x_n. As a minimal polynomial G is irreducible in $K(Y)[x_n]$ and especially square-free, so since $\operatorname{char}(k) = 0$, G can not vanish identically on Y. So $\{(x_1, \ldots, x_{n-1}) \in Y\colon\ D(x_1, \ldots, x_{n-1}) = 0\}$ is a proper subvariety of Y, and since G has degree d the set $\{(x_1, \ldots, x_{n-1}) \in Y\colon\ a_d(x_1, \ldots, x_{n-1}) = 0\}$ is a proper subvariety as well. On the complement of their union (which is non-empty and open), the fibres of f consist of exactly d points, which are the d roots of G.

- x_n is transcendental over $K(Y)$: For any polynomial $G(x_1, \ldots, x_{n-1}) \in I(X)$, which we may write in the form

$$G(x_1, \ldots, x_n) = a_d x_n^d + a_{d-1} x_n^{d-1} + \ldots + a_0$$

with $a_i \in K[x_1, \ldots, x_{n-1}]$, the coefficients a_i must vanish identically on Y, otherwise x_n would not be transcendental. So X contains the entire fibre of the projection f over any point $p \in Y$ and f is not generically finite. □

3.3 Group Constructions and the Main Theorem

We now connect the previous sections Section 3.1 and Section 3.2 by defining both a monodromy group and a Galois group for a dominant morphism of finite degree between complex varieties in Construction 3.12. The foundations we have already laid will then enable us to show in Theorem 3.13 that the two mentioned groups are equal.

Lemma 3.11. *Let $X \subseteq \mathbb{A}^n(k)$ be an affine variety. Then for distinct $p, q \in X$ there exists a rational function $g \in K(X)$ with distinct values on q and p.*

Proof. As $q = (q_1, \ldots, q_n)$ and $p = (p_1, \ldots, p_n)$ are distinct, they differ in at least one coordinate, so there exists an $i \in \{1, \ldots, n\}$ with $q_i \neq p_i$. So the polynomial x_i in the coordinate ring $\Gamma(X)$ has different image on p and q, and so has $x_i \in K(X)$. □

Construction 3.12. [Har79, p. 688f] Let $X \subseteq \mathbb{A}^n(\mathbb{C})$ and $Y \subseteq \mathbb{A}^m(\mathbb{C})$ be complex affine varieties of the same dimension and let $f\colon X \to Y$ be a dominant morphism of degree $d \in \mathbb{Z}_{\geq 1}$. Let $p \in Y$ be a general point of Y and $F = f^{-1}(p) = \{q_1, \dots, q_d\}$ the fibre of f over Y. We may now define two subgroups of the permutation group on d elements S_d of F.

1. As $f\colon X \to Y$ is a dominant morphism, it especially is a dominant rational map. With Lemma 3.9 we have an inclusion of function fields $f^*\colon K(Y) \hookrightarrow K(X)$ induced by f. As $\operatorname{char}(K(Y)) = 0$ and by Theorem 3.10 the extension $K(X)/K(Y)$ is a finite separable field extension of degree d. The primitive element theorem [Bos09, Satz 12, p. 119] implies that $K(X) = (K(Y))(\xi)$ for a $\xi \in K(X)$ satisfying a polynomial of degree d, i.e.

$$P(f) = \xi^d + g_{d-1}\xi^{d-1} + \ldots + g_0 = 0,$$

where $g_1, \dots, g_d \in K(Y)$. Now let $\mathcal{M}$ be the field of germs of meromorphic functions around p, which is defined in the following way: a meromorphic germ around p is an equivalence class $[(U, s)]$, where $p \in U \subseteq \mathbb{C}^m$ is open and s is a meromorphic function on U, i.e. s is a function $U \setminus A \to \mathbb{C}$, where $A \subseteq U$ is an analytic hypersurface, such that for every point $y \in U$ there exist holomorphic functions s_1 and s_2 on an open neighbourhood $y \in U(y) \subseteq U$ such that the vanishing set of s_2 is contained in $U(y) \cap A$ and $s = \frac{s_1}{s_2}$ on $U(y) \setminus A$. A subset $A \subseteq U$ is an analytic hypersurface, if for each point $y \in U$ there is a connected open neighbourhood $y \in U(y) \subseteq U$ and a holomorphic function $h\colon U(y) \to \mathbb{C}$ such that $U(y) \cap A$ is the vanishing set of h. Two such pairs (U, s) and (U', s') are said to be equivalent if there exists a smaller open set $p \in V \subseteq U \cap U'$ such that $s|_V = s'|_V$. Define $\mathcal{M}$ as the set of these germs. Then the field structure on $\mathcal{M}$ is given by $[(U, s)] + [(U', s')] := [(V,\, s|_V + s'|_V)]$ and $[(U, s)] \cdot [(U', s')] := [(V,\, s|_V \cdot s'|_V)]$, where $p \in V \subseteq U \cap U'$ is open. The neutral elements are given by the equivalence classes of the zero- and the constant one-mapping. Analogously let $\mathcal{M}_\alpha$ be the field of germs of meromorphic functions around q_α for $\alpha \in \{1, \dots, d\}$.
With Proposition 3.1 we can assume without restriction that for every q_α the differential df_{q_α} is surjective, as passing over to an open subset does not change the function fields. So as a linear map of vector spaces of the same dimension df_{q_α} is invertible. We can assume f to be holomorphic on a neighbourhood of q_α which is small enough. The Inverse mapping theorem [FG02, 7.5 Inverse mapping theorem, p. 33] for holomorphic functions then provides (smaller) neighbourhoods N_{q_α} of q_α and N_p of p such that $\tilde{f}_\alpha := f|_{N_{q_\alpha}}\colon N_{q_\alpha} \to N_p$ is biholomorphic. Then composing $\tilde{f}_\alpha^{-1}$ with a germ in $\mathcal{M}_\alpha$ yields a germ

in $\mathcal{M}$, composing $\tilde{f}_\alpha$ with this germ again gives the original germ in $\mathcal{M}_\alpha$. As germs of meromorphic functions are used this does not depend on the concrete neighbourhoods on which f is biholomorphic. So we have an isomorphism

$$f_\alpha\colon \mathcal{M}_\alpha \xrightarrow{\sim} \mathcal{M}$$

induced by f restricted to a neighbourhood of q_α for every α. Let

$$\phi\colon K(Y) \hookrightarrow \mathcal{M}$$

be the inclusion obtained by restricting rational functions to a neighbourhood of p and define the field $K := \phi(K(Y))$. Let

$$\phi_\alpha\colon K(X) \hookrightarrow \mathcal{M}_\alpha \xrightarrow{f_\alpha} \mathcal{M}$$

be the inclusion obtained by restricting rational functions to a neighbourhood of q_α composed with f_α, define L as the subfield of $\mathcal{M}$ generated by the subfields $K_\alpha := \phi_\alpha(K(X))$.
Let $\tilde{g}_i := \phi(g_i)$, $\tilde{\xi}_\alpha := \phi_\alpha(\xi)$. Now each element $\tilde{\xi}_\alpha$ satisfies the polynomial

$$\tilde{P}(\tilde{\xi}_\alpha) = \tilde{\xi}_\alpha^d + \tilde{g}_{d-1}\tilde{\xi}_\alpha^{d-1} + \ldots + \tilde{g}_0 = 0.$$

The $\tilde{\xi}_\alpha$ are distinct: because ξ generates $K(X)$ over $K(Y)$, every element in $K(X)$ can be expressed as a linear combination

$$h_m\xi^m + h_{m-1}\xi^{m-1} + \ldots + h_0$$

with $h_i \in K(Y)$. For distinct $\alpha, \beta \in \{1, \ldots, d\}$ the evaluation of h_i in q_α and q_β is given as the evaluation in p by the inclusion $f^*\colon K(Y) \hookrightarrow K(X)$. So ξ has different evaluation in q_α and q_β, because otherwise every element in $K(X)$ the evaluation in q_α and q_β would be equal, which contradicts Lemma 3.11. So ξ must have distinct values in all the points q_α, the images $\tilde{\xi}_\alpha$ are distinct and the $\tilde{\xi}_\alpha$ have to be all the roots of $\tilde{P}$. The field $L \subseteq \mathcal{M}$ is then the normal closure of the extension $K(X)/K(Y) = K_\alpha/K$. So the extension L/K is normal and separable, hence a Galois extension. The Galois group $G := \mathrm{Gal}(L/K)$ then acts on the roots $\tilde{\xi}_\alpha$ of $\tilde{P}$, which yields an inclusion $G \hookrightarrow S_d$.

2. By Theorem 3.5 and Remark 3.6 there is a nonempty (Zariski-)open subset $U \subseteq Y$ such that for $V := f^{-1}(U)$ the restriction $f|_V\colon V \to U$ is a covering map of topological spaces with respect to the analytic topology. By considering $p \in U$ as a base point we obtain the monodromy action $f^{-1}(p) = F \curvearrowleft \pi_1(U, p)$ (Definition 2.16). Let $M \leq S_d$ be the monodromy group of $f|_V\colon V \to U$, i.e. the image of the induced homomorphism $\pi_1(U, p) \to S_d$. In this specific

context the monodromy group M is obtained in the following way: for a loop $\gamma\colon I \to U$ with base point p there is a unique lifting $\tilde{\gamma}_\alpha$ to a path in V with $\tilde{\gamma}_\alpha(0) = q_\alpha \in f^{-1}(p)$ for every $\alpha \in \{1, \dots, d\}$. This defines an automorphism of F by considering $\tilde{\gamma}_\alpha(1)$ as the image of q_α under γ. The homomorphism above then sends each q_α to the endpoint of the lifted path $\tilde{\gamma}_\alpha$ and depends only on the homotopy class of γ.

We now prove the announced main theorem of Section 3.

Theorem 3.13. *[Har79, Proposition, p. 689] For the morphism $f\colon X \to Y$ from Construction 3.12, the monodromy group M equals the Galois group $G = \mathrm{Gal}(L/K)$.*

Proof. $M \leq G$: Let γ be a loop in U with base point p, $\tilde{\gamma}_\alpha$ the lifting of γ to V with $\tilde{\gamma}_\alpha(0) = q_\alpha$, and $\tau \in S_d$ the induced permutation by γ on $F = f^{-1}(p)$, i.e. $\tilde{\gamma}_\alpha(1) = q_{\tau(\alpha)}$. We construct an automorphism $\sigma \in \mathrm{Gal}(L/K) = G$ and show that τ is induced by σ. This is done by analytic continuation of a germ $h \in L \subseteq \mathcal{M}$ along the path γ.
For any germ $h_\alpha \in K_\alpha \subseteq L$ of a meromorphic function on X at a point $q_\alpha \in F$ there is an analytic continuation along the lifting $\tilde{\gamma}_\alpha$ from q_α to $q_{\tau(\alpha)}$. In the one-dimensional case we have a meromorphic function in one variable which has at most countably many isolated poles. In this case $\tilde{\gamma}_\alpha$ can be altered such that it does not cross poles, but the homotopy class of the corresponding loop γ is not changed. Set $\sigma(h_\alpha)$ as the germ in L obtained by this continuation. Since L is generated by the $K_\alpha = \phi_\alpha(K(X))$, every $h \in L$ is a polynomial in such germs, and we have defined $\sigma(h)$ by defining σ on the fields K_α. Then this analytic continuation along γ defines an automorphism of the field L which fixes K, since by definition of σ continuation of a germ in K is just the continuation along the loop γ with base point p, which yields the same germ. By construction σ sends a root $\tilde{\xi}_\alpha$ of $\tilde{P}$ to the root $\tilde{\xi}_{\tau(\alpha)}$, so the permutation $\tau \in S_d$ is induced by σ and in the Galois group $G = \mathrm{Gal}(L/K)$.
It now suffices to show that any automorphism $\sigma \in G$ of L fixing K is obtained by analytic continuation along some path γ in U. For that we show that the subfield of L fixed under the subgroup $M \leq G$ is K, i.e. that any function element $h \in L$ fixed under analytic continuation along every loop γ in U with base point p is in fact the germ of a meromorphic function on Y. So let $h \in L$ be such an element and define a meromorphic function $\tilde{h}$ on U by choosing for every $r \in U$ a path $\delta\colon I \to U$ from p to r and letting the germ of $\tilde{h}$ at r be the analytic continuation of h along δ. Choosing a different path $\delta'\colon I \to U$ from p to r must yield the same germ since the continuation of h along the loop $\delta^{-1} * \delta'$ is again h by assumption. We can write $h = Q(h_1, \dots, h_d)$ where h_α is the germ in $\mathcal{M} \cong \mathcal{M}_\alpha$ of a meromorphic function $\tilde{h}_\alpha$ on X and Q is a polynomial. So we see that $\tilde{h}$ as a linear combination of germs

of meromorphic functions can not have essential singularities and thus extends to a meromorphic function on Y with germ h at p. $\square$

In the following we discuss two examples for Construction 3.12. Both of them construct the Galois extension as well as the monodromy action explicitly. The first one (Example 3.14) considers a cubic map from $\mathbb{A}^1(\mathbb{C})$ to $\mathbb{A}^1(\mathbb{C})$, the second one (Example 3.15) a projection from $\mathbb{A}^2(\mathbb{C})$ to $\mathbb{A}^1(\mathbb{C})$.

Example 3.14. Consider the surjective morphism

$$f\colon X := \mathbb{A}^1(\mathbb{C}) \to \mathbb{A}^1(\mathbb{C}) =: Y,\ z \mapsto z^3.$$

The morphism f has degree $d = 3$: let ζ_3 be a primitive third root of unity and $c \in \mathbb{C} \setminus \{0\}$. Taking $d \in \mathbb{C}$ such that $d^3 = c$, there are exactly three solutions d, $\zeta_3 d$ and $\zeta_3^2 d$ for the equation $z^3 = c$. Define $U := \mathbb{A}^1(\mathbb{C}) \setminus \{0\}$, which is the open set on which the fibre of f has cardinality $d = 3$, and set $V := f^{-1}(U) = \mathbb{A}^1(\mathbb{C}) \setminus \{0\}$. As df_x is surjective for every $x \in V$ the map $f|_V\colon V \to U$ is a covering by Theorem 3.5. Let $p := 1 \in U$, so $F := f^{-1}(p) = \{q_1, q_2, q_3\}$ with $q_1 := 1,\ q_2 := \exp\left(\frac{2}{3}\pi i\right),\ q_3 := \exp\left(\frac{4}{3}\pi i\right)$, the three complex roots of unity.

We now follow the construction of the Galois extension from Construction 3.12: the field extension $K(X)/f^*(K(Y))$ given by $f^*\colon K(Y) \hookrightarrow K(X)$, $\frac{g(z)}{h(z)} \mapsto \frac{g(z^3)}{h(z^3)}$ is generated by $z \in K(X) = \mathbb{C}(z)$ over $f^*(K(Y))$, the minimal polynomial being

$$P := x^3 - z^3 \in (f^*(K(Y)))[x].$$

Let $\mathcal{M}$ be the field of germs of meromorphic functions around p and $\mathcal{M}_\alpha$ the field of germs of meromorphic functions around q_α for $\alpha \in \{1, 2, 3\}$. Then let

$$f_\alpha\colon \mathcal{M}_\alpha \xrightarrow{\sim} \mathcal{M},\ g_\alpha \mapsto g_\alpha \circ \tilde{f}_\alpha^{-1}$$

be the isomorphism induced by f restricted to a neighbourhood of q_α for every α, where g_α is a germ of a meromorphic function at q_α and $\tilde{f}_\alpha^{-1}$ is an inverse of f biholomorphic from a neighbourhood of p to one of q_α taking the respective third root. Let $\phi\colon K(Y) \hookrightarrow \mathcal{M}$ be the inclusion obtained by restricting rational functions to a neighbourhood of p, define $K := \phi(K(Y)) \subseteq \mathcal{M}$. Let $\phi_\alpha\colon K(X) \xhookrightarrow{\mathrm{res}_\alpha} \mathcal{M}_\alpha \xrightarrow{f_\alpha} \mathcal{M}$ be the inclusion obtained by composing f_α with rational functions restricted to a neighbourhood of q_α and define L as the subfield of $\mathcal{M}$ generated by the subfields $K_\alpha := \phi_\alpha(K(X))$.
Identifying z^3 with the germ $\phi(z^3) \in \mathcal{M}$ and defining $\tilde{\xi}_\alpha := \phi_\alpha(z) \in \mathcal{M}$, each element $\tilde{\xi}_\alpha$ satisfies the polynomial

$$\tilde{P}(\tilde{\xi}_\alpha) = \tilde{\xi}_\alpha^3 - z^3 = (f_\alpha(\mathrm{res}_\alpha(z)))^3 - z^3 = (\mathrm{res}_\alpha(z) \circ \tilde{f}_\alpha^{-1})^3 - z^3 = 0.$$

As we have seen before the $\tilde{\xi}_\alpha$ are all the roots of $\tilde{P}$. The field $L \subseteq \mathcal{M}$ is then the normal closure of the extension $K(X)/f^*(K(Y)) = K_\alpha/K$. The discriminant of $\tilde{P}$ is $-27\left(-z^3\right)^2 = -27z^6$, which has the square root $\sqrt{-27}z^3$ in $\phi(K(Y)) = K$, so by [Bos09, p. 160 (2)] the degree of the Galois extension L/K is 3. Thus, the Galois group $G := \mathrm{Gal}(L/K)$ is Z_3, the cyclic group with 3 elements, acting on the set $\{\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3\}$ of the roots of $\tilde{P}$. As A_3 is the only subgroup with three elements of S_3, G can be identified with the even permutations of the set of three elements.

Again following Construction 3.12 we consider the monodromy action in this setting: as U is a surface with one puncture in 0, the fundamental group $\pi_1(U,p)$ is infinitely cyclic and isomorphic to $\mathbb{Z}$, because it is generated by the homotopy class $[\alpha]$, represented by a loop α around 0 with base point p. Without restriction, we can assume that α runs counterclockwise, e.g. $\alpha\colon I \to U,\ t \mapsto \exp(2\pi i t)$. We now consider different paths between fibre elements, which are liftings of loops in U with base point p. This is essentially the same approach as in the proof of transitivity of the monodromy action in Lemma 2.18.

- $q_1 \rightsquigarrow q_2$: The path $\tilde{\alpha}_{1,2}\colon I \to V,\ t \mapsto \exp\left(\frac{2\pi i}{3}t\right)$ from q_1 to q_2 projects to $f \circ \tilde{\alpha}_{1,2}\colon I \to U,\ t \mapsto \exp\left(\frac{2\pi i}{3}t\right)^3$. This loop starts in $p = 1$ and traverses an ellipse intersecting the imaginary axis for $t = \frac{1}{4}$ in i, the real axis for $t = \frac{1}{2}$ in -1, again the imaginary axis for $t = \frac{3}{4}$ in $-i$, finally ending in $p = 1$. So $f \circ \tilde{\alpha}_{1,2}$ is a counterclockwise loop around 0 with base point p, so homotopic to α and thus in $[\alpha] \in \pi_1(U,p)$. Conversely, the path $\tilde{\alpha}_{2,1}$ from q_2 to q_1 defined analogously projects to $[\alpha^{-1}] = [\alpha]^{-1}$.

- $q_2 \rightsquigarrow q_3$: The path $\tilde{\alpha}_{2,3}\colon I \to V,\ t \mapsto \exp\left(\frac{2\pi i}{3}t + \frac{2\pi i}{3}\right)$ from q_2 to q_3 projects to $f \circ \tilde{\alpha}_{2,3}\colon I \to U,\ t \mapsto \exp\left(\frac{2\pi i}{3}t + \frac{2\pi i}{3}\right)^3$, which is a counterclockwise loop around 0 with base point p as above, so homotopic to α and thus in $[\alpha] \in \pi_1(U,p)$. The path $\tilde{\alpha}_{3,2}$ from q_3 to q_2 defined analogously projects to $[\alpha^{-1}] = [\alpha]^{-1}$.

- $q_3 \rightsquigarrow q_1$: The path $\tilde{\alpha}_{3,1}\colon I \to V,\ t \mapsto \exp\left(\frac{2\pi i}{3}t + \frac{4\pi i}{3}\right)$ from q_3 to q_1 projects to $f \circ \tilde{\alpha}_{3,1}\colon I \to U,\ t \mapsto \exp\left(\frac{2\pi i}{3}t + \frac{4\pi i}{3}\right)^3$, which again is a counterclockwise loop around 0 with base point p, so homotopic to α and thus in $[\alpha] \in \pi_1(U,p)$. Again, the path $\tilde{\alpha}_{1,3}$ from q_1 to q_3 defined analogously projects to $[\alpha^{-1}] = [\alpha]^{-1}$.

So the generator $[\alpha]$ of $\pi_1(U,p)$ corresponds to the permutation (123) in the monodromy group $M \leq S_3$ via the monodromy $\pi_1(U,p) \to S_3$, which implies that M is generated by (123). Thus, we have $M = \langle(123)\rangle = \{\mathrm{id}, (123), (132)\} = A_3 \leq S_3$, just like we saw above.

Example 3.15. Consider the affine variety

$$X := V(y^4 - x^3 - x^2 - 1) \subseteq \mathbb{A}^2(\mathbb{C})$$

of dimension $\dim(X) = 1$ and the projection on the first component

$$f\colon X \to \mathbb{A}^1(\mathbb{C}) := Y,\ (p_1, p_2) \mapsto p_1.$$

For every $p_1 \in \mathbb{C}$ there are the four solutions

$$p_2 = \pm\sqrt[4]{p_1^3 + p_1^2 + 1},\ p_2 = \pm i\sqrt[4]{p_1^3 + p_1^2 + 1}$$

to the equation $p_2^4 = p_1^3 + p_1^2 + 1$. As there are exactly three complex solutions

$$s_1 \approx 0.23 - 0.79i,\ s_2 \approx 0.23 + 0.79i,\ s_3 \approx -1.47$$

to $x^3 + x^2 + 1 = 0$, the roots $\pm\sqrt[4]{p_1^3 + p_1^2 + 1}$ and $\pm i\sqrt[4]{p_1^3 + p_1^2 + 1}$ from above are distinct for $p_1 \notin \{s_1, s_2, s_3\}$ and f is a surjective morphism. Define $U := \mathbb{A}^1(\mathbb{C}) \setminus \{s_1, s_2, s_3\}$, the open set on which the fibre of f has cardinality $d = 4$, and $V := f^{-1}(U) = X \setminus \{(s_1, 0), (s_2, 0), (s_3, 0)\}$. As $df_{(p_1,p_2)}$ is surjective for every $(p_1, p_2) \in V$, $f|_V\colon V \to U$ is a covering map by Theorem 3.5. Let $p := 0 \in U$ and let $F := f^{-1}(p) = \{q_1, q_2, q_3, q_4\} \subseteq V$ with $q_1 := (0, 1),\ q_2 := (0, -1),\ q_3 := (0, i),\ q_4 = (0, -i)$ be the fibre of p under f.

We follow the construction of the Galois extension from Construction 3.12: the field extension $K(X)/f^*(K(Y))$ given by

$$f^*\colon K(Y) \hookrightarrow K(X),\ \frac{g(x)}{h(x)} \mapsto \frac{g(f(x,y))}{h(f(x,y))} = \frac{g(x)}{h(x)}$$

is then generated by $y \in K(X) = \mathrm{Quot}(\Gamma(X))$ over $f^*(K(Y))$, where $\Gamma(X)$ is the coordinate ring of X. The minimal polynomial of the extension is

$$P := z^4 - x^3 - x^2 - 1 \in (f^*(K(Y)))[z].$$

Let $\mathcal{M}$ be the field of germs of meromorphic functions around p and $\mathcal{M}_\alpha$ the field of germs of meromorphic functions around q_α for $\alpha \in \{1, 2, 3, 4\}$. Then let

$$f_\alpha\colon \mathcal{M}_\alpha \xrightarrow{\sim} \mathcal{M},\ g_\alpha \mapsto g_\alpha \circ \tilde{f}_\alpha^{-1}$$

be the isomorphism induced by f restricted to a neighbourhood of q_α for every α, where g_α is a germ of a meromorphic function at q_α and $\tilde{f}_\alpha^{-1}$ is an inverse of f biholomorphic from a neighbourhood of p to one of q_α. Let $\phi\colon K(Y) \hookrightarrow \mathcal{M}$ be the inclusion obtained by restricting rational functions to a neighbourhood of p, define

$K := \phi(K(Y)) \subseteq \mathcal{M}$. Let $\phi_\alpha \colon K(X) \xhookrightarrow{\mathrm{res}_\alpha} \mathcal{M}_\alpha \xrightarrow{f_\alpha} \mathcal{M}$ be the inclusion obtained by composing f_α with rational functions restricted to a neighbourhood of q_α and define L as the subfield of $\mathcal{M}$ generated by the subfields $K_\alpha := \phi_\alpha(K(X))$.
Identifying x^3, x^2, 1 with the germs $\phi(x^3)$, $\phi(x^2)$, $\phi(1) \in \mathcal{M}$ respectively and defining $\tilde{\xi}_\alpha := \phi_\alpha(y) \in \mathcal{M}$, each element $\tilde{\xi}_\alpha$ satisfies the polynomial

$$\tilde{P}(\tilde{\xi}_\alpha) = \tilde{\xi}_\alpha^4 - x^3 - x^2 - 1 = (\mathrm{res}_\alpha(y) \circ \tilde{f}_\alpha^{-1})^4 - x^3 - x^2 - 1 = 0.$$

As we have seen before the $\tilde{\xi}_\alpha$ are all the roots of $\tilde{P}$. The field $L \subseteq \mathcal{M}$ is then the normal closure of the extension $K(X)/f^*(K(Y)) = K_\alpha/K$. Since the coefficient of the cubic term in $\tilde{P}$ is zero, we can calculate the resolvent cubic:

$$r(t) \coloneqq 8t^3 - 8\left(-x^3 - x^2 - 1\right)t = 8t^3 + \left(8x^3 + 8x^2 + 8\right)t \in K[t] = (\phi(K(Y))[t].$$

Then r has the root $t = 0$ in K. Set

$$\begin{aligned}\underline{t} &:= \{t_1 := \tilde{\xi}_1\tilde{\xi}_2 + \tilde{\xi}_3\tilde{\xi}_4,\ t_2 := \tilde{\xi}_1\tilde{\xi}_3 + \tilde{\xi}_2\tilde{\xi}_4,\ t_3 := \tilde{\xi}_1\tilde{\xi}_4 + \tilde{\xi}_2\tilde{\xi}_4\} \\ &= \{t_1 = 0,\ t_2 = 2i\sqrt{x^3 + x^2 + 1},\ t_3 = 0\} \\ &= \{2i\sqrt{x^3 + x^2 + 1},\ 0\}.\end{aligned}$$

We now have

$$\begin{aligned}&\frac{1}{4}(2z^2 + t_2)(2z^2 - t_2) = \frac{1}{4}\left(\left(2z^2\right)^2 - \left(2i\sqrt{x^3 + x^2 + 1}\right)^2\right) \\ =& \frac{1}{4}(4z^4 + 4(x^3 + x^2 + 1)) = z^4 + x^3 + x^2 + 1 = \tilde{P}(z)\end{aligned}$$

in $(K(\underline{t}))[z] = (K(t_2))[z]$, so $\tilde{P}$ is reducible over $K(\underline{t})$. Therefore, by [KW89, Theorem 1 (iv), p. 134] the Galois group $G = \mathrm{Gal}(L/K)$, which is the Galois group of the polynomial $\tilde{P}$, is isomorphic to $C_4 \leq S_4$, a cyclic group of order 4 in S_4. A cyclic subgroup of order 4 of S_4 has two 4-cycles. As there are six 4-cycles in S_4 there are three transitive cyclic subgroups of order 4 (and all of them are isomorphic). So we have, for instance, $G \cong \langle(1324)\rangle = \{\mathrm{id}, (1324), (12)(14), (1423)\} \leq S_4$.

We will now consider the monodromy action of $\pi_1(U, p)$ on F. Let $\mu \colon \pi_1(U, p) \to \mathrm{Aut}(F) \cong S_4$ be the monodromy of $f|_V \colon V \to U$ and set $M \coloneqq \mathrm{im}(\mu)$.
We give an argument for the fact that the fundamental group of the complex plane with $n > 0$ distinct points removed is F_n, the free group on n generators. Let $R_n \coloneqq \mathbb{C} \setminus \{r_1, \dots, r_n\}$ be the complex plane with n distinct punctures. The punctured plane R_1 deformation retracts to S^1, so it has an infinitely cyclic fundamental group $\pi_1(R_1) = \mathbb{Z} \cong F_1$. Now for R_{n+1} we can find a line in the complex plane of the form $|z - z_1| = |z - z_2|$ for $z_1,\ z_2 \in \mathbb{C},\ z_1 \neq z_2$ separating R_{n+1} into two open half planes W and W', such that only one puncture is in W'. Without restriction let $r_1, \dots, r_n \in W$

and $r_{n+1} \in W'$. Extending W and W' so that there is an open stripe as a nonempty overlap of W and W', such that the overlap does not contain a puncture, we have that $R_{n+1} = W \cup W'$ with $W, W', W \cap W'$ open and path-connected, $W \cap W' \neq \emptyset$ and simply connected. As $\pi_1(W) = \pi_1(R_n) = F_n$ by induction and $\pi_1(W') = \pi_1(R_1) = F_1$ as above it follows with Seifert-Van Kampen in the form of [Bre93, Corollary 9.5, p. 161] that $\pi_1(R_{n+1}) = \pi_1(W) * \pi_1(W') = F_n * F_1 \cong F_{n+1}$, the free group on $n+1$ generators.
So $\pi_1(U, p)$ is free on three generators, e.g. on the set $\{a, b, c\}$ of three letters. Without restriction let a, b and c be the homotopy classes of counterclockwise loops around s_1, s_2 and s_3 respectively with base point p like in Figure 3.

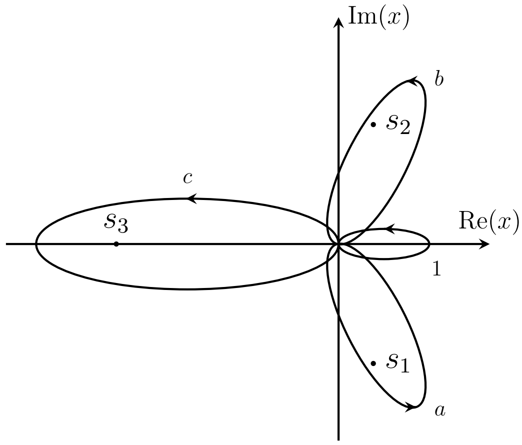


Figure 3: The loops generating $\pi_1(U)$.

Explicitly we could represent a, b and c by three counterclockwise loops α, β and γ with base point p around s_1, s_2 and s_3 respectively, i.e.

$$\alpha\colon I \to U,\ t \mapsto \sqrt{2}\exp(\pi i(2t + \frac{5}{4})) + 1 - i,$$
$$\beta\colon I \to U,\ t \mapsto \sqrt{2}\exp(\pi i(2t + \frac{3}{4})) + 1 + i, \text{ and}$$
$$\gamma\colon I \to U,\ t \mapsto \exp(2\pi i t) - 1.$$

Instead of directly calculating the liftings we again consider different paths between fibre elements, which are liftings of loops in U with base point p. Since the fibre elements $q_1, \dots, q_4$ are all 0 in the first component we can choose paths connecting the second components in the complex plane $\mathbb{C}$ and determine the first component of the path in $\mathbb{C}^2$ by continuity and the constraint that the path has to be in V. The possible solutions to the equation $y^4 - x^3 - x^2 - 1 = 0$ are given by $x = v_1(y),\ v_2(y),\ v_3(y)$ for $y \in \mathbb{C}$:

$$
\begin{aligned}
v_1(y) = \frac{1}{6}\Bigg(& 2^{\frac{2}{3}}\sqrt[3]{27y^4 + 3\sqrt{81y^8 - 174y^4 + 93} - 29} \\
& + \frac{2\sqrt[3]{2}}{\sqrt[3]{27y^4 + 3\sqrt{81y^8 - 174y^4 + 93} - 29}} - 2\Bigg), \\
v_2(y) = \frac{1}{12}\Bigg(& i2^{\frac{2}{3}}(\sqrt{3} + i)\sqrt[3]{27y^4 + 3\sqrt{81y^8 - 174y^4 + 93} - 29} \\
& - \frac{2i\sqrt[3]{2}(\sqrt{3} - i)}{\sqrt[3]{27y^4 + 3\sqrt{81y^8 - 174y^4 + 93} - 29}} - 4\Bigg), \\
v_3(y) = \frac{1}{12}\Bigg(& - 2^{\frac{2}{3}}(1 + i\sqrt{3})\sqrt[3]{27y^4 + 3\sqrt{81y^8 - 174y^4 + 93} - 29} \\
& + \frac{2i\sqrt[3]{2}(\sqrt{3} + i)}{\sqrt[3]{27y^4 + 3\sqrt{81y^8 - 174y^4 + 93} - 29}} - 4\Bigg).
\end{aligned}
$$

For example, if we would like to find out which element of $\pi_1(U,p)$ induces the mapping $q_1 \mapsto q_3$ we would connect the points $q_1 = (0,1)$ and $q_3 = (0,i)$ with a path in V by first choosing a parametrization of a quarter of the unit circle $t \mapsto \exp(\frac{\pi i}{2}t),\ t \in I$ which connects 1 and i. Then we use the formulas for v_1, v_2 and v_3 to find a path $t \mapsto w(t)$ such that $\exp(\frac{\pi i}{2}t)^4 - w(t)^3 - w(t)^2 - 1 = 0$ and $w(t) \notin \{s_1, s_2, s_3\}$ hold for all $t \in I$, i.e. the path $t \mapsto (w(t),\ \exp(\frac{\pi i}{2}t))$ lies in V. Explicitly, we calculate and plot $v_i(\exp(\frac{\pi i}{2}t)),\ t \in I$ for $i \in \{1,2,3\}$ and use the plot to define $w(t)$ as $v_i(\exp(\frac{\pi i}{2}t))$ for an $i \in \{1,2,3\}$ depending on $t \in I$, so that $t \mapsto f(w(t),\ \exp(\frac{\pi i}{2}t)) = w(t)$ is a (continuous) loop in U with base point p. Then w belongs to some equivalence class in $\pi_1(U,p)$, so we could conclude that this element of $\pi_1(U,p)$ induces some permutation of F which sends q_1 to q_3.
For such quatercircle paths in the second component like above there are three possible ways of using two solutions v_i and v_j of $y^4 - x^3 - x^2 - 1 = 0$ to obtain a continuous path in V (which means using v_i for the parameter $t \in [0, \frac{1}{2})$ and v_j for $t \in [\frac{1}{2}, 1]$). But only in two of these cases the projections onto U will form a loop with base point $p = 0$:

- $(0,i) = q_3 \rightsquigarrow q_2 = (0,-1)$: The path $\delta_{3,2}\colon I \to V,\ t \mapsto (w(t),\ \exp(\frac{\pi i}{2}t + \frac{\pi i}{2}))$ from q_3 to q_2, where $w(t)$ is determined by continuity and $w(t)^3 + w(t)^2 + 1 = (\exp(\frac{\pi i}{2}t + \frac{\pi i}{2}))^4$ for every $t \in [0,1]$. Using v_2 and then v_1 from above for w, the path projects to $f \circ \delta^1_{3,2}\colon I \to U,\ t \mapsto w(t)$, which is a counterclockwise loop around s_1 with base point p, so homotopic to α and thus in $[\alpha] = a \in \pi_1(U,p)$. Using v_1 and then v_3 from above for w, the path projects to $f \circ \delta^2_{3,2}\colon I \to$

$U,\ t \mapsto w(t)$, which is a counterclockwise loop around s_2 with base point p, so homotopic to β and thus in $[\beta] = b \in \pi_1(U,p)$.

- $(0,-1) = q_2 \rightsquigarrow q_4 = (0,-i)$: The path $\delta_{2,4}\colon I \to V,\ t \mapsto (w(t),\ \exp(\frac{\pi i}{2}t + \pi i))$ from q_2 to q_4, where $w(t)$ is determined by continuity and $w(t)^3 + w(t)^2 + 1 = (\exp(\frac{\pi i}{2}t + \pi i)^4$ for every $t \in [0,1]$. As above, using v_2 and then v_1 for w, the path projects to $f \circ \delta^1_{2,4}\colon I \to U,\ t \mapsto w(t)$, which is the same counterclockwise loop around s_1 with base point p as above, so homotopic to α and thus in $[\alpha] = a \in \pi_1(U,p)$. Again using v_1 and then v_3 for w, the path projects to $f \circ \delta^2_{2,4}\colon I \to U,\ t \mapsto w(t)$, which is the same counterclockwise loop around s_2 with base point p as above, so homotopic to β and thus in $[\beta] = b \in \pi_1(U,p)$.
- $(0,-i) = q_4 \rightsquigarrow q_1 = (0,1)$: The path $\delta_{4,1}\colon I \to V,\ t \mapsto (w(t),\ \exp(\frac{\pi i}{2}t + \frac{3\pi i}{2}))$ from q_4 to q_1, where $w(t)$ is determined by continuity and $w(t)^3 + w(t)^2 + 1 = (\exp(\frac{\pi i}{2}t + \frac{3\pi i}{2}))^4$ for every $t \in [0,1]$. The projections $f \circ \delta^1_{4,1}\colon I \to U$ and $f \circ \delta^2_{4,1}\colon I \to U$ are exactly as above.
- $(0,1) = q_1 \rightsquigarrow q_3 = (0,i)$: The path $\delta_{1,3}\colon I \to V,\ t \mapsto (w(t), \exp(\frac{\pi i}{2}t))$ from q_1 to q_3, where $w(t)$ is determined by continuity and $w(t)^3 + w(t)^2 + 1 = (\exp(\frac{\pi i}{2}t))^4$ for every $t \in [0,1]$. Again the projections $f \circ \delta^1_{1,3}\colon I \to U$ and $f \circ \delta^2_{1,3}\colon I \to U$ are as above.

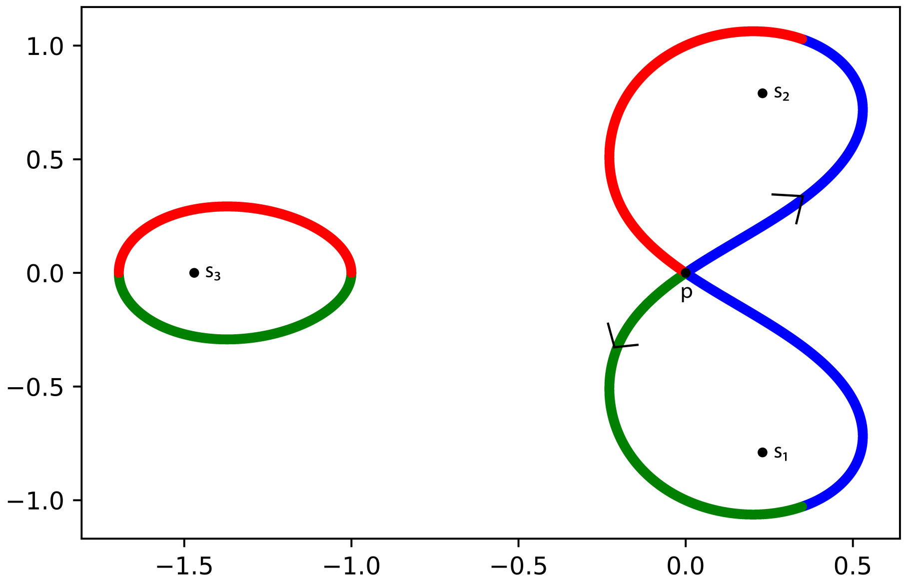


Figure 4: The first components of the paths $\delta_{r,s}$ form loops in U with counterclockwise orientation. All possible usages of v_1, v_2 and v_3 are shown. (Appendix A.1)

Figure 4 shows the projections of the loops $\delta_{r,s}$ from above onto U, which are identical for all four considered cases. The usage of v_1 is displayed blue, v_2 green and v_3 red. The loop on the left is the mentioned third case and can be ignored, as the use of v_3 and then v_2 does not lead to a loop with base point $p = 0$. So for each of the considered paths in the second component we obtain the same two loops in U with base point p.

As we know that the liftings of (representatives of) a and b, given a fixed starting point in the fibre, are uniquely determined up to homotopy and have the same endpoint (Corollary 2.5), the liftings we considered completely determine the permutations given by a and b in the monodromy group $M \leq S_4$. Namely, a and b induce the same permutation $q_1 \mapsto q_3 \mapsto q_2 \mapsto q_4 \mapsto q_1$ which can be identified with (1324) in S_4. It remains to show that the third generator c of $\pi_1(U, p)$ induces the same permutation in M.

This can again be done by specifying the second component of a path, in this case a loop at a fibre point, and calculating the first component w, i.e. the projection onto U, by using v_1, v_2 and v_3 so that the original loop is in V. This time, the usage of v_1, v_2 and v_3 alternates. We consider the following:

- $q_1 = (0, 1)$: The loop $\delta_1 \colon I \to V,\ t \mapsto (w(t),\ 2\exp(2\pi i t) - 1)$ at q_1.
- $q_3 = (0, -1)$: The loop $\delta_3 \colon I \to V,\ t \mapsto (w(t),\ 2\exp(2\pi i t + \pi i) + 1)$ at q_3.
- $q_2 = (0, i)$: The loop $\delta_2 \colon I \to V,\ t \mapsto (w(t),\ 2\exp(2\pi i t + \frac{\pi i}{2}) - i)$ at q_2.
- $q_4 = (0, -i)$: The loop $\delta_4 \colon I \to V,\ t \mapsto (w(t),\ 2\exp(2\pi i t + \frac{3\pi i}{2}) + i)$ at q_2.

For all the second components $y(t)$ from above the plots of $v_i(y(t)),\ i \in \{1, 2, 3\}$, are identical. So we can choose the same first component w in all four cases from the plots, e.g. the counterclockwise loop in U with base point p around s_2, s_3, s_1 and again s_2 as highlighted with yellow in Figure 5. We could also choose another (mirror-symmetric) loop with base point p around s_1, s_2, s_3 and again s_1 as a first component w, leading to the same result. The loop on the right has not got base point p, so it can be omitted. The yellow loop is homotopic to $\beta * \gamma * \alpha * \beta$ and thus in $bcab \in \pi_1(U, p)$. With the identification $\mathrm{Aut}(F) \cong S_4,\ q_i \leftrightarrow i$ it follows $\mu(bcab)(i) = i \Leftrightarrow (\mu(b) \circ \mu(c) \circ \mu(a) \circ \mu(b))(i) = i \Leftrightarrow ((1324) \circ \mu(c) \circ (1324)^2)(i) = i$ for each $i \in \{1, 2, 3, 4\}$. It follows $\mu(c) = (1324)$, as for instance $((1324) \circ \mu(c) \circ (1324)^2)(1) = 1 \Leftrightarrow ((1324) \circ \mu(c))(2) = 1 \Leftrightarrow \mu(c)(2) = (1324)^{-1}(1) \Leftrightarrow \mu(c)(2) = 4$.
As a, b and c generate the free group $\pi_1(U, p)$ we conclude that the monodromy group M is generated by the permutation (1324), so we obtain the group $M = \langle (1324) \rangle = \{\mathrm{id}, (1324), (12)(14), (1423)\} \leq S_4$, a cyclic subgroup of order 4 of S_4, as expected.

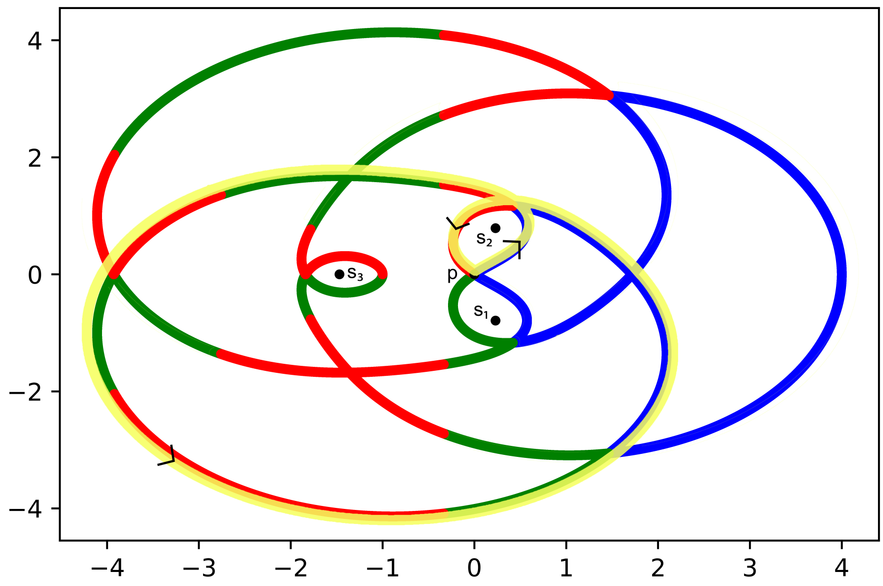


Figure 5: The first components of the loops δ_r form loops in U with counterclockwise orientation. Again all possible usages of v_1 (blue), v_2 (green) and v_3 (red) are shown. (Appendix A.1)

4 Applications of Geometric Coverings

We can now apply the established theory by showing that every finite group can be obtained as a monodromy group of a covering map of affine varieties (Section 4.1). We then consider deck transformations in a geometric context (Section 4.2).

4.1 Every Finite Group is a Monodromy Group

We prove that every finite group is a monodromy group of a covering map obtained by restricting a dominant morphism of suitable affine varieties. This uses the usual construction one does to prove that every finite group is a Galois group, and we do not need further topological considerations.

Lemma 4.1. *Let X and Y be complex affine varieties and let $f\colon X \to Y$ be a dominant morphism of degree $d > 0$. If the inclusion of function fields $f^*\colon K(Y) \hookrightarrow K(X)$ induced by f (Lemma 3.9) is a Galois extension, then the Galois extension L/K constructed in Construction 3.12 equals the extension $K(X)/K(Y)$. In particular* $\operatorname{Gal}(K(X)/K(Y)) = \operatorname{Gal}(L/K)$.

Proof. Using the notation from Construction 3.12, the field $L \subseteq \mathcal{M}$ is the normal closure of the extension $K(X)/K(Y) = K_\alpha/K$, which is already a normal extension by assumption, so $K(X)/K(Y) = L/K$. □

Consider the situation of Lemma 3.9: For a dominant rational map $f\colon X \dashrightarrow Y$ of affine varieties X and Y defined over k we obtain an inclusion of function fields $f^*\colon K(Y) \hookrightarrow K(X)$. The map $f \mapsto f^*$ is a bijection:

Theorem 4.2. *[Har77, Theorem 4.4, p. 25] The mapping $f \mapsto f^*$ from above is a bijection between*

(i) dominant rational maps $f\colon X \dashrightarrow Y$, and

(ii) k-algebra homomorphisms $K(Y) \to K(X)$.

Proof. Let $X \subseteq \mathbb{A}^n(k)$ and $Y \subseteq \mathbb{A}^m(k)$. We construct an inverse to the mapping $f \mapsto f^*$. Let $\varphi\colon K(Y) \to K(X)$ be a homomorphism of k-algebras, and let $y_1|_Y, \ldots, y_m|_Y \in \Gamma(Y)$ be the restricted coordinate functions which generate $\Gamma(Y)$ as a k-algebra. Then $\varphi(y_1|_Y), \ldots, \varphi(y_m|_Y) \in K(X)$ are rational functions on X. As X is irreducible we can find a nonempty open subset $U \subseteq X$ such that the functions $\varphi(y_i|_Y)$ are all regular on U. Then φ defines an injective homomorphism of k-algebras $\Gamma(Y) \to \mathcal{O}_X(U)$, corresponding to a regular map

$$f_U\colon U \to Y,\ p \mapsto (\varphi(y_1|_Y)(p), \ldots, \varphi(y_m|_Y)(p))$$

by [Har77, Proposition 3.5, p. 19]. This regular map gives a rational map $f_\varphi\colon X \dashrightarrow Y$ represented by (U, f_U) which is dominant as φ is injective. The map $\varphi \mapsto f_\varphi$ then is the desired inverse:
In the above situation the induced inclusion of function fields

$$f_\varphi^*\colon K(Y) \hookrightarrow K(X),\ (V, g_V) \mapsto (f_U^{-1}(V), g_V \circ f_U|_{f_U^{-1}(V)})$$

satisfies $f_\varphi^*((D(y_i|_Y), y_i|_Y)) = (f_U^{-1}(D(y_i|_Y)), \varphi(y_i|_Y)|_{f_U^{-1}(D(y_i|_Y))})$, so f_φ^* agrees with φ on $y_1|_Y, \ldots, y_m|_Y$, which implies $f_\varphi^* = \varphi$ because $y_1|_Y, \ldots, y_m|_Y$ generate $K(Y)$ as a field.
For a dominant rational map $f\colon X \dashrightarrow Y$ represented by (V, f_V) we consider

$$f^*\colon K(X) \hookrightarrow K(Y),\ (V', g_{V'}) \mapsto (f_V^{-1}(V'), g_{V'} \circ f_V|_{f_V^{-1}(V')}).$$

With the notation from above we obtain the injective homomorphism of k-algebras $\Gamma(Y) \to \mathcal{O}_X(U)$ defined on the generators of $\Gamma(Y)$ through $y_i|_Y \mapsto y_i|_Y \circ f_V$. The corresponding regular map is

$$f_U\colon U \to Y,\ p \mapsto ((y_1|_Y \circ f_V)(p), \ldots, (y_m|_Y \circ f_V)(p)) = f_V(p),$$

so $f_U = f_V$ on $U \cap V$ which implies $f = f_{f^*}$. □

Remark 4.3. The above Theorem 4.2 also holds for arbitrary varieties. Furthermore, the correspondence (with consideration of functoriality) shows that the category of varieties with dominant rational maps and the category of finitely generated field extensions of k are anti-equivalent.

Corollary 4.4. *[Har77, Corollary 4.5, p. 26] For affine varieties X and Y defined over k the following are equivalent:*

(i) X and Y are birational,

(ii) there are nonempty open subsets $U \subseteq X$ and $V \subseteq Y$ with $U \cong V$,

(iii) $K(X) \cong K(Y)$ as k-algebras.

Proof. $(i) \Rightarrow (ii)$: Let $f\colon X \dashrightarrow Y$ and $g\colon Y \dashrightarrow X$ be rational maps which are inverse to each other. Let f be represented by (U, f_U) and g by (V, g_V). Then $g \circ f$ is represented by $(f_U^{-1}(V), g_V \circ f_U)$, and since $g \circ f = \mathrm{id}_X$ as a rational map it follows $g_V \circ f_U = \mathrm{id}_{f_U^{-1}(V)}$. Similarly, $f_U \circ g_V = \mathrm{id}_{g_V^{-1}(U)}$. Then $f_U^{-1}(g_V^{-1}(U)) \cong g_V^{-1}(f_U^{-1}(V))$ via f_U and g_V.
$(ii) \Rightarrow (iii)$: This follows from the definition of the function field for prevarieties which coincides with the definition of being the quotient field of the coordinate ring in the affine case.
$(iii) \Rightarrow (i)$: Follows from Theorem 4.2. □

In order to show that every finite group is a monodromy group we will use the fact that every finite group is a Galois group of an extension of the field of symmetric rational functions.

Definition 4.5 (Elementary symmetric polynomials)**.** The *elementary symmetric polynomials* in n variables $x_1, \ldots, x_n$ are defined as follows:

$$\begin{aligned}
\sigma_0 &:= 1,\\
\sigma_1 &:= \sum_{1 \le i \le n} x_i,\\
\sigma_2 &:= \sum_{1 \le i < j \le n} x_i x_j,\\
\sigma_3 &:= \sum_{1 \le i < j < l \le n} x_i x_j x_l,\\
&\;\vdots\\
\sigma_n &:= x_1 \cdots x_n.
\end{aligned}$$

Lemma 4.6. *The morphism*

$$\chi_n\colon \mathbb{A}^n(\mathbb{C}) \to \mathbb{A}^n(\mathbb{C}),\ (p_1, \ldots, p_n) \mapsto (\sigma_1(p_1, \ldots, p_n), \ldots, \sigma_n(p_1, \ldots, p_n))$$

is surjective and has Galois group S_n (in the sense of Construction 3.12).

Proof. For any $a = (a_1, \dots, a_n) \in \mathbb{C}^n$ the polynomial $x^n + \sum_{i=1}^n a_i x^i \in \mathbb{C}[x]$ can be written as a product of linear factors $\prod_{i=1}^n (x + b_i)$ since $\mathbb{C}$ is algebraically closed. Expanding the right side and comparing coefficients shows $a_i = \sigma_i(b_1, \dots, b_n)$ for all $i \in \{1, \dots, n\}$, so χ_n is surjective.
Let $L := \mathbb{C}(x_1, \dots, x_n) = \mathrm{Quot}(\mathbb{C}[x_1, \dots, x_n])$ be the function field of $\mathbb{A}^n(\mathbb{C})$, the field of rational functions in n variables. Consider the inclusion of function fields

$$\chi_n^*\colon L \hookrightarrow L,\ \frac{g}{h}(\underline{x}) \mapsto \frac{g}{h}(\sigma_1(\underline{x}), \dots, \sigma_n(\underline{x})).$$

The image consists of all rational functions $\frac{\tilde{g}}{\tilde{h}}$ for $\tilde{g}, \tilde{h} \in \mathbb{C}[x_1, \dots, x_n]$ such that $\tilde{g}$ and $\tilde{h}$ are polynomials in the elementary symmetric polynomials. Hence, $\chi_n^*(L) = \mathbb{C}(\sigma_1, \dots, \sigma_n) =: K$, the *field of symmetric rational functions* [Bos09, Satz 3, p. 164]. So the field extension induced by χ_n is $L/\mathbb{C}(\sigma_1, \dots, \sigma_n) = L/K$. By [Bos09, p. 163 (4)] this is precisely the extension L/L^{S_n}, where $L^{S_n} = K$ is the fixed field obtained by automorphisms of L of the form

$$L \to L,\ \frac{g(x_1, \dots, x_n)}{h(x_1, \dots, x_n)} \mapsto \frac{g(x_{\pi(1)}, \dots, x_{\pi(n)})}{h(x_{\pi(1)}, \dots, x_{\pi(n)})}$$

for $\pi \in S_n$. So L/K is a Galois extension of degree $n!$ with Galois group $\mathrm{Gal}(L/K) = S_n$. Using Lemma 4.1 the Galois group of the map χ_n in the sense of Construction 3.12 is also S_n. □

The next result is classic as well as handy: every finite group can be viewed as a subgroup of a symmetric group.

Lemma 4.7 (Cayley's Theorem)**.** *[Bos09, p. 14 (2)] Let G be a finite group of n elements. Then G is isomorphic to a subgroup of S_n.*

Proof. For $a \in G$ define $\tau_a \in S(G)$ by $\tau_a\colon G \to G,\ g \mapsto ag$, where $S(G)$ is the group of bijections $G \to G$. Then $G \to S(G),\ a \mapsto \tau_a$ is an injective group homomorphism, so G is isomorphic to a subgroup of $S(G) \cong S_n$. □

We now have all prerequisites needed to show that a finite group G, regarded as a subgroup of S_n, induces a Galois extension corresponding to a morphism of affine varieties.

Theorem 4.8. *Let G be a finite group. Then there exist affine varieties of the same dimension X and Y defined over $\mathbb{C}$ and a dominant morphism $\chi_G\colon X \to Y$ of finite degree such that G is the Galois group of χ_G (in the sense of Construction 3.12 and Theorem 3.13).*

Proof. Let $n := |G|$. By Lemma 4.7 we can identify G with a subgroup of S_n of order n, so write $G \leq S_n$. By Lemma 4.6 the map $\chi_n \colon \mathbb{A}^n(\mathbb{C}) \to \mathbb{A}^n(\mathbb{C})$ induces the finite Galois extension $L/K = \mathbb{C}(x_1, \ldots, x_n)/\mathbb{C}(\sigma_1, \ldots, \sigma_n)$ with Galois group $\operatorname{Gal}(L/K) = S_n$. The fundamental theorem of Galois theory [Bos09, Theorem 6, p. 142] implies that the fixed field $E := L^G$ is an intermediate field of the extension L/K such that L/E is a finite Galois extension with Galois group $\operatorname{Gal}(L/E) = \operatorname{Gal}(L/L^G) = G$.
As L/K is finite the extension E/K is finite, and by the primitive element theorem [Bos09, Satz 12, p. 119] there exists an $f \in E$ such that

$$E = K(f) = (\mathbb{C}(\sigma_1, \ldots, \sigma_n))(f) = \mathbb{C}(\sigma_1, \ldots, \sigma_n, f).$$

Let $Y \subseteq \mathbb{A}^{n+1}(\mathbb{C})$ be the affine variety with coordinate ring $\Gamma(Y) = \mathbb{C}[\sigma_1, \ldots, \sigma_n, f]$, where $\mathbb{C}[\sigma_1, \ldots, \sigma_n, f]$ is a reduced $\mathbb{C}$-Algebra which is finitely generated (as a ring) by $\sigma_1, \ldots, \sigma_n$ and f. We then have

$$K(Y) = \operatorname{Quot}(\Gamma(Y)) = \operatorname{Quot}(\mathbb{C}[\sigma_1, \ldots, \sigma_n, f]) = \mathbb{C}(\sigma_1, \ldots, \sigma_n, f) = E.$$

By setting $X := \mathbb{A}^n(\mathbb{C})$ it follows $K(X) = \mathbb{C}(x_1, \ldots, x_n) = L$, and we have an inclusion $\varphi \colon K(Y) = \mathbb{C}(\sigma_1, \ldots, \sigma_n, f) \hookrightarrow \mathbb{C}(x_1, \ldots, x_n) = K(X)$ giving the Galois extension L/E. The corresponding dominant rational map $\chi_G \colon X \dashrightarrow Y$ from Theorem 4.2 then satisfies $K(X)/\chi_G^*(K(Y)) = L/E$. Following the proof of Theorem 4.2 the images of the generators $\sigma_1, \ldots, \sigma_n, f$ of $\Gamma(Y)$ under φ are all rational on X, so χ_G is a dominant morphism of finite degree (using Theorem 3.10 and the fact that L/E is finite). By Lemma 4.1 the Galois group of χ_G (in the sense of Construction 3.12) is $\operatorname{Gal}(L/E) = G$. □

The statement about monodromy groups is an immediate consequence of Theorem 3.13.

Corollary 4.9. *Every finite group is a monodromy group of a covering map of smooth affine varieties defined over* $\mathbb{C}$ *(and hence of topological manifolds).*

Proof. Let G be a finite group. By Theorem 4.8 there exist affine varieties of the same dimension X and Y over $\mathbb{C}$ (where $X = \mathbb{A}^n(\mathbb{C})$ for $n = |G|$) and a dominant morphism $\chi_G \colon X \to Y$ such that G is the Galois group of χ_G. Use Theorem 3.5 and Remark 3.6 to restrict χ_G to a covering map of smooth affine varieties, which does not change the function fields. Then with Theorem 3.13 it follows that G is the monodromy group of the restriction of χ_G. □

Example 4.10. Consider the Klein four-group $V_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$, which we can identify with the subgroup $\{\mathrm{id}, (12)(34), (13)(24), (14)(23)\} \leq S_4$. The morphism

$$\chi_n \colon \mathbb{A}^4(\mathbb{C}) \to \mathbb{A}^4(\mathbb{C}),\ (p_1, p_2, p_3, p_4) \mapsto (\sigma_1(p_1, p_2, p_3, p_4), \ldots, \sigma_4(p_1, p_2, p_3, p_4))$$

$$= (p_1 + p_2 + p_3 + p_4,\ p_1p_2 + p_1p_3 + p_1p_4 + p_2p_3 + p_2p_4 + p_3p_4,$$
$$p_1p_2p_3 + p_1p_2p_4 + p_1p_3p_4 + p_2p_3p_4,\ p_1p_2p_3p_4)$$

induces the Galois extension $L/K = \mathbb{C}(x_1, x_2, x_3, x_4)/\mathbb{C}(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ with Galois group $\mathrm{Gal}(L/K) = S_4$. The chosen representative of V_4 in S_4 is normal, so L^{V_4}/K is Galois with $\mathrm{Gal}(L^{V_4}/K) = S_4/V_4 \cong S_3$. We can find a primitive element for the extension L^{V_4}/K by considering the complement of all intermediate fields.
The intermediate subgroups $V_4 \leq H \leq S_4$ are the three dihedral groups ($\cong D_8$)

$$H_1 := \{\mathrm{id}, (1234), (13)(24), (1432), (12)(34), (14)(23), (13), (24)\},$$
$$H_2 := \{\mathrm{id}, (1324), (12)(34), (1423), (13)(24), (14)(23), (12), (34)\},$$
$$H_3 := \{\mathrm{id}, (1243), (14)(23), (1342), (12)(34), (13)(24), (14), (23)\},$$

and the alternating group ($\cong A_4$)

$$H_4 := \langle (12)(34), (123) \rangle,$$

so using $V_4 \subseteq H \Leftrightarrow L^H \subseteq L^{V_4}$ it suffices to find an $f \in L^{V_4}$ which is not in the union of the intermediate fields $\bigcup_{i=1}^4 L^{H_i}$, i.e. an $f \in \mathbb{C}(x_1, x_2, x_3, x_4)$ satisfying

$$f = f^\pi \text{ for } \pi \in V_4, \text{ and}$$
$$f \neq f^\pi \text{ for some } \pi \in H_i \text{ for each } i \in \{1, \ldots, 4\},$$

where $f = f(x_1, \ldots, x_4)$ and $f^\pi := f(x_{\pi(1)}, \ldots, x_{\pi(4)})$. Choosing

$$f := \frac{1}{2}(x_1 - x_2)(x_3 - x_4) + (x_1 - x_3)(x_2 - x_4) + (x_1 - x_4)(x_2 - x_3)$$
$$= x_1x_2 - x_3x_2 - x_1x_4 + x_3x_4$$

we clearly have $f = f^\pi$ for every $\pi \in V$, but

$$f^{(13)} = x_3x_2 - x_1x_2 - x_3x_4 + x_1x_4 \neq f \text{ for } (13) \in H_1,$$
$$f^{(12)} = x_1x_2 - x_3x_1 - x_2x_4 + x_3x_4 \neq f \text{ for } (12) \in H_2,$$
$$f^{(14)} = x_4x_2 - x_3x_2 - x_1x_4 + x_3x_1 \neq f \text{ for } (14) \in H_3,$$
$$f^{(123)} = x_2x_3 - x_1x_3 - x_2x_4 + x_1x_4 \neq f \text{ for } (123) \in H_4,$$

so $f \in L^{V_4} \setminus \bigcup_{i=1}^4 L^{H_i}$ and $E := L^{V_4} = K(f) = \mathbb{C}(\sigma_1, \sigma_2, \sigma_3, \sigma_4, f)$. Following the proof of Theorem 4.8 we can set $X := \mathbb{A}^4(\mathbb{C})$ and $Y \subseteq \mathbb{A}^5(\mathbb{C})$ such that $\Gamma(Y) = \mathbb{C}[\sigma_1, \sigma_2, \sigma_3, \sigma_4, f]$. Explicitly, we could consider the surjective $\mathbb{C}$-Homomorphism $\psi\colon \mathbb{C}[x_1, \ldots, x_5] \to \mathbb{C}[\sigma_1, \sigma_2, \sigma_3, \sigma_4, f]$ with $\psi(x_i) = \sigma_i,\ i \in \{1, 2, 3, 4\}$, and $\psi(x_5) = f$. Then set $Y := V(\ker(\psi)) = V(F) \subseteq \mathbb{A}^5(\mathbb{C})$, where

$$F := x_1^2x_2^2x_3^2 - 4\,x_1^3x_3^3 - 4\,x_1^2x_2^3x_4 + 18\,x_1^3x_2x_3x_4 - 27\,x_1^4x_4^2 - x_2^4x_5^2 + 6\,x_1x_2^2x_3x_5^2$$

$$
\begin{aligned}
&- 9\,x_1^2x_3^2x_5^2 + 2\,x_2^2x_5^4 - 6\,x_1x_3x_5^4 - x_5^6 - 4\,x_2^3x_3^2 + 18\,x_1x_2x_3^3 + 16\,x_2^4x_4 \\
&- 80\,x_1x_2^2x_3x_4 - 6\,x_1^2x_3^2x_4 + 144\,x_1^2x_2x_4^2 - 24\,x_2^2x_4x_5^2 + 72\,x_1x_3x_4x_5^2 \\
&+ 24\,x_4x_5^4 - 27\,x_3^4 + 144\,x_2x_3^2x_4 - 128\,x_2^2x_4^2 - 192\,x_1x_3x_4^2 - 144\,x_4^2x_5^2 + 256\,x_4^3,
\end{aligned}
$$

calculated with [GS, Macaulay2].
Now the extension $L/E = \mathbb{C}(x_1, x_2, x_3, x_4)/\mathbb{C}(\sigma_1, \sigma_2, \sigma_3, \sigma_4, f)$ is given by the inclusion of function fields

$$K(Y) \hookrightarrow K(X),\ \frac{g}{h}(x_1, \dots, x_4) \mapsto \frac{g}{h}(x_1, \dots, x_4),$$

as E does not contain x_5. This inclusion by Theorem 4.2 corresponds to the dominant morphism

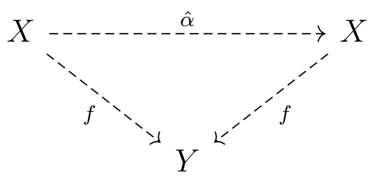
$$\chi_{V_4}\colon X \to Y,\ (p_1, \dots, p_4) = \underline{p} \mapsto (\sigma_1(\underline{p}), \dots, \sigma_4(\underline{p}), f(\underline{p})).$$

So V_4 is the Galois group of χ_{V_4} (in the sense of Construction 3.12), and by Corollary 4.9 it is the monodromy group of a covering map obtained by restricting $\chi_{V_4}\colon X \to Y$.

4.2 Geometric Deck Transformations

In Section 2.3 we already saw that for a given covering map $p\colon X \to Y$ with fibre $F = p^{-1}(y_0)$ of the base point $y_0 \in Y$ the deck transformation group Δ can be viewed as a subgroup of the automorphism group $\mathrm{Aut}(F)$ via $\delta\colon \Delta \hookrightarrow \mathrm{Aut}(F),\ D \mapsto D|_F$. We will now consider deck transformations of geometric coverings and connect them to the Galois group of extensions of function fields.

Proposition 4.11. *Let X and Y be affine varieties defined over k and $f\colon X \dashrightarrow Y$ be a dominant rational map. Let $\alpha \in \mathrm{Aut}(K(X)/f^*(K(Y)))$ be an automorphism of $K(X)$ fixing $f^*(K(Y))$. Then α induces a birational map $\hat{\alpha}\colon X \dashrightarrow X$ satisfying $f \circ \hat{\alpha} = f$ on a nonempty open subset $U \subseteq X$.*



Proof. As α fixes $K(Y)$ it is especially a k-algebra isomorphism $K(X) \xrightarrow{\sim} K(X)$. Following the proof of Theorem 4.2 and its notation let $x_1|_X, \dots, x_n|_X$ be the restricted coordinate functions which are generators of $\Gamma(X)$ as a k-algebra. We obtain the following rational maps: $\hat{\alpha}\colon X \dashrightarrow X$ represented by

$$\hat{\alpha}_{U_1}\colon U_1 \to X,\ p \mapsto (\alpha(x_1|_X)(p), \dots, \alpha(x_n|_X)(p))$$

on a nonempty open set $U_1 \subseteq X$ and similarly $\bar{\alpha}\colon X \dashrightarrow X$ represented by

$$\bar{\alpha}_{U_2}\colon U_2 \to X,\ p \mapsto (\alpha^{-1}(x_1|_X)(p), \ldots, \alpha^{-1}(x_n|_X)(p))$$

on a nonempty open set $U_2 \subseteq X$. Then Theorem 4.2 shows $\bar{\alpha}_{U_2} \circ \hat{\alpha}_{U_1} = \mathrm{id}_X$ on a nonempty open subset of $\hat{\alpha}_{U_1}^{-1}(U_2)$ and $\hat{\alpha}_{U_1} \circ \bar{\alpha}_{U_2} = \mathrm{id}_X$ on a nonempty open subset of $\bar{\alpha}_{U_2}^{-1}(U_1)$. So $\hat{\alpha}$ is a birational map.
Consider the inclusion of function fields $f^*\colon K(Y) \hookrightarrow K(X)$. As α fixes $f^*(K(Y))$ by assumption, it follows $\alpha \circ f^* = f^*$. The corresponding rational map of f^* is f, the one of $\alpha \circ f^*$ is $f \circ \hat{\alpha}$ because of $(f \circ \hat{\alpha})^* = \hat{\alpha}^* \circ f^* = \alpha \circ f^*$ and uniqueness from Theorem 4.2. Using the correspondence of Theorem 4.2 it follows $f \circ \hat{\alpha} = f$ as rational maps. □

Using the notation from above we have a map

$$\mathrm{Aut}(K(X)/K(Y)) \to \{\text{Birational maps } X \dashrightarrow X\},\ \alpha \mapsto \hat{\alpha}.$$

In order to show that $\hat{\alpha}$ induces a deck transformation we still need to show that it is an automorphism of some open set. This will also require to restrict the covering map obtained by f to make both maps f and $\hat{\alpha}$ compatible.

Theorem 4.12. *Let X and Y be affine varieties of the same dimension defined over $\mathbb{C}$ and let $f\colon X \to Y$ be a dominant morphism of degree $d > 0$ (which restricts to a covering map $f|_V\colon V \to U$ by Theorem 3.5 and Remark 3.6).*
Let $\alpha \in \mathrm{Aut}(K(X)/f^*(K(Y)))$. Then there exist nonempty open subsets $\mathcal{U} \subseteq U$ and $\mathcal{V} \subseteq V$ such that the restriction $\hat{\alpha}|_{\mathcal{V}}\colon \mathcal{V} \to \mathcal{V}$ of $\hat{\alpha}$ from Proposition 4.11 is a deck transformation of the restricted covering map $f|_{\mathcal{V}}\colon \mathcal{V} \to \mathcal{U}$.

Proof. Using Proposition 4.11 we obtain $\hat{\alpha}\colon X \dashrightarrow X$ represented by $\hat{\alpha}\colon W \to X$ on a nonempty open subset $W \subseteq X$, satisfying $f \circ \hat{\alpha} = f$ on W. By possibly replacing with $W \cap V$ we can assume $W \subseteq V$. Using [GW20, Theorem 10.19, p. 251] we can find a nonempty open set $W' \subseteq \hat{\alpha}(W) \subseteq X$. By restricting $\hat{\alpha}$ to $\hat{\alpha}^{-1}(W')$ we can assume without restriction that $\hat{\alpha}\colon W \to W'$ is surjective. By replacing W' with $W \cap W'$ and restricting to $\hat{\alpha}^{-1}(W \cap W')$ we can assume that $\hat{\alpha}$ is defined on W'.
As f has finite degree $d > 0$ the extension $K(X)/f^*(K(Y))$ has degree d by Theorem 3.10, so

$$n := |\mathrm{Aut}(K(X)/K(Y))| \leq [K(X) : K(Y)] = d.$$

Then $\alpha^n = \mathrm{id}_{K(X)}$ implies $\alpha^{-1} = \alpha^{n-1}$, and Proposition 4.11 gives the birational map $\widehat{\alpha^{n-1}}\colon X \dashrightarrow X$ we can represent by the $(n-1)$-fold composition $\hat{\alpha}^{n-1}\colon W \to W'$: As $\hat{\alpha}$ is defined on W' the composition is well-defined and corresponds to α^{n-1}.

Using [GW20, Theorem 10.19, p. 251] we find an open set $\emptyset \neq W'' \subseteq \hat{\alpha}^{n-1}(W) \subseteq W'$. By restricting both $\hat{\alpha}$ and $\hat{\alpha}^{n-1}$ to the preimage $(\hat{\alpha}^{n-1})^{-1}(W'')$ we can assume $\hat{\alpha}\colon W \to W'$ and $\hat{\alpha}^{n-1}\colon W \to W'$ to be surjective. By replacing W' with $W \cap W'$ and restricting to $(\hat{\alpha}^{n-1})^{-1}(W \cap W')$ we can also assume both maps to be defined on W'.
Using $\alpha^{-1} = \alpha^{n-1}$ and Theorem 4.2 it follows $\hat{\alpha} \circ \hat{\alpha}^{n-1} = \mathrm{id}_W = \hat{\alpha}^{n-1} \circ \hat{\alpha}$. Then for $p \in W \cap W'$ we have $\hat{\alpha}(p) \in W'$ and $\hat{\alpha}^{n-1}(\hat{\alpha}(p)) = p \Rightarrow \hat{\alpha}(p) \in (\hat{\alpha}^{n-1})(p) \subseteq W$, so $\hat{\alpha}(W \cap W') \subseteq W \cap W'$ and similarly $\hat{\alpha}^{n-1}(W \cap W') \subseteq W \cap W'$. With the identity from above it follows that the restriction $\hat{\alpha}|_{\mathcal{W}}\colon \mathcal{W} \to \mathcal{W}$ is a bijection of the open subset $\mathcal{W} := W \cap W'$ onto itself.
We now have to ensure that the open sets on which the bijection $\hat{\alpha}$ and the covering map f are defined on coincide. Again using [GW20, Theorem 10.19, p. 251] we can find a nonempty open subset $\mathcal{U} \subseteq f(\mathcal{W})$. Set $\mathcal{V} := f|_{\mathcal{W}}^{-1}(\mathcal{U})$, which is a nonempty open subset of $\mathcal{W}$. As we have $f \circ \hat{\alpha} = f$ on W it also holds on $\mathcal{W} \subseteq W$. Then it follows $f|_{\mathcal{W}}(\hat{\alpha}(\mathcal{V})) = f|_{\mathcal{W}}(\mathcal{V}) = \mathcal{U} \Rightarrow \hat{\alpha}(\mathcal{V}) \subseteq f|_{\mathcal{W}}^{-1}(\mathcal{U}) = \mathcal{V}$ and analogously $f \circ \hat{\alpha}^{n-1} = f$ on $\mathcal{W}$ shows $\hat{\alpha}^{n-1}(\mathcal{V}) \subseteq \mathcal{V}$. So by restricting we obtain a bijection $\hat{\alpha}|_{\mathcal{V}}\colon \mathcal{V} \to \mathcal{V}$. As $\hat{\alpha}|_{\mathcal{V}}$ and $\hat{\alpha}^{n-1}|_{\mathcal{V}}$ are polynomial maps they are continuous with respect to the analytic topology. So we have an automorphism $\hat{\alpha}|_{\mathcal{V}}\colon \mathcal{V} \xrightarrow{\sim} \mathcal{V}$ which is a deck transformation with respect to the restricted covering map $f|_{\mathcal{V}}\colon \mathcal{V} \to \mathcal{U}$. □

We now want to embed $\mathrm{Aut}(K(X)/K(Y))$ into a deck transformation group Δ. As every element of $\mathrm{Aut}(K(X)/K(Y))$ yields a deck transformation compatible with a possibly different covering map, we have to show that a *common* covering map exists. As the functor from finitely generated field extensions of k to the varieties with dominant rational maps is contravariant we have to consider the opposite group $\mathrm{Aut}(K(X)/K(Y))^{\mathrm{op}}$ in order to obtain a homomorphic embedding.

Theorem 4.13. *In the situation of Theorem 4.12 there exist nonempty open subsets $\mathcal{U} \subseteq Y$ and $\mathcal{V} \subseteq X$ such that for every $\alpha \in \mathrm{Aut}(K(X)/f^*(K(Y)))$ the restriction $\hat{\alpha}|_{\mathcal{V}}\colon \mathcal{V} \to \mathcal{V}$ is a deck transformation of the restricted covering map $f|_{\mathcal{V}}\colon \mathcal{V} \to \mathcal{U}$. Furthermore, the map*

$$\lambda\colon \mathrm{Aut}(K(X)/K(Y))^{\mathrm{op}} \to \Delta(f|_{\mathcal{V}}),\ \alpha \mapsto \hat{\alpha}|_{\mathcal{V}},$$

is an injective group homomorphism.

Proof. As f has finite degree $\mathrm{Aut}(K(X)/K(Y))$ has finite order, and we set $n := |\mathrm{Aut}(K(X)/K(Y))|$. So we can write $\mathrm{Aut}(K(X)/K(Y)) = \{\alpha_1, \dots, \alpha_n\}$. With Theorem 4.12 we obtain nonempty open subsets $V_1, \dots, V_n \subseteq X$ and $U_1, \dots, U_n \subseteq Y$ such that the $\hat{\alpha}_i|_{V_i}\colon V_i \xrightarrow{\sim} V_i$ are deck transformations of the covers $f|_{V_i}\colon V_i \to U_i$. Set $\mathcal{U} := \bigcap_{i=1}^n U_i$ and $\mathcal{V} := \bigcap_{i=1}^n V_i$. Then $\mathcal{U}$ and $\mathcal{V}$ are nonempty, open and $f|_{\mathcal{V}}$

maps to $\mathcal{U}$. As $\mathcal{V} = f|_{\mathcal{V}}^{-1}(\mathcal{U})$ we obtain a restricted covering $f|_{\mathcal{V}} \colon \mathcal{V} \to \mathcal{U}$. As $f \circ \hat{\alpha}_i = f$ holds on each V_i it holds on $\mathcal{V} \subseteq \tilde{\mathcal{V}}$. With the same argument as in the proof of Theorem 4.12 it follows $\hat{\alpha}_i(\mathcal{V}) \subseteq \mathcal{V}$ and analogously $\hat{\alpha}_i{}^{n-1}(\mathcal{V}) \subseteq \mathcal{V}$ follows from $f \circ \hat{\alpha}_i{}^{n-1} = f$ which also holds on $\mathcal{V}$. So we obtain deck transformations $\hat{\alpha}_i|_{\mathcal{V}} \colon \mathcal{V} \xrightarrow{\sim} \mathcal{V}$ for every $i \in \{1, \dots, n\}$.
Now consider $\operatorname{Aut}(K(X)/K(Y))^{\mathrm{op}}$ as the group $\operatorname{Aut}(K(X)/K(Y))$ with opposite group law $\alpha \circledcirc \beta \coloneqq \beta \circ \alpha$. With Theorem 4.2 and Remark 4.3 it follows $\widehat{\beta \circ \alpha} = \hat{\alpha} \circ \hat{\beta}$ as birational maps. So we have

$$\lambda(\alpha \circledcirc \beta) = \lambda(\beta \circ \alpha) = (\widehat{\beta \circ \alpha})|_{\mathcal{V}} = (\hat{\alpha} \circ \hat{\beta})|_{\mathcal{V}} = \hat{\alpha}|_{\mathcal{V}} \circ \hat{\beta}|_{\mathcal{V}} = \lambda(\alpha) \circ \lambda(\beta)$$

and λ is a homomorphism. If $\hat{\alpha}|_{\mathcal{V}} = \lambda(\alpha) = \mathrm{id}_{\mathcal{U}}$ then $\hat{\alpha}$ is represented by the identity on an open set, so it is equal to id_X as a rational map. With the bijection of Theorem 4.2 it follows $\alpha = \mathrm{id}_{K(X)}$ and λ is injective. □

Like in Section 2.3 we can restrict deck transformations to the fibre of a base point in order to further embed $\operatorname{Aut}(K(X)/K(Y))^{\mathrm{op}}$ into the automorphisms of the fibre.

Corollary 4.14. *In the situation of Theorem 4.13 choose a base point $y_0 \in \mathcal{V}$ and let $F \coloneqq f|_{\mathcal{V}}^{-1}(y_0)$ be its fibre. Then composing λ with the restriction res_F onto F yields an injective group homomorphism*

$$\operatorname{Aut}(K(X)/K(Y))^{\mathrm{op}} \xrightarrow{\lambda} \Delta(f|_{\mathcal{V}}) \xrightarrow{\mathrm{res}_F} \operatorname{Aut}(F) \cong S_{|F|},\ \alpha \mapsto \hat{\alpha}|_F.$$

Corollary 4.15. *In the situation of Theorem 4.13 let $y_0 \in \mathcal{V}$ be a base point with fibre $F \coloneqq f|_{\mathcal{V}}^{-1}(y_0)$. If the extension $K(X)/f^*(K(Y))$ is a Galois extension (like in Lemma 4.1) we obtain an injective group homomorphism embedding the opposite Galois Group $\operatorname{Gal}(K(X)/K(Y))^{\mathrm{op}}$ into $\operatorname{Aut}(F)$ in the following way:*

$$\operatorname{Gal}(K(X)/K(Y))^{\mathrm{op}} \hookrightarrow \operatorname{Aut}(K(X)/K(Y))^{\mathrm{op}} \xrightarrow{\lambda} \Delta(f|_{\mathcal{V}}) \xrightarrow{\mathrm{res}_F} \operatorname{Aut}(F) \cong S_{|F|}.$$

Corollary 4.16. *Consider the situation of Corollary 4.15. As $\operatorname{Gal}(K(X)/K(Y))$ is (isomorphic to) a transitive subgroup of $S_{|F|}$ the deck transformation group $\Delta(f|_{\mathcal{V}})$ is transitive as well and the covering $f|_{\mathcal{V}} \colon \mathcal{V} \to \mathcal{U}$ is regular (Definition 2.29). Using Corollary 2.34 and Proposition 2.35 it follows for $x_0 \in F$:*

(i) $\Delta(f|_{\mathcal{V}}) \cong \pi_1(\mathcal{U}, y_0)/\pi_1(\mathcal{V}, x_0)$,

(ii) $\Delta(f|_{\mathcal{V}}) \cong \pi_1(\mathcal{U}, y_0)$ *if $\mathcal{V}$ is simply connected,*

(iii) if $\mathcal{V}$ is simply connected the actions of $\pi_1(\mathcal{U}, y_0)$ on F through $\Delta(f|_{\mathcal{V}})$ and the monodromy coincide if and only if $\pi_1(\mathcal{U}, y_0)$ is abelian.

We will reconsider an example from Section 3.3 and calculate the embedding of $\mathrm{Aut}(K(X)/K(Y))^{\mathrm{op}}$ into the fibre-automorphisms and the deck transformation group $\Delta(f|_{\mathcal{V}})$.

Example 4.17. Consider Example 3.14, i.e. the surjective morphism

$$f\colon X := \mathbb{A}^1(\mathbb{C}) \to \mathbb{A}^1(\mathbb{C}) =: Y,\ z \mapsto z^3.$$

We already saw that the corresponding extension of function fields is given by $f^*\colon K(Y) \hookrightarrow K(X),\ \frac{f(z)}{g(z)} \mapsto \frac{f(z^3)}{g(z^3)}$ and generated by $z \in K(X) = \mathbb{C}(z)$ over $K(Y) = \mathbb{C}(z)$ with minimal polynomial $P := x^3 - z^3 \in (f^*(K(Y)))[x]$. As $|K(X)\colon K(Y)| = \deg(P) = 3$ it follows $|\mathrm{Aut}(K(X)/K(Y))| \leq 3$. The automorphisms

$$\alpha_1\colon \mathbb{C}(z) \to \mathbb{C}(z),\ z \mapsto z,$$
$$\alpha_2\colon \mathbb{C}(z) \to \mathbb{C}(z),\ z \mapsto \exp\left(\frac{2}{3}\pi i\right) z \text{ and}$$
$$\alpha_3\colon \mathbb{C}(z) \to \mathbb{C}(z),\ z \mapsto \exp\left(\frac{4}{3}\pi i\right) z$$

of $K(X)$ fix $K(Y)$, so we have $\mathrm{Aut}(K(X)/K(Y)) = \{\alpha_1 = \mathrm{id}_{\mathbb{C}(z)},\ \alpha_2,\ \alpha_3\}$.
Setting $\mathcal{U} := \mathbb{A}^1(\mathbb{C}) \setminus \{0\}$ and $\mathcal{V} := f^{-1}(\mathcal{U}) = \mathbb{A}^1(\mathbb{C}) \setminus \{0\}$ we obtain automorphisms of $\mathcal{V}$ from the elements of $\mathrm{Aut}(K(X)/K(Y))$:

$$\hat{\alpha_1}|_{\mathcal{V}}\colon \mathcal{V} \to \mathcal{V},\ x \mapsto x,$$
$$\hat{\alpha_2}|_{\mathcal{V}}\colon \mathcal{V} \to \mathcal{V},\ x \mapsto \exp\left(\frac{2}{3}\pi i\right) x \text{ and}$$
$$\hat{\alpha_3}|_{\mathcal{V}}\colon \mathcal{V} \to \mathcal{V},\ x \mapsto \exp\left(\frac{4}{3}\pi i\right) x$$

compatible with the covering map $f|_{\mathcal{V}}\colon \mathcal{V} \to \mathcal{U},\ x \mapsto x^3$, i.e. $f \circ \hat{\alpha}_i = f$ on $\mathcal{V}$ for every $i \in \{1,2,3\}$. So the $\hat{\alpha}_i|_{\mathcal{V}}$ are deck transformations.
Choosing a base point $p := 1 \in \mathcal{U}$ with fibre $F := f^{-1}(p) = \{q_1, q_2, q_3\}$ where $q_1 := 1,\ q_2 := \exp\left(\frac{2}{3}\pi i\right),\ q_3 := \exp\left(\frac{4}{3}\pi i\right)$, we can embed $\mathrm{Aut}(K(X)/K(Y))^{\mathrm{op}}$ into $\Delta(f|_{\mathcal{V}})$ and $\mathrm{Aut}(F) \cong S_3$:

$$\mathrm{Aut}(K(X)/K(Y))^{\mathrm{op}} \xrightarrow{\lambda} \Delta(f|_{\mathcal{V}}) \xrightarrow{\mathrm{res}_F} \mathrm{Aut}(F) \cong S_3,\ \alpha_i \mapsto \hat{\alpha}_i|_F.$$

We have $\mathrm{res}_F \circ \lambda(\alpha_1) = \mathrm{id},\ \mathrm{res}_F \circ \lambda(\alpha_2) = (123)$ and $\mathrm{res}_F \circ \lambda(\alpha_3) = (132)$, so we conclude $\mathrm{Aut}(K(X)/K(Y))^{\mathrm{op}} \cong A_3 \leq S_3$.

Example 4.18. In Example 3.14 and Example 4.17 we saw that

$$\mathrm{Aut}(K(X)/K(Y)) = (\mathrm{Aut}(K(X)/K(Y))^{\mathrm{op}})^{\mathrm{op}} \overset{4.17}{\cong} (A_3)^{\mathrm{op}}$$

$$= A_3 \overset{3.14}{\cong} \mathrm{Gal}(K(X)/K(Y)),$$

and the extension $K(X)/f^*(K(Y))$ is Galois. With the notation from Example 4.17 above the fundamental groups $\pi_1(\mathcal{V}, q_1)$ and $\pi_1(\mathcal{U}, p)$ are fundamental groups of the plane with one puncture. So they are free on one generator and thus infinitely cyclic and isomorphic to $\mathbb{Z}$.

We can represent the generating loop in 1 around 0 by $\gamma\colon I \to U,\ t \mapsto \exp(2\pi i t)$. Then $f_{\#}(\pi_1(\mathcal{V}, q_1)) = f_{\#}(\langle[\gamma]\rangle) = \langle[f\circ\gamma]\rangle \trianglelefteq \langle[\gamma]\rangle = \pi_1(\mathcal{U}, p)$ as the covering is regular. We have $(f\circ\gamma)(t) = \exp(2\pi i t)^3 = \exp(6\pi i t)$ for $t \in I$, and thus $(f\circ\gamma) \simeq \gamma^3$ rel ∂I. So $f_{\#}(\pi_1(\mathcal{V}, q_1)) = \langle[\gamma^3]\rangle$. With Corollary 4.16 we see $\Delta(f|_{\mathcal{V}}) \cong \pi_1(\mathcal{U}, y_0)/\pi_1(\mathcal{V}, x_0) = \langle[\gamma]\rangle/\langle[\gamma^3]\rangle \cong \mathbb{Z}/3\mathbb{Z}$.

A Appendix

A.1 Code used for Examples

The following code written in [VD09, Python 3.8] was used for calculations and the visualization of Figure 4 and Figure 5 in Example 3.15.

```
import numpy as np
import matplotlib.pyplot as plt

def v1(y): return (1/6)*((2**(2/3))*(27*y**4
    + 3*np.sqrt(81*y**8 - 174*y**4 + 93)
    - 29)**(1/3) + (2*2**(1/3))/((27*y**4
    + 3*np.sqrt(81*y**8 - 174*y**4 + 93)
    - 29)**(1/3)) - 2)

def v2(y): return (1/12) * (1j*(2**(2/3))*(np.sqrt(3)
    + 1j)*((27*y**4 + 3*np.sqrt(81*y**8
    - 174*y**4 + 93) - 29)**(1/3))
    - (2*1j*(2**(1/3))*(np.sqrt(3)- 1j))
    / ((27*(y**4) + 3*np.sqrt(81*y**8
    - 174*(y**4) + 93) - 29)**(1/3)) - 4)

def v3(y): return (1/12)*((-2**(2/3))*(1 + 1j*np.sqrt(3))
    * ((27*y**4 + 3*np.sqrt(81*y**8 - 174*y**4 + 93)
    - 29)**(1/3)) + (2*1j*(2**(1/3))*(np.sqrt(3) + 1j))
    / ((27*(y**4) + 3*np.sqrt(81*y**8 - 174*(y**4)
    + 93) - 29)**(1/3)) - 4)

def y(t): return np.exp((1/2)*np.pi*1j*t)
    # np.exp((1/2)*np.pi*1j*t + (1/2)*np.pi*1j)
    # np.exp((1/2)*np.pi*1j*t + np.pi*1j)
    # np.exp((1/2)*np.pi*1j*t + (3/2)*np.pi*1j)

def y2(t): return 2*np.exp(2*np.pi*1j*t) - 1
    # 2*np.exp(2*np.pi*1j*t + np.pi*1j) + 1
    # 2*np.exp(2*np.pi*1j*t + (3/2)*np.pi*1j) + 1j
    # 2*np.exp(2*np.pi*1j*t + (1/2)*np.pi*1j) - 1j

p = [0.23+0.79*1j,0.23-0.79*1j,-1.47,0]
```

```python
re = [x.real for x in p]
im = [x.imag for x in p]
r1 = []
i1 = []
r2 = []
i2 = []
r3 = []
i3 = []
res = 0

t = 0
while t <= 1:
    print(t)

    res = v1(y(t))
    print(res)
    r1.append(res.real)
    i1.append(res.imag)

    res = v2(y(t))
    print(res)
    r2.append(res.real)
    i2.append(res.imag)

    res = v3(y(t))
    print(res)
    r3.append(res.real)
    i3.append(res.imag)

    t += 0.0001

plt.figure(dpi=1200)
plt.plot(r1,i1,'.', color = 'blue')
plt.plot(r2,i2,'.', color = 'green')
plt.plot(r3,i3,'.', color = 'red')
plt.plot(re, im, '.', color = 'black')
plt.show()
```

References

[BCR98] Jacek Bochnak, Michel Coste and Marie-Francoise Roy. *Real Algebraic Geometry*. Ergebnisse der Mathematik und ihrer Grenzgebiete - A Series of Modern Surveys in Mathematics. Springer-Verlag Berlin Heidelberg GmbH, 1998. ISBN: 978-3-642-08429-4.

[Bos09] Siegfried Bosch. *Algebra*. Springer-Lehrbuch. Springer-Verlag Berlin Heidelberg, 2009. ISBN: 978-3-540-92811-9.

[Bre93] Glen E. Bredon. *Topology and Geometry*. Graduate Texts in Mathematics. Springer Science+Business Media New York, 1993. ISBN: 978-1-4419-3103-0.

[FG02] Klaus Fritzsche and Hans Grauert. *From Holomorphic Functions to Complex Manifolds*. Graduate Texts in Mathematics. Springer-Verlag New York. Inc, 2002. ISBN: 978-1-4419-2983-9.

[GS] Daniel R. Grayson and Michael E. Stillman. *Macaulay2, a software system for research in algebraic geometry*. URL: `http://www.math.uiuc.edu/Macaulay2/`.

[GW20] Ulrich Görtz and Torsten Wedhorn. *Algebraic Geometry I: Schemes*. Springer Studium Mathematik – Master. Springer Fachmedien Wiesbaden GmbH, 2020. ISBN: 978-3-658-30732-5.

[Har77] Robin Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics. Springer Science+Business Media, Inc., 1977. ISBN: 978-1-4419-2807-8.

[Har79] Joe Harris. 'Galois groups of enumerative problems'. In: *Duke Mathematical Journal* 46.4 (1979), pp. 685–724. DOI: `10.1215/S0012-7094-79-04635-0`. URL: `https://doi.org/10.1215/S0012-7094-79-04635-0`.

[Har92] Joe Harris. *Algebraic Geometry - A first course*. Graduate Texts in Mathematics. Springer Science+Business Media New York, 1992. ISBN: 978-1-4419-3099-6.

[Hat02] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2002. ISBN: 0-521-79540-0.

[Jän01] Klaus Jänich. *Vector Analysis*. Undergraduate Texts in Mathematics. Springer Science+Business Media New York, 2001. ISBN: 978-1-4419-3144-3.

[KW89] Luise-Charlotte Kappe and Bette Warren. 'An Elementary Test for the Galois Group of a Quartic Polynomial'. In: *The American Mathematical Monthly* 96.2 (1989), pp. 133–137. ISSN: 00029890, 19300972. URL: `http://www.jstor.org/stable/2323198`.

[Sha88] Igor Shafarevich. *Basic Algebraic Geometry 2*. Springer-Verlag Berlin Heidelberg, 1988. ISBN: 978-3-540-57554-2.

[VD09] Guido Van Rossum and Fred L. Drake. *Python 3 Reference Manual*. Scotts Valley, CA: CreateSpace, 2009. ISBN: 1441412697.